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P.3

# Pure Maths- 3

## Differentiation and Integration.
## Ex. 1(b). Solution (Revision)

| | | | | |
|---|---|---|---|---|
| S-22 | S-23 | W-20 | W-21 | W-22 |

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1. Let $I=\int_0^3 \frac{27}{(9+x^2)^2}\,dx$

(a) Using the substitution $x=3\tan\theta$, show that $I=\int_0^{\frac{1}{4}\pi}\cos^2\theta\,d\theta$ ---[4]

(b) Hence find the exact value of $I$. --[4]

[S-22/31/Q6]

Solution: $I=\int_0^3 \frac{27}{(9+x^2)}\,dx$

(a) $=\int_0^{\frac{\pi}{4}} \frac{27\cdot\sec^2\theta\,d\theta}{(9+(3\tan\theta)^2)^2}$

$\left\{\begin{array}{l}\text{Put } x=3\tan\theta \\ \text{Diff } \frac{dx}{d\theta}=3\sec^2\theta \Rightarrow dx=3\sec^2\theta\,d\theta\end{array}\right.$

adjust the limits

$=\int_0^{\frac{\pi}{4}} \frac{27\times3\sec^2\theta\,d\theta}{[9(1+\tan^2\theta)]^2}$

$\left\{\begin{array}{l}x=3\tan\theta \\ x=0\Rightarrow\tan\theta=0\Rightarrow\theta=0 \\ x=3\Rightarrow\tan\theta=1\Rightarrow\theta=\frac{1}{4}\pi\end{array}\right.$

$=\int_0^{\frac{\pi}{4}} \frac{81\sec^2\theta\,d\theta}{81\cdot\sec^4\theta}$ $=\int_0^{\frac{\pi}{4}}\frac{1}{\sec^2\theta}\,d\theta=\int_0^{\frac{\pi}{4}}\cos^2\theta\,d\theta$ ✓

(b) $I=\int_0^{\frac{\pi}{4}}\cos^2\theta\,d\theta=\frac{1}{2}\int_0^{\pi}(1+\cos2\theta)\,d\theta$

$=\frac{1}{2}\left[\theta+\frac{\sin2\theta}{2}\right]_0^{\frac{\pi}{4}}=\frac{1}{2}\left[\left(\frac{\pi}{4}+\frac{1}{2}\sin\frac{\pi}{2}\right)-(0+0)\right]$

$=\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\times1\right)=\frac{1}{8}(\pi+2)$ ✓

2. The equation of the curve is $x^3+y^3+2xy+8=0$

(a) Find $\frac{dy}{dx}$ in terms of $x$ and $y$. ---[4]

The tangent to the curve at the point where $x=0$ and the tangent at the point where $y=0$ intersect at the acute angle $\alpha$.

(b) Find the exact value of $\tan\alpha$. ---[5]

[S-22/31/Q8]

Solution: $x^3+y^3+2xy+8=0$ ---(1)

(a) Diff. w.r.t $x$.

$3x^2+3y^2\frac{dy}{dx}+2\left(x\frac{dy}{dx}+y\times1\right)=0$

$\frac{dy}{dx}(3y^2+2x)+(3x^2+2y)=0$

$\frac{dy}{dx}=-\frac{(3x^2+2y)}{3y^2+2x}$ ✓ ——(2)

(b) At $x=0$ from (1) $y=-2$; $(0,-2)$

at $(0,-2)$ from (2) gradient $=\frac{1}{3}$ ✓ —— (3)

At $y=0$ from (1) $x=-2$, $(-2,0)$

Gradient $=3$ ✓ ⟶ at $(-2,0)$

$\tan A=3$, $\tan B=\frac{1}{3}$

$\tan\alpha=\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\cdot\tan B}$

$=\frac{3-\frac{1}{3}}{1+3\times\frac{1}{3}}=\frac{4}{3}$

$\therefore\tan\alpha=\frac{4}{3}$ ✓

3. The equation of a curve is $y = \cos^3 x . \sqrt{\sin x}$. It is given that the curve has one stationary point in the interval $0 < x < \frac{1}{2}\pi$. Find the x-coordinate of this stationary point, giving your answer correct to 3 significant figures. ---[6]

[S-22/32/Q4]

**Solution:**

$y = \cos^3 x . \sqrt{\sin x}$ ---(1)

$\frac{dy}{dx} = \cos^3 x \times \frac{1}{2\sqrt{\sin x}} \times \cos x + \sqrt{\sin x} \times 3\cos^2 x . (-\sin x)$

$= \frac{\cos^4 x - \sin^2 x . \cos^2 x}{2\sqrt{\sin x}}$ ---(2)

for any stationary point $\frac{dy}{dx} = 0$

$\Rightarrow$ from (2)

$\cos^4 x - \sin^2 x \cos^2 x = 0$

$\cos^2 x(\cos^2 x - \sin^2 x) = 0 \Rightarrow \cos^2 x = 0$ or $\tan^2 x = \frac{1}{6}$

$\Rightarrow \cos x = 0, \ \tan x = \frac{1}{\sqrt{6}}$

$\Rightarrow x = \frac{\pi}{2}$ ✗ ; $0 < x < \pi/2$

$x = \tan^{-1}\frac{1}{\sqrt{6}} = \tan^{-1}(0.4082) = 0.388$ radians

4. The equation of the curve is $x^3 + 3x^2 . y - y^3 = 3$.

(a) Show that $\frac{dy}{dx} = \frac{x^2 + 2xy}{y^2 - x^2}$ ---[4]

(b) Find the coordinates of the points on the curve where the tangent is parallel to the x-axis. ---[5]

[S-22/32/Q7]

**Solution:** Curve: $x^3 + 3x^2 . y - y^3 = 3$ ---(1)

(a) Diff. w.r.t $x$:

$3x^2 + 3[x^2 . \frac{dy}{dx} + y . 2x] - 3y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx}(3x^2 - 3y^2) = -(3x^2 + 6xy)$

$\frac{dy}{dx} = \frac{-3(x^2 + 2xy)}{-3(y^2 - x^2)}$

$\frac{dy}{dx} = \frac{x^2 + 2yx}{y^2 - x^2}$ ---(2)

(b) Tangent is parallel to x-axis

$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow \frac{x^2 + 2y}{y^2 - x^2} = 0$

$\Rightarrow x^2 + 2xy = 0$

$x(x + 2y) = 0$

$x = 0, \quad x = -2y$

from (1) $x = 0 \Rightarrow -y^3 = 3 \Rightarrow y = -\sqrt[3]{3}$

$(0, -\sqrt[3]{3})$

and at from (1)

$x = -2y \Rightarrow -8y^3 + 12y^3 - y^3 = 3$

$3y^3 = 3 \Rightarrow y = 1$

$x = -2$

$\therefore (-2, 1)$ and $(0, -\sqrt[3]{3})$

5 Let $f(x)=\dfrac{x^2+9x}{(3x-1)(x^2+3)}$

(a) Express $f(x)$ in partial fractions. ---[5]

(b) Hence find $\int_1^3 f(x)\,dx$, giving your answer in a simplied exact form

[S-22/32/Q 8] --[5]

Solution:

(a) $\dfrac{x^2+9x}{(3x-1)(x^2+3)} = \dfrac{a}{(3x-1)} + \dfrac{bx+c}{x^2+3}$ ---(1)

multiply (1) by $(3x-1) \Rightarrow \dfrac{x^2+x}{(x^2+3)} = a + \dfrac{(bx+c)}{x^2+3}$ —— (2)

Put $3x-1=0 \Rightarrow x=1/3$ in (2) $\dfrac{(\frac{1}{3})^2+\cdot\times\frac{1}{3}}{(\frac{1}{3})^2+3} = a \Rightarrow a=\dfrac{\frac{1}{9}+3}{\frac{1}{9}+3}=1$ ✓

multiply (1) by $D^r \to (3x-1)(x^2+3)$ and replace $a=1$

$\Rightarrow x^2+9x = 1(x^2+3)+(bx+c)(3x-1)$ —— (3)

Comparing coeff of $x^2 \to 1=1+3b = b=0$ ✓

Comparing Constant term $\to 0=3-C \Rightarrow C=0$ ✓

$\therefore$ from (1) Required Partial fractions: $f(x)=\dfrac{1}{3x-1}+\dfrac{3}{x^2+3}$ ✓ (4)

(b) $\int_1^3 f(x)\,dx = \int_1^3 \dfrac{x^2+9x}{(3x-1)(x^2+3)}\,dx$

$= \int_1^3 \left(\dfrac{1}{3x-1}+\dfrac{3}{x^2+3}\right)dx$

$= \int_1^3 \left(\dfrac{1}{3x-1}+\dfrac{3}{3(1+\frac{x^2}{3})}\right)dx$

$= \int_1^3 \left(\dfrac{1}{3x-1}+\dfrac{1}{1+(\frac{x}{\sqrt{3}})^2}\right)dx$

$= \left[\dfrac{1}{3}\ln(3x-1)+\sqrt{3}\tan^{-1}\dfrac{x}{\sqrt{3}}\right]_1^3$ $\left[\because \int \dfrac{1}{1+x^2}dx = \tan^{-1}x\right]$

$= \left[\left(\frac{1}{3}\ln 8+\sqrt{3}\tan^{-1}\sqrt{3}\right)-\left(\frac{1}{3}\ln 2+\sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}}\right)\right]$

$= \frac{1}{3}\ln 2^3 - \frac{1}{3}\ln 2 + \sqrt{3}\cdot\frac{\pi}{3} - \sqrt{3}\cdot\frac{\pi}{6}$

$= \ln 2 - \frac{1}{3}\ln 2 + \sqrt{3}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$

$= \frac{2}{3}\ln 2 + \frac{\sqrt{3}\pi}{6}$ ✓

6. The curve $y = e^{-4x} \cdot \tan x$ has two stationary points in $0 \leq x < \frac{1}{2}\pi$

(a) Obtain an expression for $\frac{dy}{dx}$ and show it can be written in the form $\sec^2 x \cdot (a + b \sin 2x) \cdot e^{-4x}$, where $a$ and $b$ are constants. ---[4]

(b) Hence find the exact x-coordinates of the two stationary points ---[3]

[S-22/33/Q4]

**Solution:** $y = e^{-4x} \cdot \tan x$ ---- ①

(a) $\frac{dy}{dx} = e^{-4x} \cdot \sec^2 x + \tan x \cdot e^{-4x} \cdot (-4)$

$= e^{-4x} \left[\sec^2 - 4 \frac{\sin x}{\cos x}\right]$

$= e^{-4x} \left[\frac{1 - 4 \sin x \cos x}{\cos^2 x}\right]$

$= e^{-4x} \left[1 - 2 \sin 2x\right] \cdot \sec^2 x$ - ②

(b) for stationary points $\frac{dy}{dx} = 0$

from ② $e^{-4x} [1 - 2 \sin 2x] \cdot \sec^2 x = 0$

$\Rightarrow \sin 2x = \frac{1}{2}$ ; $\{0 \leq x < \frac{1}{2}\pi$, $0 \leq 2x < \pi\}$

$2x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$2x = \frac{\pi}{6}, \frac{5\pi}{6}$

Stationary Points

$\therefore x = \pi/12$ ; $x = 5\pi/12$

7. The parametric equations of a curve are:

$x = \frac{1}{\cos t}$ ; $y = \ln \tan t$, where $0 < t < \frac{1}{2}\pi$

(a) Show that: $\frac{dy}{dx} = \frac{\cos t}{\sin^2 t}$ ---[5]

(b) Find the equation of the tangent to the curve at the point where $y = 0$ ---[3]

[S-22/33/Q6]

**Solution:** $x = \frac{1}{\cos t} = \sec t$ ---- ①

(a) $\frac{dx}{dt} = \sec t \tan t$ -- ②

and $y = \ln \tan t$ --- ③

$\frac{dy}{dt} = \frac{1}{\tan t} \cdot \sec^2 t$ --- ④

Now $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$

$= \frac{\sec^2 t}{\tan t} / \frac{\sec t \cdot \tan t}{1}$

$= \frac{\sec^2 t}{\tan t} \times \frac{1}{\sec t \cdot \tan t}$

$\frac{dy}{dx} = \frac{\sec t}{\tan^2 t} = \frac{1}{\cos t} \times \frac{\cos^2 t}{\sin^2 t} = \frac{\cos t}{\sin^2 t}$ - ⑤

(b) for $y = 0 \Rightarrow \ln \tan t = 0$

$\Rightarrow \tan t = 1$ [$\because \ln 1 = 0$]

$\Rightarrow t = \pi/4$

from ① $x = \sec \frac{\pi}{4}$ [at $t = \frac{\pi}{4}$]

$= \sqrt{2}$

$\therefore$ Point $(\sqrt{2}, 0)$,

from ⑤ gradient at $t = \frac{\pi}{4}$

$= \cos\frac{\pi}{4} / \sin^2\frac{\pi}{4} = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{2}}$

$m = \sqrt{2}$

$\therefore$ Equation of tangent at $(\sqrt{2}, 0)$

$y - 0 = \sqrt{2}(x - \sqrt{2})$

$\Rightarrow y = \sqrt{2}x - 2$

8. The equation of a curve is $x^2y - ay^2 = 4a^3$, where a is a non-zero constant.

(a) Show that $\frac{dy}{dx} = \frac{2xy}{2ay - x^2}$ ---[4]

(b) Hence find the coordinates of the points where the tangent to the curve is parallel to the y-axis. --[4]

S-23|31|Q5

Solution (a) Curve: $x^2y - ay^2 = 4a^3$ --- ①

Diff. w.r.t $x$.

$2xy + x^2\frac{dy}{dx} - 2ay\frac{dy}{dx} = 0$

$\frac{dy}{dx}(x^2 - 2ay) = -2xy$

$\Rightarrow \frac{dy}{dx} = \frac{-2xy}{(x^2-2ay)} = \frac{+2xy}{+(2ay-x^2)}$ ✓

(b) from part ⓐ $\frac{dy}{dx} = \frac{2xy}{(2ay-x^2)}$ --- ②

for tangent parallel to y-axis, $\frac{dy}{dx} = \infty$ (Not or defined)

from ②

$2ay - x^2 = 0 \Rightarrow x^2 = 2ay$ --- ③

from ③ in ①

$2ay \times y - ay^2 = 4a^3 \Rightarrow ay^2 = 4a^3$

$\Rightarrow y^2 = 4a^2 \Rightarrow y = \pm 2a$

from ③ $x^2 = 2a \times 2a = 4a^2$ [from $x = 2a$]

$x = \pm 2a$ [for $y = -2a$, ③ is not defined, $x^2 = -4a^2$]

∴ Required points are: ✓

$(2a, 2a), (-2a, 2a)$ ✓

9. Let $f(x) = \frac{3-3x^2}{(2x+1)(x+2)^2}$

(a) Express $f(x)$ in partial fractions. ---[5]

(b) Hence find the exact value of $\int_0^4 f(x)\,dx$, giving your answer in the form $a + b\ln c$, where a, b and c are in integers. ---[5]

S-23|31|Q8

Solution: $f(x) = \frac{3-3x^2}{(2x+1)(x+2)^2} = \frac{A}{(2x+1)} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)}$ --- ①

(a) multiply ① by the denominator $(2x+1)(x+2)^2 \rightarrow$

$3 - 3x^2 = A(x+2)^2 + B(2x+1) + C(x+2)(2x+1)$ --- ②

To find the value of A, $2x+1=0 \Rightarrow$ put $x = -\frac{1}{2}$ in ②: $3 - 3(-\frac{1}{2})^2 = A(-\frac{1}{2}+2)^2 + 0 + 0$

$\Rightarrow 9/4 = 9/4 A \Rightarrow$ $A = 1$ ✓

Again to find the value of B, $x+2=0 \Rightarrow$ put $x=-2$ in ②: $3 - 3(-2)^2 = 0 + B(2\times(-2)+1)$

$\Rightarrow -9 = -3B \Rightarrow$ $B = 3$ ✓

Put the values $A=1$ and $B=3$ in ②

$\Rightarrow 3 - 3x^2 = (x+2)^2 + 3(2x+1) + C(2x+1)(x+2)$ --- ③

Put $x=0$ in ③ $\Rightarrow$ $3 = 4 + 3 + 2C \Rightarrow C = -2$ ✓

Put the value of $A=1$, $B=3$ and $C=-2$ in ①

The required partial fractions: $f(x) = \frac{1}{(2x+1)} + \frac{3}{(x+2)^2} + \frac{-2}{(x+2)}$ ✓

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9 (b) $\int_0^4 f(x)\,dx = \int_0^4 \left(\frac{1}{2x+1} + \frac{-2}{(x+2)} + 3(x+2)^{-2}\right)dx$

$= \left[\frac{1}{2}\ln(2x+1) - 2\ln(x+2) + -3(x+2)^{-1}\right]_0^4$

$= \left(\frac{1}{2}\ln 9 - 2\ln 6 - \frac{3}{6}\right) - \left(\frac{1}{2}\ln 1 - 2\ln 2 - \frac{3}{2}\right)$

$= \ln 3 - \ln 36 + \ln 4 - \frac{1}{2} + \frac{3}{2} = \ln\frac{3\times 4}{36} + 1 = \ln\frac{1}{3} + 1$

$= 1 - \ln 3$ ✓

10. The equation of a curve is $3x^2 + 4xy + 3y^2 = 5$

(a) Show that: $\frac{dy}{dx} = -\frac{3x+2y}{2x+3y}$ --- [4]

(b) Hence find the exact coordinates of the two points on the curve at which the tangent is parallel to $y + 2x = 0$ --- [5]

S-23/32/Q7

Solution (a) Curve: $3x^2 + 4xy + 3y^2 = 5$ --- ①

diff. w.r.t $x$,

$6x + 4\left(x\cdot\frac{dy}{dx} + 1\times y\right) + 6y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx}(4x+6y) = -(6x+4y)$ ✓

$\Rightarrow \frac{dy}{dx} = -\frac{2(3x+2y)}{2(2x+3y)} = -\frac{3x+2y}{2x+3y}$ --- ②

(b) Line: $y + 2x = 0$ --- ③ (Gradient of line ③ = −2)

Gradient of tangent = −2 --- ④ (Tangents to the curve are parallel to line ③)

for tangent is parallel to line ③

from ② & ③ $-\frac{3x+2y}{2x+3y} = -2$

$\Rightarrow 3x + 2y = 2(2x+3y)$

$\Rightarrow 4y = -x$

$\Rightarrow y = -\frac{x}{4}$ --- ⑤

from ① and ⑤

$3x^2 + 4x\left(-\frac{x}{4}\right) + 3\left(-\frac{x}{4}\right)^2 = 5$

$\Rightarrow 3x^2 - x^2 + \frac{3x^2}{16} = 5$

$\Rightarrow \frac{35x^2}{16} = 5 \Rightarrow x^2 = \frac{16}{7}$

$x = \pm\frac{4}{\sqrt{7}}$ ✓   from ⑤ $x = \frac{4}{\sqrt{7}}, y = -\frac{1}{\sqrt{7}}$ ; $x = -\frac{4}{\sqrt{7}}, y = \frac{1}{\sqrt{7}}$

∴ Required coordinates $\left(\frac{4}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right)$ & $\left(-\frac{4}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$ ✓

11. Let $f(x) = \dfrac{2x^2+17x-17}{(1+2x)(2-x)^2}$

(a) Express $f(x)$ in partial fractions. ....[5]

(b) Hence show that $\int_0^1 f(x)\,dx = \frac{5}{2} - \ln 72$. ....[5]

S-23|32|Q9

**Solution(a)** $f(x) = \dfrac{2x^2+17x-17}{(1+2x)(2-x)^2} = \dfrac{A}{1+2x} + \dfrac{B}{(2-x)^2} + \dfrac{C}{(2-x)}$ ---- ①

multiply ① by denominator: $2x^2+17x-17 = A(2-x)^2 + B(1+2x) + C(1+2x)(2-x)$ --- ②

To get the value of A, $1+2x=0 \Rightarrow$ put $x=-\frac{1}{2}$ in ② $\Rightarrow 2\left(-\frac{1}{2}\right)^2 + 17\left(-\frac{1}{2}\right) - 17 = A\left(2+\frac{1}{2}\right)^2 + 0 + 0$

$\Rightarrow -5/2 = \frac{25}{4}A \Rightarrow$ $A = -4$ ✓

To get B, $2-x=0$, put $x=2$ in ② $\Rightarrow 2(-2)^2 + 17\times 2 - 17 = 0 + B(1+2\times 2) + 0$

$\Rightarrow 25 = 5b \Rightarrow$ $B = 5$

Put the values of A and B in ②

$2x^2+17x-17 = -4(2-x)^2 + 5(1+2x) + C(1+2x)(2-x)$ --- ③

Put $x=0$ in ③ $\Rightarrow -17 = -4\times 4 + 5\times 1 + 2C \Rightarrow C = -3$ ✓

Put $A=-4$, $B=5$, $C=-3$ in ①

The required partial fractions: $f(x) = \dfrac{-4}{1+2x} + \dfrac{5}{(2-x)^2} + \dfrac{-3}{(2-x)}$ --- ④

(b) $\int_0^1 f(x)\,dx = \int_0^1 \left( \dfrac{-4}{1+2x} + \dfrac{-3}{(2-x)} + \dfrac{5}{(2-x)^2} \right) dx$ ----- ⑤

$= \left[ -\frac{4}{2}\ln(1+2x) - \frac{3}{-1}\ln(2-x) + \frac{5}{-1\times -1(2-x)} \right]_0^1$

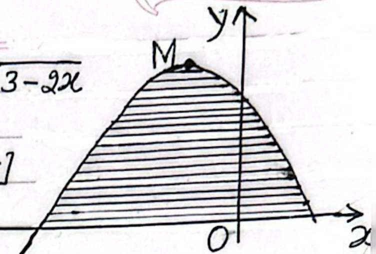
$= \left[ -2\ln(1+2x) + 3\ln(2-x) + \frac{5}{(2-x)} \right]_0^1$

$= (-2\ln 3 + 3\ln 1 + 5) - (-2\ln 1 + 3\ln 2 + \frac{5}{2})$

$= -\ln 9 + 0 + 5 - \frac{5}{2} + 0 - \ln 8$

$= \frac{5}{2} - (\ln 9 + \ln 8)$

$= \frac{5}{2} - \ln 72.$ ✓




12. The diagram shows the curve $y=(x+5)\sqrt{3-2x}$ and its maximum point M.

(a) Find the exact coordinates of M. ... [5]

(b) Using the substitution $u=3-2x$, find by integration the area of the shaded region bounded by the curve and the x-axis. Give your answer in the form $a\sqrt{13}$, where $a$ is a rational number. [5]

S-23/32/Q10

**Solution** (a) Curve: $y=(x+5)\sqrt{3-2x}$ ... ①

$\Rightarrow \frac{dy}{dx} = 1\times(3-2x)^{1/2} + (x+5)\times\frac{1\times -2}{2\sqrt{3-2x}}$

$= \frac{(3-2x)-(x+5)}{-4\sqrt{3-2x}}$

$= \frac{3-2x-x-5}{\sqrt{3-2x}}$

$= \frac{-2-3x}{\sqrt{3-2x}}$ ---- ②

For the point of Maximim,

$\frac{dy}{dx}=0 \Rightarrow -2-3x=0$ fm ②

$\Rightarrow x=-2/3$

from ① $y=\left(-\frac{2}{3}+5\right)\sqrt{3-2\left(-\frac{2}{3}\right)}$

$y=\frac{13}{3}\cdot\sqrt{\frac{13}{3}}=\frac{13\sqrt{39}}{9}$ ✓

$\therefore M\left(-\frac{2}{3}, \frac{13}{9}\sqrt{39}\right)$ ✓

(b) Curve intersect the x-axis at $y=0$

from ① $(x+5)\sqrt{3-2x}=0$

$\Rightarrow -5,\ x=3/2$.

$\therefore$ Area of the shaded region.

$= \int_{-5}^{3/2} y\,dx = \int_{-5}^{3/2}(x+5)\sqrt{3-2x}\,dx$ --- ③

Put $u=3-2x$ ✓ -- ④

$x=\frac{3-u}{2}$

$\frac{du}{dx}=-2$

$\Rightarrow dx=-\frac{1}{2}du$ ✓

$x+5=\frac{3-u}{2}+5$

$=\frac{13-u}{2}$ ✓ -- ⑤

also limits

$x=-5 \Rightarrow u=13$

$x=\frac{3}{2} \quad u=0$

hence from ③, ④, ⑤,

Area $= \int_{13}^{0}\left(\frac{13-u}{2}\right)\cdot u^{1/2}\times -\frac{1}{2}du$

$= \frac{1}{4}\int_0^{13}(13u^{1/2}-u^{3/2})\,du$

$= \frac{1}{4}\left[13\frac{u^{3/2}}{3/2}-\frac{u^{5/2}}{5/2}\right]_0^{13}$

$= \frac{1}{4}\left[\frac{26}{3}(13)^{3/2}-\frac{2}{5}(13)^{5/2}-0\right]$

$= \frac{1}{4}\times(13)^{3/2}\left[\frac{26}{3}-\frac{2}{5}\times 13\right]$

$= \frac{1}{4}\times 13\sqrt{13}\left(\frac{52}{15}\right)$

$= \frac{169}{15}\sqrt{13}$ ✓

13. The parametric equations of a curve are,

$x = \frac{\cos\theta}{2-\sin\theta}$, $y = \theta + 2\cos\theta$; Show that: $\frac{dy}{dx} = (2-\sin\theta)^2$ ....[5]

S-23/33/Q4

**Solution:** $y = \theta + 2\cos\theta$ ; $x = \frac{\cos\theta}{2-\sin\theta}$

$\Rightarrow \frac{dy}{d\theta} = 1 - 2\sin\theta$ ---(1) and $\frac{dx}{d\theta} = \frac{-(2-\sin\theta)\cdot\sin\theta - \cos\theta(-\cos\theta)}{(2-\sin\theta)^2}$

$= \frac{-2\sin\theta + \sin^2\theta + \cos^2\theta}{(2-\sin\theta)^2}$

$= \frac{(1-2\sin\theta)}{(2-\sin\theta)^2}$ ---(2)

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$ from (1) & (2)

$= (1-2\sin\theta) \div \frac{(1-2\sin\theta)}{(2-\sin\theta)^2}$

$= (1-2\sin\theta) \times \frac{(2-\sin\theta)^2}{(1-2\sin\theta)} \Rightarrow \frac{dy}{dx} = (2-\sin\theta)^2$ ✓

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14(a) Use the substitution $u = \cos x$ to show that:

$\int_0^{\pi} \sin 2x\, e^{2\cos x}\, dx = \int_{-1}^{1} 2u\, e^{2u}\, du$ ---[4]

(b) Hence find the exact value of: $\int_0^{\pi} \sin 2x \cdot e^{2\cos x}\, dx$ ---[4]

S-23/33/Q7

**Solution (a)** $\int_0^{\pi} \sin 2x \cdot e^{2\cos x}\, dx$

$= \int_0^{\pi} 2\sin x\cos x\, dx$

$= -\int_0^{\pi} 2\cos x \cdot e^{2\cos x} \cdot (-\sin x)\, dx$

$= -\int_1^{-1} 2u\, e^{2u}\, du = \int_{-1}^{1} 2u\, e^{2u}\, du$ ✓

Put $u = \cos x$
$du = -\sin x\, dx$
adjust the limits
$x = 0 \rightarrow u = 1$
$x = \pi \rightarrow u = -1$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(b) $\int_0^{\pi} \sin 2x\, e^{2\cos x}\, dx = \int_{-1}^{1} 2u\, e^{2u}\, du$ (Part(a)) --(1)

Let $\frac{dv}{du} = e^{2u}$
$\Rightarrow v = \frac{e^{2u}}{2}$

Consider $2\int u \cdot e^{2u}\, du = 2\left[u \cdot \frac{e^{2u}}{2} - \int \frac{e^{2u}}{2} \times \frac{d}{du} u\, du\right]$

$= 2\left[u\frac{e^{2u}}{2} - \frac{1}{2}\int e^{2u}\, du\right] = 2\left[\frac{1}{2}ue^{2u} - \frac{1}{2} \times \frac{e^{2u}}{2}\right] = ue^{2u} - \frac{1}{2}e^{2u}$ ---(2)

Hence from (1) and (2) $\int_{-1}^{1} 2u\, e^{2u}\, du = \left[ue^{2u} - \frac{1}{2}e^{2u}\right]_{-1}^{1} = \left(e^2 - \frac{1}{2}e^2\right) - \left(-e^{-2} - \frac{1}{2}e^{-2}\right)$

$= \frac{1}{2}e^2 + \frac{3}{2}e^{-2}$

15 The parametric equations of a curve are

$$x = 3 - \cos 2\theta, \quad y = 2\theta + \sin 2\theta \text{ for } 0 < \theta < \tfrac{1}{2}\pi$$

Show that $\frac{dy}{dx} = \cot\theta$. ---[5]

[W-20/31/Q 3]

**Solution:** $x = 3 - \cos 2\theta$ and $y = 2\theta + \sin 2\theta$

$\Rightarrow \frac{dx}{d\theta} = +\sin 2\theta \times 2$ ; $\frac{dy}{d\theta} = 2 + 2\cos 2\theta$ ✓

$= 2\sin 2\theta$ ✓

Now $\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$

$= \frac{2 + 2\cos 2\theta}{2\sin 2\theta} = \frac{2(1+\cos 2\theta)}{2\sin 2\theta}$

$= \frac{2\cos^2\theta}{2\sin\theta \cdot \cos\theta}$

$\therefore \frac{dy}{dx} = \frac{\cos\theta}{\sin\theta} = \cot\theta$ ✓

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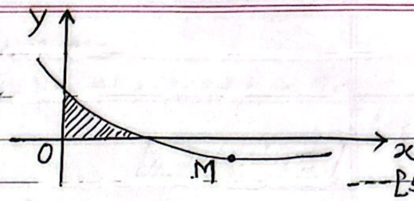
16. The diagram shows the curve $y = (2-x)e^{-\frac{1}{2}x}$, and its minimum point M.




(a) Find the exact coordinates of M. ---[5]

(b) Find the area of the shaded region bounded by the curve and the axes. Give your answer in terms of e. [W-20/31/Q10] ---[5]

<u>Solution:</u> $y = (2-x)\cdot e^{-\frac{1}{2}x}$ —— ①

(a) diff w.r.t $x$; $\frac{dy}{dx} = -e^{-\frac{1}{2}x} + (2-x)e^{-\frac{1}{2}x} \times (-\frac{1}{2})$

$= e^{-\frac{1}{2}x}[-1 - \frac{1}{2}(2-x)]$

$\frac{dy}{dx} = (\frac{x}{2} - 2)e^{-\frac{1}{2}x}$ —— ②

for any max/Min $\frac{dy}{dx} = 0 \Rightarrow (\frac{x}{2}-2)e^{-\frac{1}{2}x} = 0$

$\Rightarrow \frac{x}{2} - 2 = 0 \Rightarrow x = 4$

from ① $y = (2-4)e^{-\frac{1}{2}\times 4} = -2e^{-2}$

$\therefore M(4, -2e^{-2})$ ✓

(b) The curve ① intersects the x-axis $\Rightarrow (2-x)e^{-\frac{1}{2}x} = 0 \Rightarrow x = 2$ ✓

Now $\int (2-x)e^{-\frac{1}{2}x}dx = (2-x)\int e^{-\frac{1}{2}x}dx - \int\left(\frac{d}{dx}(2-x)\cdot\int e^{-\frac{1}{2}x}\right)dx$

$= (2-x)\cdot\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}} - \int(-1)\left(\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}\right)dx$

$= 2(x-2)e^{-\frac{1}{2}x} - 2\int e^{-\frac{1}{2}x}$

$= 2(x-2)e^{-\frac{1}{2}x} - 2\times\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}$

$= 2(x-2)e^{-\frac{1}{2}x} + 4e^{-\frac{1}{2}x}$

$= e^{-\frac{1}{2}x}[2x - 4 + 4] = 2x\cdot e^{-\frac{1}{2}x}$ ✓

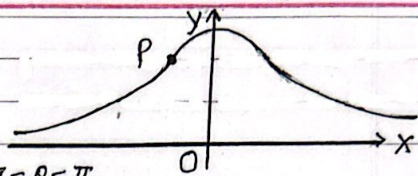
$\therefore$ Required Area $= \int_0^2 (2-x)e^{-\frac{1}{2}x}dx = \left[2xe^{-\frac{1}{2}x}\right]_0^2$

$= 4e^{-1} - 0$

$\therefore$ Area $= 4e^{-1}$ ✓

17. The diagram shows the curve with parametric equations:

$x = \tan\theta \;,\; y = \cos^2\theta$




for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

(a) Show that the gradient of the curve at the point with parameter $\theta$ is $-2\sin\theta\cos^3\theta$. ---[3]

The gradient of the curve has its maximum value at the point P.

(b) Find the exact value of the x-coordinate of P. ---[4]

[W-20/32/Q5]

**Solution** (a) $x = \tan\theta \;;\; y = \cos^2\theta$

$\frac{dx}{d\theta} = \sec^2\theta \qquad \frac{dy}{d\theta} = -2\sin\theta\cos\theta$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{-2\sin\theta\cos\theta}{\sec^2\theta}$

$\therefore$ gradient of the curve $= \frac{-2\sin\theta\cos\theta}{\frac{1}{\cos^2\theta}}$

$= -2\sin\theta\cos^3\theta$ ✓

(b) Gradient of the curve $m = -2\sin\theta\cos^3\theta$

$\frac{dm}{d\theta} = -2[\cos\theta\cdot\cos^3\theta + \sin\theta\cdot 3\cos^2\theta(-\sin\theta)]$

$= -2\cos^2\theta[\cos^2\theta - 3\sin^2\theta] = 0$ for max

$\Rightarrow \tan^2\theta = \frac{1}{3} \Rightarrow \tan\theta = \pm\frac{1}{\sqrt{3}}$

Looking at the diagram $x = \tan\theta = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$ ✓

18. Let $f(x) = \dfrac{7x+18}{(3x+2)(x^2+4)}$

(a) Express $f(x)$ in partial fractions. ---[5]

(b) Hence find the exact value of $\int_0^2 f(x)\,dx$ ---[6]

W-20/32/Q9

Solution:
(a) $f(x) = \dfrac{7x+18}{(3x+2)(x^2+4)} = \dfrac{a}{(3x+2)} + \dfrac{bx+c}{x^2+4}$ —— ①

$\Rightarrow 7x+18 = a(x^2+4) + (bx+c)(3x+2)$ —— ②

[ $3x+2=0 \Rightarrow x = -2/3$

Put $x = -\frac{2}{3} \Rightarrow 7\times\left(-\frac{2}{3}\right)+18 = a\left[\left(-\frac{2}{3}\right)^2+4\right] + 0$

$\Rightarrow \frac{40}{3} = \frac{40}{9}a \Rightarrow a = 3$ ✓

multiplying by D^r of L.H.S. in ①

$7x+18 = 3(x^2+4)+(bx+c)(3x+2)$

Comparing the constant term $\rightarrow 3\times4+2c = 18 \Rightarrow c = 3$ ✓

Comparing the coefficients of $x^2 \rightarrow 0 = 3+3b \Rightarrow b = -1$ ✓

∴ Required partial fractions: $f(x) = \dfrac{3}{(3x+2)} + \dfrac{-x+3}{(x^2+4)}$ ✓

(b) $\int_0^2 f(x)\,dx = \int_0^2 \left\{ \dfrac{3}{(3x+2)} + \dfrac{-x+3}{x^2+4} \right\} dx$

$= \int_0^2 \left\{ \dfrac{3}{(3x+2)} + \dfrac{-2x}{2(x^2+4)} + \dfrac{3}{x^2+2^2} \right\} dx$

$= \left[ \dfrac{3\ln(3x+2)}{3} - \frac{1}{2}\ln(x^2+4) + 3\times\frac{1}{2}\tan^{-1}\frac{x}{2} \right]_0^2$

$= \left[ \left\{ \left(\ln 8 - \frac{1}{2}\ln 8\right) + \frac{3}{2}\tan^{-1}1 \right\} - \left\{ \left(\ln 2 - \frac{1}{2}\ln 4\right) + \frac{3}{2}\tan^{-1}0 \right\} \right]$

$= \frac{1}{2}\ln 8 + \frac{3}{2}\times\frac{\pi}{4} - 0$

$= \frac{1}{2}\ln 2^3 + \frac{3}{8}\pi$

$= \frac{3}{2}\ln 2 + \frac{3}{8}\pi$ ✓

[ ∵ $\int \dfrac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$\int \dfrac{f'(x)}{f(x)}dx = \ln(f(x))$

19 The curve with equation $y = xe^{1-2x}$ has one stationary point.

(a) Find the coordinates of this point. ---[4]

(b) Determine whether the stationary point is a max or a min. --[2]

[W-21/31/Q3]

Solution: Curve: $y = xe^{1-2x}$ ---- ①

$\frac{dy}{dx} = 1 \times e^{1-2x} + x \cdot e^{1-2x} \times -2$

$= e^{1-2x}[1-2x]$ --- ②

for station point $\frac{dy}{dx} = 0$

from ② $e^{1-2x} \cdot (1-2x) = 0$

$\Rightarrow 1-2x = 0 \rightarrow x = \frac{1}{2}$

from ① $y = \frac{1}{2}e^0 = \frac{1}{2}$

$\therefore$ Station point is $(\frac{1}{2}, \frac{1}{2})$

Now to check the nature of the stationary pt.

diff ② $\frac{d^2y}{dx^2} = -2e^{1-2x} + (1-2x)e^{1-2x} \times (-2)$

$= -2e^{1-2x}(1+1-2x)$

$= -2e^{1-2x}(2-2x)$ --- ③

at stationary point $(\frac{1}{2}, \frac{1}{2})$ from ③

$\left(\frac{d^2y}{dx^2}\right)_{x=\frac{1}{2}} = -2e^0(2-1) = -2 < 0$, Max.

$\therefore$ Maximum.

20. Using the substitution $u = \sqrt{x}$, find the exact value of $\int_3^\infty \frac{1}{(x+1)\sqrt{x}} dx$

[W-21/31/Q4] --- [6]

Solution: $\int_3^\infty \frac{1}{(x+1)\sqrt{x}} dx$

let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} dx \Rightarrow \frac{1}{\sqrt{x}} dx = 2du$

for $x = 3 \rightarrow u = \sqrt{3}$

$x = \infty \rightarrow u = \infty$

$= \int_{\sqrt{3}}^\infty \frac{1}{u^2+1} \cdot 2du$

$= 2\left[\tan^{-1} u\right]_{\sqrt{3}}^\infty$

$= 2[\tan^{-1}\infty - \tan^{-1}\sqrt{3}]$

$= 2\left[\frac{\pi}{2} - \frac{\pi}{3}\right] = 2 \times \frac{\pi}{6}$

$= \frac{1}{3}\pi$ ✓

21 (a) Using the expansion of $\sin(3x+2x)$ and $\sin(3x-2x)$, show that,

$$\frac{1}{2}(\sin 5x+\sin x) \equiv \sin 3x\cdot\cos 2x$$ ---[3]

(b) Hence show that:

$$\int_0^{\frac{1}{4}\pi} \sin 3x\cdot\cos 2x\,dx = \frac{1}{5}(3-\sqrt{2})$$ --[3]

[W-21/32/Q6]

Solution: $\sin(3x+2x) = \sin 3x\cos 2x + \cos 3x\sin 2x$ --- ①

and $\sin(3x-2x) = \sin 3x\cos 2x - \cos 3x\sin 2x$ --- ②

add ① & ② ⇒ $\sin 5x + \sin x = 2\sin 3x\cdot\cos 2x$

⇒ $\frac{1}{2}(\sin 5x+\sin x) = \sin 3x\cdot\cos 2x$ --- ③ ✓

(b) $\int_0^{\frac{1}{4}\pi} \sin 3x\cdot\cos 2x\,dx = \frac{1}{2}\int_0^{\frac{1}{4}\pi}(\sin 5x+\sin x)\,dx$ (from Part (a))

$= \frac{1}{2}\left[-\frac{\cos 5x}{5} - \cos x\right]_0^{\frac{\pi}{4}}$

$= -\frac{1}{2}\left[\left(\frac{1}{5}\cdot\cos\frac{5\pi}{4} + \cos\frac{\pi}{4}\right) - \left(\frac{\cos 0}{5}+\cos 0\right)\right]$

$= -\frac{1}{2}\left[\left(\frac{1}{5}\times\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{5}+1\right)\right] = \frac{1}{5}(3-\sqrt{2})$

$= \frac{1}{5}(3-\sqrt{2})$ ✓

22. The equation of a curve is: $ye^{2x} - y^2\cdot e^x = 2$

(a) Show that $\frac{dy}{dx} = \frac{2ye^x - y^2}{2y - e^x}$ ---[4]

(b) Find the exact coordinates of the point on the curve where the tangent is parallel to Y-axis.

[W-31/32/Q9] --[4]

Solution: C: $ye^{2x} - y^2e^x = 2$ --- ①

(a) diff. w.r.t x:

$2e^{2x}\cdot y + e^{2x}\cdot\frac{dy}{dx} - \left(2y\frac{dy}{dx}e^x + y^2e^x\right) = 0$

$\frac{dy}{dx}(e^{2x} - e^x 2y) + (2e^{2x}y - y^2e^x) = 0$

$\frac{dy}{dx} = \frac{-e^x(2ye^x - y^2)}{-e^x(2y - e^x)}$

$\frac{dy}{dx} = \frac{2ye^x - y^2}{2y - e^x}$ ✓ --- ②

(b) Tangent is parallel to Y-axis

in $\frac{dy}{dx}$ → denomenator = 0

from ②

$2y - e^x = 0 \Rightarrow e^x = 2y$ --- ③

from ① and ③

$y\cdot(e^x)^2 - y^2e^x = 2$

$\Rightarrow y\cdot(2y)^2 - y^2\cdot 2y = 2$

$2y^3 = 2 \Rightarrow y = 1$

∴ Point $(\ln 2, 1)$ ✓ ∵ $\{e^x = 2\times 1 = 2,\ x = \ln 2$

23 Find the exact value of $\int_{\frac{1}{3}\pi}^{\pi} x \sin \frac{1}{2}x \, dx$ ---[5]

[W-21/33/Q4]

**Solution:** Consider $\int x \cdot \sin \frac{x}{2} dx = x \cdot \int \sin \frac{x}{2} dx - \int \left(\frac{d}{dx}x \cdot \int \sin \frac{x}{2} dx\right) dx$

$= x \cdot \frac{(-\cos x/2)}{1/2} - \int 1 \times \frac{(-\cos x/2)}{-\frac{1}{2}} dx$

$= -2x \cos x/2 + 2\int \cos x/2 \, dx$

$= -2x \cos x/2 + 2 \times \frac{\sin x/2}{1/2}$

$= -2x \cos \frac{x}{2} + 4 \sin x/2$

$\therefore \int_{\frac{1}{3}\pi}^{\pi} x \cdot \sin \frac{x}{2} dx = \left[-2x \cos \frac{x}{2} + 4 \sin x/2\right]_{\frac{1}{3}\pi}^{\pi}$

$= \left[(0 + 4\times 1) - \left(-2\frac{\pi}{3} \times \frac{\sqrt{3}}{2} + 4 \times \frac{1}{2}\right)\right]$

$= 4 - 2 + \frac{\sqrt{3}}{3}\pi = \left(2 + \frac{\sqrt{3}}{3}\pi\right)$ ✓

---

24 The equation of a curve is $\ln(x+y) = x - 2y$

(a) Show that: $\frac{dy}{dx} = \frac{x+y-1}{2(x+y)+1}$ ---[4]

(b) Find the coordinates of the point on the curve where the tangent is parallel to the $x$-axis. [W-21/33/Q7] ---[3]

**Solution:** C: $\ln(x+y) = x - 2y$ —— ①

Diff w.r.t $x$

$\frac{1}{(x+y)}\left(1 + \frac{dy}{dx}\right) = 1 - 2\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}\left(\frac{1}{(x+y)} + 2\right) = 1 - \frac{1}{x+y}$

$\Rightarrow \frac{dy}{dx}\left(\frac{1+2(x+y)}{x+y}\right) = \frac{x+y-1}{x+y}$

$\frac{dy}{dx} = \frac{x+y-1}{x+y} \times \frac{x+y}{2(x+y)+1}$

$\frac{dy}{dx} = \frac{(x+y-1)}{2(x+y)+1}$ —— ②

(b) Tangent is parallel to x-axis

$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow$ Numerator of ② $= 0$

$\Rightarrow x + y - 1 = 0$

$\Rightarrow x + y = 1$ —— ③

or $y = 1 - x$ —— ④

from ① and ③ & ④

$\ln 1 = x - 2(1-x)$

$0 = x - 2 + 2x \Rightarrow 3x = 2$

$\Rightarrow x = 2/3$

from ④ $y = 1 - 2/3 = \frac{1}{3}$

$\therefore$ Required point $\left(\frac{2}{3}, \frac{1}{3}\right)$

25. Let $f(x) = \frac{1}{(9-x)\sqrt{x}}$

(a) Find the $x$-coordinate of the stationary point of the curve with equation $y = f(x)$. ---[4]

(b) Use the substitution $u = \sqrt{x}$, show that $\int_0^4 f(x)\,dx = \frac{1}{3}\ln 5$ --[6]

[W-21/33/Q9]

<u>Solution:</u> $f(x) = \frac{1}{(9-x)\sqrt{x}} = (9x^{1/2} - x^{3/2})^{-1}$ --- ①

Diff:

$f'(x) = -1(9x^{1/2} - x^{3/2})^{-2} \cdot \left[\frac{9}{2\sqrt{x}} - \frac{3}{2}\sqrt{x}\right]$

$= \frac{-1(9-3x)}{2\sqrt{x}(9\sqrt{x} - x\sqrt{x})^2}$ --- ②

for stationary point $\frac{dy}{dx} = 0 \Rightarrow -(9-3x) = 0$

$x$-coord. of the stationary pt. $x = 3$ ✓

ⓑ $\int_0^4 \frac{1}{(9-x)\sqrt{x}}\,dx$

Put $u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$\Rightarrow 2du = \frac{1}{\sqrt{x}}dx$

$x = 0, u = 0$

$x = 4, u = 2$

$= \int_0^2 \frac{2\,du}{9-u^2}$

$= 2\int_0^2 \frac{1}{(3+u)(3-u)}\,du$

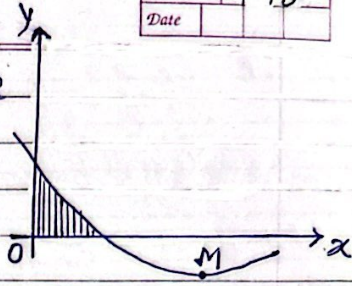
$= 2 \cdot \int_0^2 \frac{1}{6}\left[\frac{1}{3+u} + \frac{1}{3-u}\right]dx$

$= \frac{1}{3}\left[\ln(3+u) + \frac{\ln(3-u)}{-1}\right]_0^2$

$= \frac{1}{3}\left[\ln\frac{(3+u)}{(3-u)}\right]_0^2 = \frac{1}{3}[\ln 5 - \ln 1] = \frac{1}{3}(\ln 5 - 0)$

$= \frac{1}{3}\ln 5$ ✓

26. The diagram shows part of the curve $y = (3-x)e^{-\frac{1}{3}x}$ for $x \geq 0$, and its minimum point M.




(a) Find the exact coordinate of M. ---[5]

(b) Find the area of the shaded region bounded by the curve and the axes, giving your answer in terms of e. ---[5]

[W-22/31/Q9]

**Solution:**

(a) $y = (3-x)e^{-\frac{1}{3}x}$ ...① for $x \geq 0$

diff. $\frac{dy}{dx} = -1 \times e^{-\frac{1}{3}x} + (3-x) \times \left(-\frac{1}{3}e^{-\frac{1}{3}x}\right)$

$= e^{-\frac{1}{3}x}\left[-1 - \frac{1}{3}(3-x)\right]$

$= \frac{1}{3}e^{-\frac{1}{3}x}[x-6]$ ---②

for stationary point, $\frac{dy}{dx} = 0 \Rightarrow \frac{1}{3}e^{-\frac{1}{3}x}(x-6) = 0$ from ②

$\Rightarrow x - 6 = 0$ $[\because e^{-\frac{1}{3}x} \neq 0]$

Minimum at, $\Rightarrow x = 6$

$\therefore$ Minimum at $(6, -3e^{-2})$ ✓ {from ① $y = -3e^{-2}$

(b) Curve intersects x-axis at $y = 0 \Rightarrow (3-x)e^{-\frac{1}{3}x} = 0 \Rightarrow x = 3$ ✓

Hence the area of the shaded region $= \int_0^3 y\,dx$

$= \int_0^3 (3-x)e^{-\frac{1}{3}x}\,dx$

Integral by

$= (3-x)\cdot\int e^{-\frac{1}{3}x}dx - \int\left(\frac{d}{dx}(3-x)\cdot\int e^{-\frac{1}{3}x}dx\right)dx$

$= \left[(3-x)\times(-3)e^{-\frac{1}{3}x}\right]_0^3 - \int_0^3 (-1)\times(-3)e^{-\frac{1}{3}x}\,dx$

$= (0-(-9)) - 3\left[-3e^{-\frac{1}{3}x}\right]_0^3$

$= 9 + 9(e^{-1} - 1)$

$= 9 + 9e^{-1} - 9$

$= \frac{9}{e}$ ✓

27 The equation of a curve is $y = \sin x \sin 2x$. The curve has a stationary point in the interval $0 < x < \frac{1}{2}\pi$. Find the x-coordinate of this point, giving the answer correct to 3 s.f. ...[6]

[W-22/32/Q3]

Solution: $y = \sin x \cdot \sin 2x$ --- ① $\quad 0 < x < \frac{1}{2}\pi$

$\frac{dy}{dx} = \cos x \cdot \sin 2x + \sin x \times 2\cos 2x$ [Product rule of diff]

$= \cos x \times 2\sin x \cos x + 2\sin x(1 - 2\sin^2 x)$

$= 2\sin x(1 - \sin^2 x) + 2\sin x(1 - 2\sin^2 x)$

$= 2\sin x - 2\sin^3 x + 2\sin x - 4\sin^3 x$

$= 4\sin x - 6\sin^3 x$

$= 2\sin x[2 - 3\sin^2 x]$

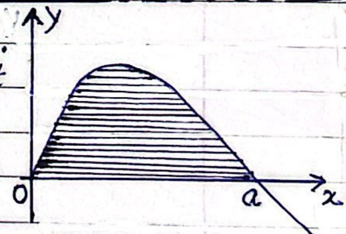
for any stationary point $\frac{dy}{dx} = 0 \Rightarrow 2\sin x(2 - 3\sin^2 x) = 0$

$\Rightarrow 2 - 3\sin^2 x = 0$ or $\sin x = 0$ ✗ (as $x \neq 0$)

$\Rightarrow \sin^2 x = 2/3 \Rightarrow \sin x = \sqrt{2/3}$, $0 < x < \pi/2$, $\sin x > 0$

$\therefore x = \sin^{-1}\sqrt{2/3} = 0.955$ radians ✓

28 The diagram shows part of the curve $y = \sin\sqrt{x}$. This part of the curve intersects the x-axis at the point where $x = a$.




(a) State the value of a. ...[1]

(b) Using the substitution $u = \sqrt{x}$, find the exact area of the shaded region in the first quadrant bounded by this part of the curve and the x-axis. ...[7]

[W-22/32/Q8]

Solution: Curve; $y = \sin\sqrt{x}$ --- ①

(a) Curve intersects x-axis at $y = 0$

for ① $\sin\sqrt{x} = 0 \Rightarrow \sqrt{x} = 0, \pi$

$x = 0, \pi^2 \Rightarrow a = \pi^2$ ✓

(b) Area of the shaded region $= \int_0^a y\,dx$

$= \int_0^{\pi^2} \sin\sqrt{x}\,dx$ $\quad$ { Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx$

$= \int_0^{\pi} 2u \sin u\,du$ $\quad$ { $\Rightarrow dx = 2u\,du$; $x = 0 \rightarrow u = 0$; $x = \pi^2 \rightarrow u = \pi$

$= 2\left[u \cdot \int \sin u\,du - \int \frac{d}{du}u \cdot \int \sin u\,du\right]_0^{\pi}$

$= 2\left[u(-\cos u) - \int 1 \times (-\cos u)\right]_0^{\pi}$

$= 2\left[-u\cos u + \sin u\right]_0^{\pi}$

$= 2\left[(-\pi\cos\pi + \sin\pi) - (0 + 0)\right]$

$= 2(-\pi \times (-1) + 0)$

$= 2\pi$ ✓

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W-22 | 32 | Q10

29 Let $f(x)=\frac{4-x+x^2}{(1+x)(2+x^2)}$

(a) Express $f(x)$ in partial fractions. ---[5]

(b) Find the exact value of $\int_0^4 f(x)\,dx$. Give answer as a single logarithm. --[5]

Solution: $f(x)=\frac{4-x+x^2}{(1+x)(2+x^2)}=\frac{a}{(1+x)}+\frac{bx+c}{(2+x^2)}$ ---- (1)

(a) multiply (1) by $(1+x) \Rightarrow \frac{4-x+x^2}{(2+x^2)}=a+\frac{(bx+c)(1+x)}{(2+x^2)}$

Put $x=-1$

$\Rightarrow \frac{6}{3}=a \Rightarrow a=2$ ✓

Now multiply (1) by the $D^r \rightarrow (1+x)(2+x^2)$ (and replace $a=2$)

$\Rightarrow \quad 4-x+x^2=2(2+x^2)+(bx+c)(1+x)$ —— (2)

Comparing coefficient of $x^2$ on both sides $\Rightarrow 1=2+b \Rightarrow b=-1$ ✓

Comparing constant term in (2) on both sides $\Rightarrow 4=4+c \Rightarrow c=0$ ✓

Now for $a=2$, $b=-1$ and $c=0$

from (1) the required partial fractions: $f(x)=\frac{2}{1+x}-\frac{1x}{(2+x^2)}$ ✓

(b) $\int_0^4 f(x)\,dx=\int_0^4\left(\frac{2}{1+x}-\frac{x}{(2+x^2)}\right)dx$

$=\int_0^4\frac{2}{(1+x)}\,dx-\int_0^4\frac{x}{(2+x^2)}\,dx$

Put $2+x^2=u$
$2x\,dx=du$
$x\,dx=\frac{1}{2}du$
$x=0\rightarrow u=2$
$x=4\rightarrow u=18$

$=\left[2\ln(1+x)\right]_0^4-\frac{1}{2}\int_2^{18}\frac{1}{u}\,du$

$=(2\ln 5-0)-\frac{1}{2}\left[\ln u\right]_2^{18}$

$=2\ln 5-\frac{1}{2}(\ln 18-\ln 2)$

$=2\ln 5-\frac{1}{2}\ln 9$

$=2\ln 5-\frac{1}{2}\times 2\ln 3$ $\quad$ $\because [\ln 9=\ln 3^2=2\ln 3]$

$=\ln 5^2-\ln 3$

$=\ln\left(\frac{25}{3}\right)$ ✓

30. Find the exact value of $\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$ ---[5]

[W-22/33/Q3]

**Solution:** $\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx = x\cdot\int\sec^2 x\,dx - \int \frac{d}{dx}x\cdot\int\sec^2 x\,dx$

$= [x\tan x]_0^{\frac{\pi}{4}} - \int_0^{\pi/4} 1\times\tan x\,dx$

$= (\frac{\pi}{4}\times 1 - 0) - [\ln\sec x]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} - (\ln\sqrt{2} - 0) = \frac{1}{4}\pi - \frac{1}{2}\ln 2$ ✓

31. The parametric equations of a curve are $x = 2t - \tan t$, $y = \ln(\sin 2t)$ for $0 < t < \frac{1}{2}\pi$, show that, $\frac{dy}{dx} = \cot t$. ---[5]

[W-22/33/Q4]

**Solution:** $y = \ln(\sin 2t) \Rightarrow \frac{dy}{dt} = \frac{1}{\sin 2t}\times 2\cos 2t = \frac{2\cos 2t}{2\sin t\cos t} = \frac{\cos 2t}{\sin t\cdot\cos t}$

$x = 2t - \tan t \Rightarrow \frac{dx}{dt} = 2 - \sec^2 t = \frac{2\cos^2 t - 1}{\cos^2 t} = \frac{\cos 2t}{\cos^2 t}$

Now $\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt} = \frac{\cos 2t}{\sin t\cdot\cos t}/\frac{\cos 2t}{\cos^2 t} = \frac{\cos 2t}{\sin t\cos t}\times\frac{\cos^2 t}{\cos 2t} = \cot t$ ✓

32. Let $f(x) = \frac{5 - x + 6x^2}{(3-x)(1+3x^2)}$

(a) Express f(x) in partial fractions. ---[5]

(b) Find the exact value of $\int_0^1 f(x)\,dx$, simplify your answer. ---[5]

[W-22/33/Q11]

**Solution:** $f(x) = \frac{5 - x + 6x^2}{(3-x)(1+3x^2)} = \frac{a}{3-x} + \frac{bx+c}{1+3x^2}$ --- ①

(a) multiply ① by $(3-x)$

$\Rightarrow \frac{(5-x+6x^2)}{(1+3x^2)} = a + \frac{(bx+c)(3-x)}{(1+3x^2)}$

Put $3 - x = 0 \Rightarrow x = 3 \Rightarrow \frac{56}{28} = a \Rightarrow a = 2$ ✓

multiply ① by $(3-x)(1+3x^2)$ and put $a = 2$

$\Rightarrow 5 - x + 6x^2 = 2(1+3x^2) + (bx+c)(3-x)$ --- ②

Comparing coeff of $x^2$ on both sides of ②

$6 = 6 - b \Rightarrow b = 0$ ✓

Comparing the constant term in ②

$5 = 2 + 3c \Rightarrow c = 1$ ✓

Hence from ① the req. partial fractions:

$f(x) = \frac{2}{(3-x)} + \frac{1}{1+3x^2}$ --- ③

(b) $\int_0^1 f(x)\,dx$

$= \int_0^1 \left(\frac{2}{(3-x)} + \frac{1}{1+3x^2}\right)dx$

$= \left[-2\ln(3-x) + \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x)\right]_0^1$

$= (-2\ln 2 + \frac{1}{\sqrt{3}}\cdot\frac{\pi}{3}) - (-2\ln 3 + 0)$

$= 2(\ln 3 - \ln 2) + \frac{1}{3\sqrt{3}}\pi$

$= 2\ln 3/2 + \frac{1}{3\sqrt{3}}\pi$ ✓