PURE MATHEMATICS -3

9709

(March (2020-2023) June (2020-2021) series with marking scheme)

DIFFERENTIATION AND INTEGRATION

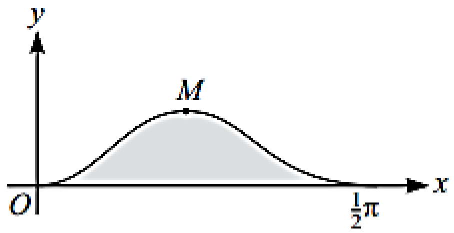
EXERCISE -1(a)

1) SP-2020 _9709_3 Q5

(a) Show that $\frac{d}{dx}(x - \tan^{-1}x) = \frac{x^2}{1+x^2}$. [2]

(b) Show that $\int_0^{\sqrt{3}} x\tan^{-1}x\,dx = \frac{2}{3}\pi - \frac{1}{2}\sqrt{3}$. [5]

2) SP-2020 _9709_3 Q9



The diagram shows the curve $y = \sin^2 2x\cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(a) Find the x-coordinate of M. [6]

(b) Using the substitution $u = \sin x$, find the area of the shaded region bounded by the curve and the x-axis. [4]

3) MARCH 2020_9709_32 Q4

Find $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} x\sec^2 x\,dx$. Give your answer in a simplified exact form. [7]

4) MARCH 2020_9709_32 Q7

The equation of a curve is $x^3 + 3xy^2 - y^3 = 5$.

(a) Show that $\frac{dy}{dx} = \frac{x^2+y^2}{y^2-2xy}$. [4]

(b) Find the coordinates of the points on the curve where the tangent is parallel to the y-axis. [5]

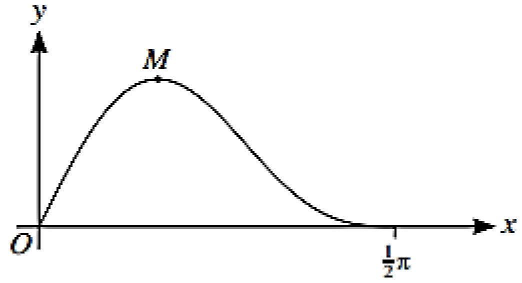
5) MARCH 2021_9709_32 Q6

Let $f(x) = \frac{5a}{(2x-a)(3a-x)}$, where a is a positive constant.

(a) Express $f(x)$ in partial fractions. [3]

(b) Hence show that $\int_a^{2a} f(x)\,dx = \ln 6$. [4]

6) MARCH 2021_9709_32 Q10



The diagram shows the curve $y = \sin 2x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(a) Using the substitution $u = \sin x$, find the exact area of the region bounded by the curve and the x-axis. [5]

(b) Find the exact x-coordinate of M. [6]

7) MARCH 2022_9709_32 Q4

The parametric equations of a curve are

$$x = 1 - \cos\theta, \qquad y = \cos\theta - \tfrac{1}{4}\cos 2\theta.$$

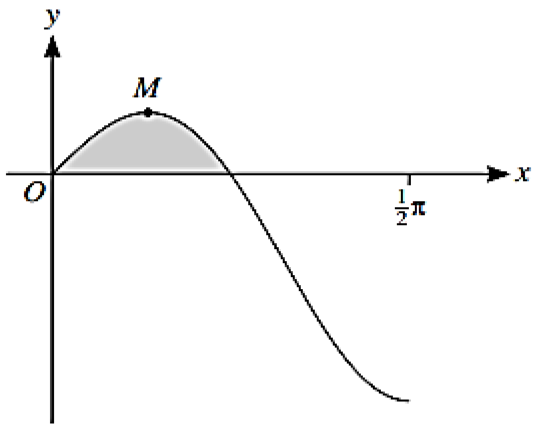
Show that $\frac{dy}{dx} = -2\sin^2(\frac{1}{2}\theta)$. [5]

8) MARCH 2022_9709_32 Q8

(a) Find the quotient and remainder when $8x^3 + 4x^2 + 2x + 7$ is divided by $4x^2 + 1$. [3]

(b) Hence find the exact value of $\int_0^{\frac{1}{2}} \frac{8x^3 + 4x^2 + 2x + 7}{4x^2 + 1}\,dx$. [5]

9) MARCH 2022_9709_32 Q11



The diagram shows the curve $y = \sin x \cos 2x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(a) Find the x-coordinate of M, giving your answer correct to 3 significant figures. [6]

(b) Using the substitution $u = \cos x$, find the area of the shaded region enclosed by the curve and the x-axis in the first quadrant, giving your answer in a simplified exact form. [5]

10) MARCH 2023_9709_32 Q5

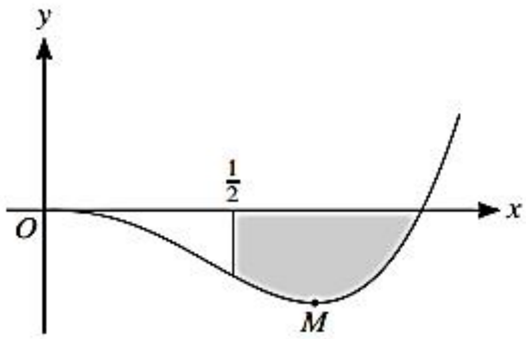
The parametric equations of a curve are

$$x = te^{2t}, \qquad y = t^2 + t + 3.$$

(a) Show that $\frac{dy}{dx} = e^{-2t}$. [3]

(b) Hence show that the normal to the curve, where $t = -1$, passes through the point $\left(0,\ 3 - \frac{1}{e^4}\right)$. [3]

11) MARCH 2023_9709_32 Q8



The diagram shows the curve $y = x^3 \ln x$, for $x > 0$, and its minimum point M.

(a) Find the exact coordinates of M. [4]

(b) Find the exact area of the shaded region bounded by the curve, the x-axis and the line $x = \frac{1}{2}$. [5]

12) MARCH 2023_9709_32 Q11

Let $f(x) = \frac{5x^2 + x + 11}{(4 + x^2)(1 + x)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence show that $\int_0^2 f(x)\,dx = \ln 54 - \frac{1}{8}\pi$. [5]

13) JUNE 2020_9709_31 Q4

The curve with equation $y = e^{2x}(\sin x + 3\cos x)$ has a stationary point in the interval $0 \leqslant x \leqslant \pi$.

(a) Find the x-coordinate of this point, giving your answer correct to 2 decimal places. [4]

(b) Determine whether the stationary point is a maximum or a minimum. [2]

14) JUNE 2020_9709_31 Q5

(a) Find the quotient and remainder when $2x^3 - x^2 + 6x + 3$ is divided by $x^2 + 3$. [3]

(b) Using your answer to part **(a)**, find the exact value of $\displaystyle\int_1^3 \frac{2x^3 - x^2 + 6x + 3}{x^2 + 3}\,dx$. [5]

15) JUNE 2020_9709_31 Q7

Let $f(x) = \dfrac{\cos x}{1 + \sin x}$.

(a) Show that $f'(x) < 0$ for all x in the interval $-\frac{1}{2}\pi < x < \frac{3}{2}\pi$. [4]

(b) Find $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} f(x)\,dx$. Give your answer in a simplified exact form. [4]

16) JUNE 2020_9709_32 Q3

Find the exact value of

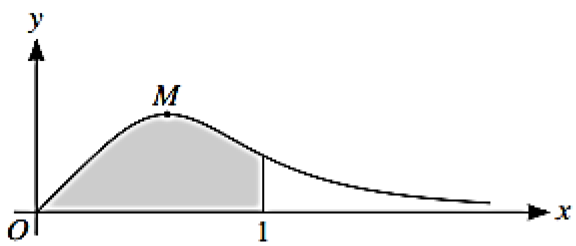
$$\int_1^4 x^{\frac{3}{2}} \ln x\,dx.$$ [5]

17) JUNE 2020_9709_32 Q4

A curve has equation $y = \cos x \sin 2x$.

Find the x-coordinate of the stationary point in the interval $0 < x < \frac{1}{2}\pi$, giving your answer correct to 3 significant figures. [6]

18) JUNE 2020_9709_32 Q6



The diagram shows the curve $y = \dfrac{x}{1 + 3x^4}$, for $x \geqslant 0$, and its maximum point M.

(a) Find the x-coordinate of M, giving your answer correct to 3 decimal places. [4]

(b) Using the substitution $u = \sqrt{3}x^2$, find by integration the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 1$. [5]

19) JUNE 2020_9709_33 Q2

Find the exact value of $\displaystyle\int_0^1 (2 - x)e^{-2x}\,dx$. [5]

20) JUNE 2020_9709_33 Q4

The equation of a curve is $y = x\tan^{-1}(\frac{1}{2}x)$.

(a) Find $\frac{dy}{dx}$. [3]

(b) The tangent to the curve at the point where $x = 2$ meets the y-axis at the point with coordinates $(0, p)$.

Find p. [3]

21) JUNE 2020_9709_33 Q7

Let $f(x) = \frac{2}{(2x-1)(2x+1)}$.

(a) Express $f(x)$ in partial fractions. [2]

(b) Using your answer to part **(a)**, show that

$$(f(x))^2 = \frac{1}{(2x-1)^2} - \frac{1}{2x-1} + \frac{1}{2x+1} + \frac{1}{(2x+1)^2}.$$ [2]

(c) Hence show that $\int_1^2 (f(x))^2\,dx = \frac{2}{5} + \frac{1}{2}\ln(\frac{5}{9})$. [5]

22) JUNE 2021_9709_31 Q4

(a) Prove that $\frac{1-\cos 2\theta}{1+\cos 2\theta} \equiv \tan^2\theta$. [2]

(b) Hence find the exact value of $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1-\cos 2\theta}{1+\cos 2\theta}\,d\theta$. [4]

23) JUNE 2021_9709_31 Q6

The parametric equations of a curve are

$$x = \ln(2+3t), \qquad y = \frac{t}{2+3t}.$$

(a) Show that the gradient of the curve is always positive. [5]

(b) Find the equation of the tangent to the curve at the point where it intersects the y-axis. [3]

24) JUNE 2021_9709_31 Q9

The equation of a curve is $y = x^{-\frac{2}{3}}\ln x$ for $x > 0$. The curve has one stationary point.

(a) Find the exact coordinates of the stationary point. [5]

(b) Show that $\int_1^8 y\,dx = 18\ln 2 - 9$. [5]

25) JUNE 2021_9709_32 Q4

Using integration by parts, find the exact value of $\int_0^2 \tan^{-1}\left(\frac{1}{2}x\right) dx$. [5]

26) JUNE 2021_9709_32 Q6

(a) Prove that $\operatorname{cosec} 2\theta - \cot 2\theta \equiv \tan \theta$. [3]

(b) Hence show that $\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} (\operatorname{cosec} 2\theta - \cot 2\theta)\, d\theta = \frac{1}{2}\ln 2$. [4]

27) JUNE 2021_9709_32 Q8

The equation of a curve is $y = e^{-5x}\tan^2 x$ for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

Find the x-coordinates of the stationary points of the curve. Give your answers correct to 3 decimal places where appropriate. [8]

28) JUNE 2021_9709_33 Q3

The parametric equations of a curve are

$$x = t + \ln(t + 2), \qquad y = (t - 1)e^{-2t},$$

where $t > -2$.

(a) Express $\frac{dy}{dx}$ in terms of t, simplifying your answer. [5]

(b) Find the exact y-coordinate of the stationary point of the curve. [2]

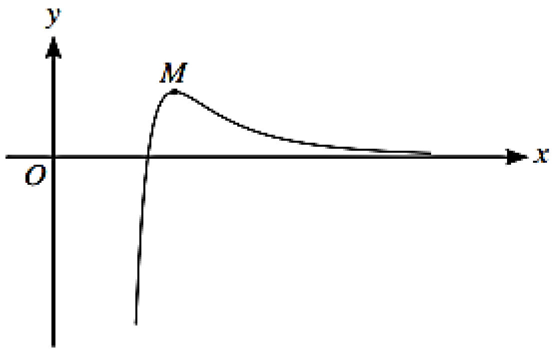
29) JUNE 2021_9709_33 Q4

Let $f(x) = \frac{15 - 6x}{(1 + 2x)(4 - x)}$.

(a) Express $f(x)$ in partial fractions. [3]

(b) Hence find $\int_1^2 f(x)\, dx$, giving your answer in the form $\ln\left(\frac{a}{b}\right)$, where a and b are integers. [4]

30) JUNE 2021_9709_33 Q8



The diagram shows the curve $y = \dfrac{\ln x}{x^4}$ and its maximum point M.

(a) Find the exact coordinates of M. [4]

(b) By using integration by parts, show that for all $a > 1$, $\displaystyle\int_1^a \frac{\ln x}{x^4}\,dx < \frac{1}{9}$. [6]

MARKING SCHEME

1) SP-2020 _9709_3 Q5

(a)	State correct derivative $1 - \dfrac{1}{1+x^2}$	1	
	Rearrange and obtain the given answer	1	
			2
(b)	Integrate by parts and reach $ax^2 \tan^{-1} x + b\displaystyle\int \frac{x^2}{1+x^2}dx$	1	
	Obtain $\frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\displaystyle\int \frac{x^2}{1+x^2}dx$, or equivalent	1	
	Obtain complete indefinite integral $\frac{1}{2}(x^2 \tan^{-1} x - x + \tan^{-1} x)$, or equivalent	1	
	Substitute limits having integrated twice	1	
	Obtain the given answer correctly	1	
			5

2) SP-2020 _9709_3 Q9

(a)	Use product rule	1	**M1**
	Obtain correct derivative in any form, e.g. $4\sin 2x\cos 2x\cos x - \sin^2 2x\sin x$	1	**A1**
	Equate derivative to zero and use a double angle formula	1	**M1***
	Reduce equation to one in a single trig function	1	**DM1**
	Obtain a correct equation in any form, e.g. $10\cos^3 x = 6\cos x$, $4 = 6\tan^2 x$, or $4 = 10\sin^2 x$	1	**A1**
	Solve and obtain $x = 0.685$	1	**A1**
		6	
(b)	Using $du = \pm\cos x\,dx$, or equivalent, express integral in terms of u and du	1	**M1**
	Obtain $\int 4u^2(1-u^2)\,du$	1	**A1**
	Use limits $u = 0$ and $u = 1$ in an integral of the form $au^3 + bu^5$	1	**M1**
	Obtain answer $\frac{8}{15}$ (or 0.533)	1	**A1**
		4	

3) MARCH 2020_9709_32 Q4

Integrate by parts and reach $ax\tan x + b\int \tan x\,dx$	**M1***
Obtain $x\tan x - \int \tan x\,dx$	**A1**
Complete the integration, obtaining a term $\pm\ln\cos x$, or equivalent	**M1**
Obtain integral $x\tan x + \ln\cos x$, or equivalent	**A1**
Substitute limits correctly, having integrated twice	**DM1**
Use a law of logarithms	**M1**
Obtain answer $\frac{5}{18}\sqrt{3}\pi - \frac{1}{2}\ln 3$, or exact simplified equivalent	**A1**
	7

4) MARCH 2020_9709_32 Q7

(a)	State or imply $3y^2 + 6xy\frac{dy}{dx}$ as derivative of $3xy^2$	**B1**
	State or imply $3y^2\frac{dy}{dx}$ as derivative of y^3	**B1**
	Equate attempted derivative of LHS to zero and solve for $\frac{dy}{dx}$	**M1**
	Obtain the given answer correctly	**A1**
		4
(b)	Equate denominator to zero	***M1**
	Obtain $y = 2x$, or equivalent	**A1**
	Obtain an equation in x or y	**DM1**
	Obtain the point $(1, 2)$	**A1**
	State the point $(\sqrt[3]{5}, 0)$	**B1**
		5

5) MARCH 2021_9709_32 Q6

(a)	Carry out a relevant method to determine constants A and B such that $\frac{5a}{(2x-a)(3a-x)} = \frac{A}{2x-a} + \frac{B}{3a-x}$	**M1**
	Obtain $A = 2$	**A1**
	Obtain $B = 1$	**A1**
		3
(b)	Integrate and obtain terms $\ln(2x-a) - \ln(3a-x)$	**B1 FT** **B1 FT**
	Substitute limits correctly in a solution containing terms of the form $b\ln(2x-a)$ and $c\ln(3a-x)$, where $bc \neq 0$	**M1**
	Obtain the given answer showing full and correct working	**A1**
		4

6) MARCH 2021_9709_32 Q10

)(a)	State or imply $du = \cos x\, dx$	B1
	Using double angle formula for sin2x and Pythagoras, express integral in terms of u and du.	M1
	Obtain integral $\int 2(u - u^3)\,du$	A1
	Use limits $u = 0$ and $u = 1$ in an integral of the form $au^2 + bu^4$, where $ab \neq 0$	M1
	Obtain answer $\frac{1}{2}$	A1
		5
)(b)	Use product rule	M1
	Obtain correct derivative in any form	A1
	Equate derivative to zero and use a double angle formula	*M1
	Obtain an equation in one trig variable	DM1
	Obtain $4\sin^2 x = 1$, $4\cos^2 x = 3$ or $3\tan^2 x = 1$	A1
	Obtain answer $x = \frac{1}{6}\pi$	A1
		6

7) MARCH 2022_9709_32 Q4

State $\frac{dx}{d\theta} = \sin\theta$ or $\frac{dy}{d\theta} = -\sin\theta + \frac{1}{2}\sin 2\theta$	B1
Use $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$	M1
Obtain correct answer in any form	A1
Use double angle correctly to obtain $\frac{dy}{dx}$ in terms of θ	M1
Obtain the given answer with no errors seen $-2\sin^2\left(\frac{1}{2}\theta\right)$	A1
	5

8) MARCH 2022_9709_32 Q8

8(a)	Commence division and reach quotient of the form $2x \pm 1$	M1
	Obtain (quotient) $2x + 1$	A1
	Obtain (remainder) 6	A1
		3
b)	Obtain terms $x^2 + x$	B1
	Obtain term of the form $a\tan^{-1}2x$	M1
	Obtain term $3\tan^{-1}2x$	A1
	Use $x = 0$ and $x = \frac{1}{2}$ as limits in a solution containing a term of the form $a\tan^{-1}2x$	M1
	Obtain final answer $\frac{3}{4}(1+\pi)$, or exact equivalent	A1
		5

9) MARCH 2022_9709_32 Q11

1(a)	Use correct product rule or chain rule	M1
	Obtain correct derivative in any form	A1
	Equate derivative to zero and use a correct double angle formula	*M1
	Obtain an equation in one trigonometric variable	DM1
	Obtain $6\sin^2 x = 1$, $6\cos^2 x = 5$ or $5\tan^2 x = 1$	A1
	Obtain final answer $x = 0.421$	A1
		6
(b)	State or imply $du = -\sin x\ dx$	B1
	Using double angle formula, express integral in terms of u and du	M1
	Integrate and obtain $\pm\left(u - \frac{2}{3}u^3\right)$	A1
	Use limits $u = 1$, $u = \frac{1}{\sqrt{2}}$ in an integral of the form $au + bu^3$, where $ab \neq 0$	M1
	Obtain $\frac{1}{3}\left(\sqrt{2}-1\right)$ or $\frac{1}{3}\sqrt{2}\ \frac{1}{3}$ or $\frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)\frac{1}{3}$ or simplified equivalent	A1
		5

10) MARCH 2023_9709_32 Q5

(a)	Obtain $\frac{dx}{dt} = e^{2t} + 2te^{2t}$	B1
	Use $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$	M1
	Obtain the given answer $\frac{dy}{dx} = e^{-2t}$	A1
		3
(b)	Obtain $x = -e^{-2}$ or $-\frac{1}{e^2}$ and $y = 3$ at $t = -1$	B1
	Obtain gradient of normal = $-e^{-2}$ or $-\frac{1}{e^2}$	B1
	$x = 0$ substituted into equation of normal or use of gradients to give $y = 3 - \frac{1}{e^4}$ with no errors	B1

11) MARCH 2023_9709_32 Q8

(a)	Use the product rule correctly	*M1
	Obtain the correct derivative in any form	A1
	Equate derivative to zero and solve exactly for x	DM1
	Obtain answer $\left(\frac{1}{\sqrt[3]{e}}, -\frac{1}{3e}\right)$ or exact equivalent	A1
(b)	Integrate by parts and reach $ax^4 \ln x + b\int(x^4 / x)dx$	*M1
	Obtain $\frac{x^4}{4}\ln x - \frac{1}{4}\int(x^4 / x)dx$	A1
	Complete integration and obtain $\frac{x^4}{4}\ln x - \frac{x^4}{16}$	A1
	Use limits of $x = \frac{1}{2}$ and $x = 1$ in the correct order, having integrated twice	DM1
	Obtain answer $\frac{15}{256} - \frac{1}{64}\ln 2$ or exact equivalent final answer	A1

12) MARCH 2023_9709_32 Q11

(a)	State or imply the form $\dfrac{Ax+B}{4+x^2}+\dfrac{C}{1+x}$	B1
	Use a correct method for finding a coefficient	M1
	Obtain one of $A=2$, $B=-1$ and $C=3$	A1
	Obtain a second value	A1
	Obtain the third value	A1
		5
(b)	Integrate and obtain terms $\left(\dfrac{A}{2}\right)\ln\left(4+x^2\right)+\dfrac{B}{2}\tan^{-1}\left(\dfrac{x}{2}\right)+C\ln(1+x)$	B1FT + B1FT + B1FT
	Substitute limits 0 and 2 correctly in an integral of the form $a\ln\left(4+x^2\right)+b\tan^{-1}\left(\dfrac{x}{2}\right)+c\ln(1+x)$, where $abc\neq 0$	M1
	Obtain answer $\ln 54-\dfrac{\pi}{8}$ after full and correct working	A1

13) JUNE 2020_9709_31 Q4

(a)	Use product rule	M1
	Obtain derivative in any correct form e.g. $2e^{2x}(\sin x+3\cos x)+e^{2x}(\cos x-3\sin x)$	A1
	Equate derivative to zero and obtain an equation in one trigonometric ratio	M1
	Obtain $x=1.43$ only	A1
		4
(b)	Use a correct method to determine the nature of the stationary point e.g. $x=1.42,\ y'=0.06e^{2.84}>0$ $x=1.44,\ y'=-0.07e^{2.88}<0$	M1
	Show that it is a maximum point	A1

14) JUNE 2020_9709_31 Q5

(a)	Commence division and reach quotient of the form $2x+k$	M1
	Obtain quotient $2x-1$	A1
	Obtain remainder 6	A1
		3
(b)	Obtain terms x^2-x (FT on quotient of the form $2x+k$)	B1FT
	Obtain term of the form $a\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$	M1
	Obtain term $\frac{6}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$ (FT on a constant remainder)	A1FT
	Use $x=1$ and $x=3$ as limits in a solution containing a term of the form $a\tan^{-1}(bx)$	M1
	Obtain final answer $\frac{1}{\sqrt{3}}\pi+6$, or exact equivalent	A1

15) JUNE 2020_9709_31 Q7

a)	Use quotient or product rule	M1
	Obtain derivative in any correct form e.g. $\frac{-\sin x(1+\sin x)-\cos x(\cos x)}{(1+\sin x)^2}$	A1
	Use Pythagoras to simplify the derivative	M1
	Justify the given statement	A1
(b)	State integral of the form $a\ln(1+\sin x)$	*M1
	State correct integral $\ln(1+\sin x)$	A1
	Use limits correctly	DM1
	Obtain answer $\ln\frac{4}{3}$	A1

16) JUNE 2020_9709_32 Q3

Commence integration and reach $ax^{\frac{5}{2}}\ln x + b\int x^{\frac{5}{2}}.\frac{1}{x}dx$	M1*
Obtain $\frac{2}{5}x^{\frac{5}{2}}\ln x - \frac{2}{5}\int x^{\frac{5}{2}}.\frac{1}{x}dx$	A1
Complete the integration and obtain $\frac{2}{5}x^{\frac{5}{2}}\ln x - \frac{4}{25}x^{\frac{5}{2}}$, or equivalent	A1
Use limits correctly, having integrated twice e.g $\frac{2}{5}\times 32\ln 4 - \frac{4}{25}\times 32 - \left(\frac{2}{5}\times 0\right) + \frac{4}{25}$	DM1
Obtain answer $\frac{128}{5}\ln 2 - \frac{124}{25}$, or exact equivalent	A1
	5

17) JUNE 2020_9709_32 Q4

Use correct product rule	M1
Obtain correct derivative in any form, e.g. $-\sin x \sin 2x + 2\cos x \cos 2x$	A1
Use double angle formula to express derivative in terms of $\sin x$ and $\cos x$	M1
Equate derivative to zero and obtain an equation in one trig function	M1
Obtain $3\sin 2x = 1$, or $3\cos 2x = 2$ or $2\tan 2x = 1$	A1
Solve and obtain $x = 0.615$	A1
	6

18) JUNE 2020_9709_32 Q6

Use quotient or product rule	M1
Obtain correct derivative in any form e.g. $\frac{(1+3x^4) - x\times 12x^3}{(1+3x^4)^2}$	A1
Equate derivative to zero and solve for x	M1
Obtain answer 0.577	A1

(b)

State or imply $du = 2\sqrt{3}x\,dx$, or equivalent	B1
Substitute for x and dx	M1
Obtain integrand $\frac{1}{2\sqrt{3}(1+u^2)}$, or equivalent	A1
State integral of the form $a\tan^{-1}u$ and use limits $u = 0$ and $u = \sqrt{3}$ (or $x = 0$ and $x = 1$) correctly	M1
Obtain answer $\frac{\sqrt{3}}{18}\pi$, or exact equivalent	A1
	5

19) JUNE 2020_9709_33 Q2

Commence integration and reach $a(2-x)e^{-2x}+b\int e^{-2x}\,dx$, or equivalent	M1*
Obtain $-\frac{1}{2}(2-x)e^{-2x}-\frac{1}{2}\int e^{-2x}\,dx$, or equivalent	A1
Complete integration and obtain $-\frac{1}{2}(2-x)e^{-2x}+\frac{1}{4}e^{-2x}$, or equivalent	A1
Use limits correctly, having integrated twice	DM1
Obtain answer $\frac{1}{4}\left(3-e^{-2}\right)$, or exact equivalent	A1
	5

20) JUNE 2020_9709_33 Q4

(a)	Use the product rule	M1
	State or imply derivative of $\tan^{-1}(\frac{1}{2}x)$ is of the form $k/(4+x^2)$, where $k = 2$ or 4, or equivalent	M1
	Obtain correct derivative in any form, e.g. $\tan^{-1}\left(\frac{1}{2}x\right)+\frac{2x}{x^2+4}$, or equivalent	A1
		3
(b)	State or imply y-coordinate is $\frac{1}{2}\pi$	B1
	Carry out a complete method for finding p, e.g. by obtaining the equation of the tangent and setting $x = 0$, or by equating the gradient at $x = 2$ to $\frac{\frac{1}{2}\pi - p}{2}$	M1
	Obtain answer $p = -1$	A1
		3

21) JUNE 2020_9709_33 Q7

(a)	State or imply the form $\frac{A}{2x-1}+\frac{B}{2x+1}$ and use a relevant method to find A or B	M1
	Obtain $A=1$, $B=-1$	A1
		2
(b)	Square the result of part (a) and substitute the fractions of part (a)	M1
	Obtain the given answer correctly	A1
		2
(c)	Integrate and obtain $-\frac{1}{2(2x-1)}-\frac{1}{2}\ln(2x-1)+\frac{1}{2}\ln(2x+1)-\frac{1}{2(2x+1)}$, or equivalent	B3, 2, 1, 0
	Substitute limits correctly	M1
	Obtain the given answer correctly	A1
		5

22) JUNE 2021_9709_31 Q4

a)	Use correct double angle formula or t-substitution twice	M1
	Obtain $\frac{1-\cos 2\theta}{1+\cos 2\theta}=\tan^2\theta$ from correct working	A1
		2
(b)	Express $\tan^2\theta$ in terms of $\sec^2\theta$	M1
	Integrate and obtain terms $\tan\theta-\theta$	A1
	Substitute limits correctly in an integral of the form $a\tan\theta+b\theta$, where $ab\neq 0$	M1
	Obtain answer $\frac{2}{3}\sqrt{3}-\frac{1}{6}\pi$	A1
		4

23) JUNE 2021_9709_31 Q6

(a)	Use correct chain rule or correct quotient rule to differentiate x or y	M1	
	Obtain $\frac{dx}{dt} = \frac{3}{2+3t}$ or $\frac{dy}{dt} = \frac{2}{(2+3t)^2}$	A1	
	Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$	M1	
	Obtain answer $\frac{2}{3(2+3t)}$	A1	
	Explain why this is always positive	A1	
	Alternative method for Question 6(a)		
	Form equation in x and y only	M1	
	Obtain $y = \frac{e^x - 2}{3e^x} \left(= \frac{1}{3} - \frac{2}{3}e^{-x} \right)$	A1	
	Differentiate	M1	
	Obtain $y' = \frac{2}{3}e^{-x}$	A1	
	Explain why this is always positive	A1	
		5	
(b)	Obtain $y = -\frac{1}{3}$ when $x = 0$	B1	
	Use a correct method to form the given tangent	M1	$\left(\frac{y + \frac{1}{3}}{x} = \frac{2}{3} \right)$
	Obtain answer $3y = 2x - 1$	A1	OE
		3	

24) JUNE 2021_9709_31 Q9

(a)	Use correct product rule or correct quotient rule	M1
	Obtain correct derivative in any form	A1
	Equate 2 term derivative to zero and solve for x	M1
	Obtain answer $x = e^{\frac{3}{2}}$	A1
	Obtain answer $y = \dfrac{3}{2e}$	A1
		5
(b)	Commence integration and reach $ax^{\frac{1}{3}}\ln x + b\int x^{\frac{1}{3}}.\frac{1}{x}dx$	*M1
	Obtain $3x^{\frac{1}{3}}\ln x - 3\int x^{\frac{1}{3}}.\frac{1}{x}dx$	A1
	Complete the integration and obtain $3x^{\frac{1}{3}}\ln x - 9x^{\frac{1}{3}}$	A1
	Use limits correctly in an expression of the form $px^{\frac{1}{3}}\ln x + qx^{\frac{1}{3}}$ $(pq \neq 0)$	DM1
	Obtain $18\ln 2 - 9$ from full and correct working	A1
		5

25) JUNE 2021_9709_32 Q4

Obtain $2\theta\tan\theta - 2\int\tan\theta\,d\theta$	A1
Complete integration and obtain $2\theta\tan\theta + 2\ln(\cos\theta)$	A1
Substitute correct limits correctly in an expression of the form $r\theta\tan\theta + s\ln(\cos\theta)$	DM1
Obtain final answer $\frac{1}{2}\pi - \ln 2$	A1
	5

26) JUNE 2021_9709_32 Q6

(a)	Express the LHS in terms of $\cos 2\theta$ and $\sin 2\theta$	B1	e.g. $\frac{1}{\sin 2\theta}-\frac{\cos 2\theta}{\sin 2\theta}$
	Use correct double angle formulae to express the LHS in terms of $\cos\theta$ and $\sin\theta$	M1	e.g. $\frac{1-\left(1-2\sin^2\theta\right)}{2\sin\theta\cos\theta}$
	Obtain $\tan\theta$ from correct working	A1	AG
	Alternative method for Question 6(a)		
	Express the LHS in terms of $\sin 2\theta$ and $\tan 2\theta$	B1	
	Use correct double angle formulae to express the LHS in terms of $\cos\theta$ and $\sin\theta$	M1	e.g. $\frac{1}{2\sin\theta\cos\theta}-\frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{2\frac{\sin\theta}{\cos\theta}}\left(=\frac{4\sin^2\theta}{4\sin\theta\cos\theta}\right)$
	Obtain $\tan\theta$ from correct working	A1	AG
	Alternative method for Question 6(a)		
	Express the LHS in terms of $\sin 2\theta$ and $\tan 2\theta$	B1	
	Use correct t substitution or rearrangement of $\sin 2\theta$ in terms of $\sec^2 2\theta$ and $\tan\theta$ to express the LHS in terms of $\tan\theta$.	M1	$\left(\frac{\sec^2\theta}{2\tan\theta}-\frac{1-\tan^2\theta}{2\tan\theta}=\right)\frac{1+\tan^2}{2\tan}-\frac{1-\tan^2}{2\tan}$
	Obtain $\tan\theta$ from correct working	A1	AG
		3	
(b)	State integral of the form $\mp\ln\cos\theta$ or $\pm\ln\sec\theta$	*M1	$\left[-\ln\cos\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ OE
	Use correct limits correctly and insert exact values for the trigonometric ratios	DM1	Need to see evidence of the substitution
	Obtain a correct expression, e.g. $-\ln\frac{1}{2}+\ln\frac{1}{\sqrt{2}}$	A1	
	Obtain $\frac{1}{2}\ln 2$ from correct working	A1	AG (must see an intermediate step)
		4	

27) JUNE 2021_9709_32 Q8

Use correct product (or quotient) rule	M1	At least 3 of 4 terms correct
Obtain $\frac{dy}{dx} = -5e^{-5x}\tan^2 x + 2e^{-5x}\tan x\sec^2 x$	A1	OE.
Equate *their* derivative to zero, use $\sec^2 x = 1 + \tan^2 x$ and obtain an equation in $\tan x$	M1	
Obtain $2\tan^2 x - 5\tan x + 2 = 0$	A1	Allow $2\tan^3 x - 5\tan^2 x + 2\tan x = 0$
State answer $x = 0$	B1	From correct derivative.
Solve a 3 term quadratic in $\tan x$ and obtain a value of x	M1	Must be in radians
Obtain answer, e.g. 0.464	A1	Must be 3 d.p. as specified in the question. Allow A1A0 if both values given to 2 d.p. or > 3 d.p.
Obtain second non-zero answer, e.g. 1.107 and no other in the given interval	A1	
Alternative method for Question 8		
Use correct product (or quotient) rule	M1	At least 3 of 4 terms correct
Obtain $\frac{dy}{dx} = -5e^{-5x}\tan^2 x + 2e^{-5x}\tan x\sec^2 x$	A1	OE
Equate *their* derivative to zero and obtain an equation in $\sin x$ and $\cos x$	M1	
Obtain $5\cos x\sin x = 2$	A1	Or simplified equivalent (i.e. cancelled)
State answer $x = 0$	B1	From correct derivative.
Use double angle formula or square both sides and solve for x	M1	Or equivalent method. Must be in radians.
Obtain answer, e.g. 0.464	A1	Must be 3 d.p. as specified in the question. Allow A1A0 if both values given to 2 d.p. or > 3 d.p.
Obtain second non-zero answer, e.g. 1.107 and no other in the given interval	A1	
	8	

28) JUNE 2021_9709_33 Q3

(a)	State $\frac{dx}{dt} = 1 + \frac{1}{t+2}$	B1
	Use product rule	M1
	Obtain $\frac{dy}{dt} = e^{-2t} - 2(t-1)e^{-2t}$	A1
	Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$	M1
	Obtain correct answer in any simplified form, e.g. $\frac{(3-2t)(t+2)}{t+3}\ e^{-2t}$	A1
		5
(b)	Equate derivative to zero and solve for t	M1
	Obtain $t = \frac{3}{2}$ and obtain answer $y = \frac{1}{2}e^{-3}$, or exact equivalent	A1
		2

29) JUNE 2021_9709_33 Q4

(a)	State or imply the form $\frac{A}{1+2x} + \frac{B}{4-x}$ and use a correct method to find a constant	M1
	Obtain one of $A = 4$ and $B = -1$	A1
	Obtain the second value	A1
		3
(b)	Integrate and obtain terms $2\ln(1+2x)+\ln(4-x)$	B1FT +B1FT
	Substitute limits correctly in an integral of the form $a\ln(1+2x)+b\ln(4-x)$, where $ab \neq 0$	M1
	Obtain final answer $\ln\left(\frac{50}{27}\right)$	A1
		4

30) JUNE 2021_9709_33 Q8

(a)	Use quotient or product rule	M1
	Obtain correct derivative in any form	A1
	Equate derivative to zero and solve for x	M1
	Obtain $x = \sqrt[4]{e}$ and $y = \frac{1}{4e}$, or exact equivalents	A1
		4
(b)	Commence integration and reach $ax^{-3}\ln x + b\int x^{-3}.\ \frac{1}{x}\,dx$	*M1
	Obtain $-\frac{1}{3}x^{-3}\ln x + \frac{1}{3}\int x^{-3}.\ \frac{1}{x}\,dx$	A1
	Complete integration and obtain $-\frac{1}{3}x^{-3}\ln x - \frac{1}{9}x^{-3}$	A1
	Substitute limits correctly, having integrated twice	DM1
	Obtain answer $\frac{1}{9} - \frac{1}{3}a^{-3}\ln a - \frac{1}{9}a^{-3}$	A1
	Justify the given statement	A1
		6