Date 10.12.23.

P. 3

Pure Maths-3

Differentiation and Integration

Ex. 1(a) Solution (Revision)

SP-20	M-20	M-22	S-20	—	—
—	M-21	M-23	S-21	—	—

Note:
Ex 1(b) - S-22, S-23
W-20, W-21, W-20

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Differentiation and Integration - Revision.

Example 1(a) Show that $\frac{d}{dx}(x - \tan^{-1}x) = \frac{x^2}{1+x^2}$ ---[2]

(b) Show that $\int_0^{\sqrt{3}} x\tan^{-1}x\,dx = \frac{2}{3}\pi - \frac{1}{2}\sqrt{3}$ ---[5]

$\boxed{SP-20/03/Q5}$

Solution.(a) $\frac{d}{dx}(x - \tan^{-1}x) = 1 - \frac{1}{1+x^2}$

$$= \frac{1+x^2-1}{(1+x^2)} = \frac{x^2}{(1+x^2)} \quad (1)$$

(b) Consider.

(Using by parts) $\int x\tan^{-1}x\,dx = \tan^{-1}x \cdot \int x\,dx - \int\left(\frac{d}{dx}\tan^{-1}x \cdot \int x\,dx\right)dx$

$$= \tan^{-1}x \cdot \frac{x^2}{2} - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2}$$

$$= \frac{1}{2}x^2 \cdot \tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x) \quad \text{from (1) Part (a)}$$

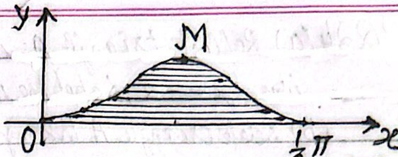
$$= \frac{1}{2}\left[x^2\tan^{-1}x - x + \tan^{-1}x\right]$$

$\therefore \int_0^{\sqrt{3}} x\tan^{-1}x\,dx = \frac{1}{2}\left[x^2\tan^{-1}x - x + \tan^{-1}x\right]_0^{\sqrt{3}}$

$$= \frac{1}{2}\left[(3\tan^{-1}\sqrt{3} - \sqrt{3} + \tan^{-1}\sqrt{3}) - 0\right]$$

$$= \frac{1}{2}\left[3 \times \frac{\pi}{3} - \sqrt{3} + \frac{\pi}{3}\right]$$

$$= \frac{1}{2}\left[\frac{4\pi}{3} - \sqrt{3}\right] = \left(\frac{2}{3}\pi - \frac{1}{2}\sqrt{3}\right)$$



Example 2: The diagram shows the curve $y=\sin^2 2x\cdot\cos x$ for $0\leq x\leq \frac{1}{2}\pi$ and its maximum point M.

(a) Find the x-coordinate of M. --- [6]

(b) Using substitution $u=\sin x$, find the area of the shaded region bounded by the curve and x-axis. --[4]

[SP-20/03/Q9]

Solution (a)

$$y=\sin^2 2x\cdot\cos x \quad \text{—①}$$

$$0\leq x\leq \tfrac{1}{2}\pi$$

$$\frac{dy}{dx}=2\sin 2x\cos 2x\times 2\cdot\cos x+\sin^2 2x(-\sin x)$$

$$=\sin 2x[4\cos 2x\cos x-\sin 2x\cdot\sin x].$$

for stationary point $\frac{dy}{dx}=0 \Rightarrow \sin 2x[4\cos 2x\cos x-\sin 2x\sin x]=0$

$\Rightarrow \sin 2x=0$ ✗ $\quad 4\cos 2x\cos x-\sin 2x\sin x=0$

$\Rightarrow 4(2\cos^2 x-1)\cdot\cos x-2\sin x\cos x\cdot\sin x=0$ $\quad\left[\begin{array}{l}\sin 2x=0\\ x=0,\frac{\pi}{2}\text{ ✗}\end{array}\right]$

$2\cos x[4\cos^2 x-2-(1-\cos^2 x)]=0$

$\cos x=0$ ✗ ; $\quad 5\cos^2 x-3=0$ $\quad\left[\begin{array}{l}\cos x=0\\ x=\frac{\pi}{2}\text{ ✗}\end{array}\right]$

$\cos^2 x=3/5$

$\cos x=\sqrt{3/5}$ $\quad 0\leq x\leq\frac{1}{2}\pi$

$\therefore$ M: $x=\underline{0.685}$ ✓

(b) Shaded Area $=\int_0^{\frac{\pi}{2}} y\,dx$

$$=\int_0^{\frac{\pi}{2}}\sin^2 2x\cos x\,dx$$

$$=\int_0^{\frac{\pi}{2}}4\sin^2 x\cos^2 x\cdot\cos x\,dx$$

$$=4\int_0^{\frac{\pi}{2}}\sin^2 x(1-\sin^2 x)\cos x\,dx$$

$$=4\int_0^1 u^2(1-u^2)\,du$$

$\left\{\begin{array}{l}\text{Let } \sin x=u\\ \cos x\,dx=du\end{array}\right.$

$\left\{\begin{array}{l}x=0\rightarrow u=0\\ x=\frac{1}{2}\pi\rightarrow u=1\end{array}\right.$

$$\text{Area}=4\int_0^1(u^2-u^4)\,du$$

$$=4\left[\frac{u^3}{3}-\frac{u^5}{5}\right]_0^1$$

$$=4\left[\left(\frac{1}{3}-\frac{1}{5}\right)-0\right]$$

$$=4\times\frac{2}{15}$$

$\therefore$ Area $=\frac{8}{15}$ (or 0.533)

Example 3: Find $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} x\sec^2 x\,dx$. Give your answer in a simplified exact form. [M-20|32|Q4] ---[7]

Solution: Consider $\int x\sec^2 x$ Int. by parts.

$$= x\int \sec^2 x\,dx - \int\left(\frac{d}{dx}x \cdot \int \sec^2 x\,dx\right)dx$$

$$= x\cdot\tan x - \int 1\times\tan x\,dx$$

$$= x\tan x - (-\ln\cos x)$$

$$\therefore \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} x\sec^2 x\,dx = \left[x\tan x + \ln\cos x\right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$$

$$= \left[\left(\frac{\pi}{3}\tan\frac{\pi}{3} + \ln\cos\frac{\pi}{3}\right) - \left(\frac{\pi}{6}\tan\frac{\pi}{6} + \ln\cos\frac{\pi}{6}\right)\right]$$

$$= \left(\frac{\pi}{3}\times\sqrt{3} + \ln\frac{1}{2}\right) - \left(\frac{\pi}{6}\times\frac{1}{\sqrt{3}} + \ln\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}\pi}{3} - \frac{\pi\sqrt{3}}{18} + \ln\frac{1}{2} - \left(\ln\sqrt{3} + \ln\frac{1}{2}\right) = \left(\frac{5\sqrt{3}}{18}\pi - \frac{1}{2}\ln 3\right)$$

Example 4: The equation of a curve is $x^3 + 3xy^2 - y^3 = 5$

(a) Show that $\frac{dy}{dx} = \frac{x^2+y^2}{y^2-2xy}$ ---[4]

(b) Find the coordinates of the points on the curve where the tangent is parallel to y-axis. [M-20|32|Q7] ---[5]

Solution: $x^3 + 3xy^2 - y^3 = 5$ —①

(a) diff. w.r.t x

$$3x^2 + 3y^2 + 3x\cdot 2y\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(-3y^2 + 6xy\right) = -3(x^2+y^2)$$

$$\frac{dy}{dx} = \frac{-3(x^2+y^2)}{-3(y^2-2xy)}$$

$$= \frac{x^2+y^2}{y^2-2xy}$$

(ii) tangent is parallel to Y-axis

gradient $\frac{dy}{dx} = \frac{x^2+y^2}{y^2-2xy}$ is not def.

$\Rightarrow y^2 - 2xy = 0$ —②

(ii) continued

$$y^2 - 2xy = 0$$

$$y(y-2x) = 0 \Rightarrow y=0 \text{ or } y=2x$$

$\Rightarrow y = 2x$ —③

$y = 0$ —④

for ① & ③

$$x^3 + 3x(2x)^2 - (2x)^3 = 5$$

$$5x^3 - 5 = 0$$

$$5(x^3-1) = 0 \Rightarrow \begin{cases} x=1 \\ y=2 \end{cases}$$

$(1,2)$

for ① & ④ $x^3 = 5$

$x = \sqrt[3]{5}$, $y=0$

$\therefore$ Points $(1,2)$, $(\sqrt[3]{5}, 0)$

5. Let $f(x) = \frac{5a}{(2x-a)(3a-x)}$, where a is a positive constant.

(a) Express $f(x)$ in partial fractions. ---[3]

(b) Hence show that $\int_a^{2a} f(x)\,dx = \ln 6$. --[4]

[M-21/32/Q6]

Solution (a) $\frac{5a}{(2x-a)(3a-x)} = \frac{A}{(2x-a)} + \frac{B}{(3a-x)}$ -----①

Multiply ① by $(2x-a) \Rightarrow \frac{5a}{(3a-x)} = A + \frac{B(2x-a)}{(3a-x)}$ --- ②

Put $2x-a=0 \rightarrow x = a/2$ in ② $\Rightarrow \frac{5a}{3a-\frac{a}{2}} = A \Rightarrow A = 2$ ✓

Again multiply ① by $(3a-x) \Rightarrow \frac{5a}{(2x-a)} = \frac{A(3a-x)}{(2x-a)} + B$ ---③

Put $(3a-x)=0 \rightarrow x = 3a$ in ③ $\Rightarrow \frac{5a}{(2\times 3a-a)} = B \Rightarrow B = 1$ ✓

∴ from ① required partial fraction:

$$\frac{2}{(2x-a)} + \frac{1}{(3a-x)}$$ ✓

(b) $\int_a^{2a} \left(\frac{2}{2x-a} + \frac{1}{3a-x}\right)dx = \left[\frac{\cancel{2}\ln(2x-a)}{\cancel{2}} + \frac{\ln(3a-x)}{-1}\right]_a^{2a}$

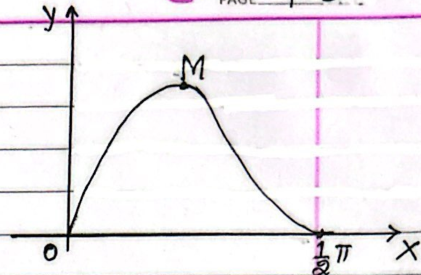
$$= \left[\ln\frac{(2x-a)}{(3a-x)}\right]_a^{2a}$$

$$= \ln\left(\frac{3a}{a}\right) - \ln\left(\frac{a}{2a}\right)$$

$$= \ln 3 - \ln\tfrac{1}{2} = \ln\left(\frac{3}{1/2}\right)$$

$$= \underline{\ln 6}$$ ✓

6. The diagram shows the curve,
$y = \sin 2x \cdot \cos^2 x$, for $0 \le x \le \frac{1}{2}\pi$,
and its maximum point M.



(a) using the substitution $u = \sin x$, find the exact area of the region bounded by the curve and the x-axis. --[5]

(b) Find the exact x-coordinate of M. --[6]

M-21	32	Q10

Solution (a) $A = \int_0^{\frac{\pi}{2}} y\,dx = \int_0^{\frac{\pi}{2}} \sin 2x \cdot \cos^2 x\,dx = \int^{\frac{\pi}{2}} 2\sin x \cdot \cos x \cdot \cos^2 x\,dx$

$= 2\int_0^{\frac{\pi}{2}} \sin x(1-\sin^2 x)\cdot \cos x\,dx$

put
$u = \sin x$
$du = \cos x\,dx$
$x = 0 \rightarrow u = 0$
$x = \frac{\pi}{2} \rightarrow u = 1$

$A = 2\int_0^1 u(1-u^2)\,du$

$= 2\int_0^1 (u-u^3)\,dx$

$= 2\left[\frac{u^2}{2} - \frac{u^4}{4}\right]_0^1 = 2\left[\left(\frac{1}{2}-\frac{1}{4}\right)-0\right] = \frac{1}{2}$ ✓

(b) $y = \sin 2x \cdot \cos^2 x$

$= 2\sin x \cos x \cdot \cos^2 x = 2\sin x \cdot \cos^3 x$

$\frac{dy}{dx} = 2[\sin x \cdot 3\cos^2 x(-\sin x) + \cos x \cdot \cos^3 x]$

$= 2[-3\sin^2 x \cdot \cos^2 x + \cos^4 x]$

$= 2\cos^2 x[-3\sin^2 x + \cos^2 x] = 0$ (for stationary point)

$\Rightarrow \cos x = 0$ or $-3\sin^2 x + \cos^2 x = 0$

$\Rightarrow x = \frac{\pi}{2}$ ✗ or $\tan^2 x = \frac{1}{3}$

$\tan x = \frac{1}{\sqrt{3}}$ ∵ $0 < x < \frac{\pi}{2}$

$x = \tan^{-1}\frac{1}{\sqrt{3}}$

∴ x-coordinate of M; $x = \frac{\pi}{6}$ ✓

7. The parametric equations of a curve are: $x = 1 - \cos\theta$, $y = \cos\theta - \frac{1}{4}\cos 2\theta$; show that $\frac{dy}{dx} = -2\sin^2\left(\frac{1}{2}\theta\right)$ ---- [5]

$\boxed{M\text{-}22/32/Q4}$

Solution: $x = 1 - \cos\theta \Rightarrow \frac{dy}{d\theta} = \sin\theta$ --- (1)

$y = \cos\theta - \frac{1}{4}\cos 2\theta$

$\Rightarrow \frac{dy}{dx} = -\sin\theta + \frac{1}{4} \times \sin 2\theta \times 2$

$= -\sin\theta + \frac{1}{4} \times 2 \times 2\sin\theta\cos\theta$

$= -\sin\theta + \sin\theta\cos\theta$

$= -\sin\theta(1 - \cos\theta)$ —— (2)

from (1) and (2)

$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{-\sin\theta(1-\cos\theta)}{\sin\theta} \Rightarrow$

$\frac{dy}{dx} = -(1 - \cos\theta)$

$= -2\sin^2\left(\frac{1}{2}\theta\right)$ ✓

8 (a) Find the quotient and remainder when, $8x^3+4x^2+2x+7$ is divided by $4x^2+1$. ---[3]

(b) Hence find the exact value of $\int_0^{1/2} \frac{8x^3+4x^2+2x+7}{4x^2+1}\,dx$ ---[5]

[M-22/32/Q8]

Solution:
(a)

$$4x^2+1 \overline{) \; 8x^3+4x^2+2x+7 \;} \; (2x+1$$
$$\underline{-8x^3 \qquad \pm 2x}$$
$$4x^2+7$$
$$\underline{-4x^2 \pm 1}$$
$$6$$

Quotient $q = 2x+1$
remainder $r = 6$

(b) $\int_0^{\frac{1}{2}} \frac{8x^3+4x^2+2x+7}{4x^2+1}$ $\left[\frac{A(x)}{B(x)} = q(x) + \frac{r(x)}{B(x)}\right.$

$= \int_0^{\frac{1}{2}} \left\{(2x+1) + \frac{6}{4x^2+1}\right\} dx$

$= \int_0^{1/2} \left\{(2x+1) + 6 \times \frac{1}{1+(2x)^2}\right\} dx$ $\because \left\{\int \frac{1}{1+x^2} dx = \tan^{-1}x\right.$

$= \left[(x^2+x) + 6 \times \frac{1}{2} \tan^{-1}(2x)\right]_0^{1/2}$

$= \left\{\left(\frac{1}{4}+\frac{1}{2}\right) + 3\tan^{-1}1\right\} - \left\{0 + 3\tan^{-1}0\right\}$

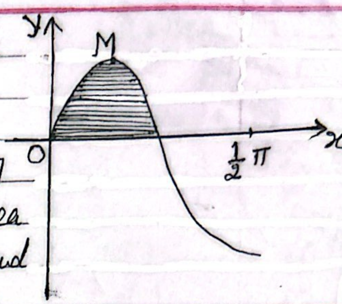
$= \frac{3}{4} + 3 \times \frac{\pi}{4} - 0$

$= \frac{3}{4}(1+\pi)$ ✓

9. The diagram shows the curve $y = \sin x \cos 2x$ for $0 \le x \le \frac{1}{2}\pi$, and its maximum point M.

(a) Find the x-coordinate of M, giving your answer to three significant figures. ---[6]

(b) Using the substitution $u = \cos x$, find the area of the shaded region enclosed by the curve and the x-axis in the first quadrant, giving your answer in a simplified exact form. ---[5]



[M-22/32/Q11]

Solution:

(a) $y = \sin x \cos 2x \; : \; 0 \le x \le \frac{1}{2}\pi$

$\frac{dy}{dx} = \sin x \cdot 2 \cdot (-\sin 2x) + \cos x \cdot \cos 2x$

$= -2\sin x \cdot 2\sin x \cos x + \cos x(1 - 2\sin^2 x)$

$= -4\sin^2 x \cos x + (1 - 2\sin^2 x)\cdot \cos x$

$= (-4\sin^2 x + 1 - 2\sin^2 x)\cdot \cos x$

for Max $\frac{dy}{dx} = 0 \Rightarrow \cos x(1 - 6\sin^2 x) = 0 \Rightarrow \cos x = 0,\; 6\sin^2 x = 1$

$\Rightarrow \sin^2 x = \frac{1}{6}$ or $\cos x = 0$

$\Rightarrow \sin x = \frac{1}{\sqrt{6}}$ or $x = \frac{\pi}{2}$ × $\left[x = \frac{\pi}{2} \Rightarrow y = -1 \right]$ for M_x ×

$\Rightarrow x = \sin^{-1}\left(\frac{1}{\sqrt{6}}\right) = 0.421^c$ ✓

(b) Area of the shaded region:

$A = \int_0^{\frac{\pi}{4}} y \, dx$

$= \int_0^{\frac{\pi}{4}} \sin x \cdot \cos 2x \, dx$

$= \int_0^{\frac{\pi}{4}} \sin x (2\cos^2 x - 1)\, dx$

$= \int_1^{\frac{1}{\sqrt{2}}} (2u^2 - 1)(-du)$

$= \int_{\frac{1}{\sqrt{2}}}^{1} (2u^2 - 1)\, du$

$\left\{ \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \right.$

$\left\{ \begin{array}{l} y = \sin x \cdot \cos 2x \\ \text{intersects } X\text{-axis for } y = 0 \\ \sin x \cdot \cos 2x = 0 \\ \Rightarrow \sin x = 0 \text{ or } \cos 2x = 0 \\ x = 0 \text{ or } 2x = \frac{\pi}{2} \\ x = 0, \frac{\pi}{4}. \quad (0 \le x \le \frac{1}{2}\pi) \end{array} \right.$

$\left\{ \begin{array}{l} \text{Let } \cos x = u \\ \Rightarrow -\sin x \, dx = du \\ \Rightarrow \sin x \, dx = -du \end{array} \right.$

$= \left[\frac{2u^3}{3} - u \right]_{\frac{1}{\sqrt{2}}}^{1}$

$\left\{ \begin{array}{l} \text{adjust the limits.} \\ x = 0 \Rightarrow u = 1 \\ x = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}} \end{array} \right.$

$= \left(\frac{2}{3} - 1\right) - \left(\frac{2}{3} \times \frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = \frac{1}{3}(\sqrt{2} - 1)$ ✓

10. The parametric equations of a curve are: $x = t\cdot e^{2t}$; $y = t^2+t+3$

(a) Show that $\frac{dy}{dx} = e^{-2t}$ ---[3]

(b) Hence Show that the normal to the curve, where $t=-1$, passes through the point $(0,\ 3-\frac{1}{e^4})$ ---[3]

$\boxed{M\text{-}23 / 32 / Q5}$

Solution (a) $x = t\cdot e^{2t}$; $y = t^2+t+3$

$\frac{dx}{dt} = 1\times e^{2t} + t\cdot e^{2t}\times 2$; $\frac{dy}{dt} = (2t+1)$ --- ②

$= e^{2t}(1+2t)$ --- ① ;

Hence $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{(1+2t)}{(2t+1)\,e^{2t}} = e^{-2t}$ ✓

$\Rightarrow \frac{dy}{dx} = e^{-2t}$ --- ③

(b) $\left(\frac{dy}{dx}\right)_{t=-1} = e^{-2(-1)} = e^2$

$\Rightarrow$ gradient of normal $= -\frac{1}{e^2} = -e^{-2}$ ✓

Now at $t=-1$, $x = -e^{-2}$ and $y = 3$ ✓

∴ Equation of Normal at $t=-1$, $y-3 = -e^{-2}(x+e^{-2})$

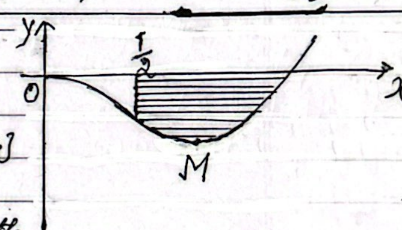
$\Rightarrow y = -e^{-2}(x+e^{-2})+3$ --- ④

Now put $x=0$ in ④ $y = -e^{-2}(0+e^{-2})+3 = 3-e^{-4}$

$= (3-\frac{1}{e^4})$

The the normal ④ passes through the point $(0,\ 3-\frac{1}{e^4})$ ✓

11 The diagram shows the curve $y = x^3 \ln x$, for $x>0$, and its minimum point M.



(a) Find the exact coordinates of M. ---[4]

(b) Find the exact area of the shaded region bounded by the curve, the x-axis and the line $x=\frac{1}{2}$. ---[5]

$\boxed{M\text{-}23 / 32 / Q8}$

Solution (a) $y = x^3 \cdot \ln x$ ---- ① for $x>0$;

$\frac{dy}{dx} = 3x^2\cdot \ln x + x^3 \times \frac{1}{x}$

$= x^2(3\ln x+1)$

for Minimum $\rightarrow \frac{dy}{dx} = 0$

$x^2(3\ln x+1) = 0$

$\Rightarrow 3\ln x+1 = 0$ or $x = 0$ ✗

$\Rightarrow \ln x = -\frac{1}{3} \Rightarrow x = e^{-\frac{1}{3}}$

from ① $y = (e^{-\frac{1}{3}})^3 \cdot \ln e^{-1/3} = -\frac{1}{3}e^{-1}$ $\Big\} \Rightarrow$ Min. at $\left(\frac{1}{\sqrt[3]{e}},\ \frac{-1}{3e}\right)$ ✓

(b) Curve intersects x-axis at $x^3 \ln x = 0$

$\Rightarrow x=0$ ✗ $\ln x = 0 \Rightarrow x=1$ ✓

Consider:

$\int x^3 \ln x\,dx = \ln x \cdot \int x^3 dx - \int \frac{d}{dx}\ln x \cdot \int x^3 dx$

$= \ln x \cdot \frac{x^4}{4} - \int \frac{1}{x}\cdot \frac{x^4}{4}\,dx$

$= \frac{x^4}{4}\ln x - \frac{1}{16}x^4$ ✓

Req. Area $= \left[\frac{x^4}{4}\left(\ln x - \frac{1}{4}\right)\right]_{1/2}^{1} = \left|-\frac{15}{256} + \frac{1}{64}\ln 2\right|$

$= \left(\frac{15}{256} - \frac{1}{64}\ln 2\right)$ ✓

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12 Let $f(x)=\dfrac{5x^2+x+11}{(4+x^2)(1+x)}$ (a) Express $f(x)$ in partial fractions. ---[5]

(b) Hence show that $\int_0^2 f(x)\,dx = \ln 54 - \frac{1}{8}\pi$ ---[5]

| M-23 | 32 | Q11 |

Solution (a) $\dfrac{5x^2+x+11}{(4+x^2)(1+x)} = \dfrac{Ax+B}{4+x^2} + \dfrac{C}{1+x}$ ---- ①

multiply ① by $(1+x) \Rightarrow \dfrac{5x^2+x+11}{(4+x^2)} = \dfrac{(Ax+B)(x+1)}{(4+x^2)} + C$ --- ②

$(x+1)=0 \Rightarrow x=-1$

Put $x=-1$ in ② $\dfrac{5-1+11}{4+1} = 0+C \Rightarrow \underline{C=3}$ --- ✓

Now multiply ① by $(4+x^2)(1+x)$ and put $C=3$

$\Rightarrow 5x^2+x+11 = (Ax+B)(x+1) + 3(4+x^2)$ --- ③

Comparing the coefficient of $x^2 \Rightarrow 5 = A+3 \Rightarrow \underline{A=2}$ ✓

Comparing the constant terms $\Rightarrow 11 = B+12 \Rightarrow \underline{B=-1}$ ✓

Hence from ① the required partial fractions of $f(x)$;

$f(x) = \dfrac{2x-1}{4+x^2} + \dfrac{3}{(1+x)}$ ---- ④ ✓

(b) $\int_0^2 f(x)\,dx = \int_0^2 \left(\dfrac{2x-1}{4+x^2} + \dfrac{3}{1+x}\right) dx$ --- ⑤

Consider $\int \dfrac{2x-1}{4+x^2}dx = \int \dfrac{2x}{(4+x^2)}dx - \int \dfrac{1}{2^2+x^2}dx$

$= \ln(4+x^2) - \frac{1}{2}\tan^{-1}\frac{x}{2}$ --- ⑥

$\left[\int \dfrac{f'(x)}{f(x)}dx = \ln f(x)\right.$

$\int \dfrac{1}{a^2+x^2}dx = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$

$\left[\int \dfrac{3}{1+x}dx = 3\ln(1+x)\right.$

from ⑤ & ⑥

$\left[\ln(4+x^2) - \frac{1}{2}\tan\frac{x}{2} + 3\ln(1+x)\right]_0^2$

$= \left[\left(\ln 8 - \frac{1}{2}\tan^{-1}1 + 3\ln 3\right) - \left(\ln 4 - \frac{1}{2}\tan^{-1}0 + 3\ln 1\right)\right]$

$= \left[\left(\ln 8 + \ln 3^3 - \frac{1}{2}\times\frac{\pi}{4}\right) - \left(\ln 4 - 0 + 0\right)\right]$

$= \ln 8 + \ln 27 - \ln 4 - \frac{1}{8}\pi$

$= \ln\left(\dfrac{8\times 27}{4}\right) - \dfrac{\pi}{8}$

$= \left(\ln 54 - \dfrac{\pi}{8}\right)$ ✓

Example 13: The curve with equation $y = e^{2x}(\sin x + 3\cos x)$ has a stationary point in the interval $0 \leq x \leq \pi$

(a) Find the x-coordinate of this point, giving your answer correct to 2 decimal places. --[4]

(b) Deter whether the stationary point is a maximum or a minimum. [S-20/31/Q4] --[2]

Solution: $y = e^{2x}(\sin x + 3\cos x)$ —— (1) $\quad 0 \leq x \leq \pi$

(a) $\frac{dy}{dx} = e^{2x} \cdot 2(\sin x + 3\cos x) + e^{2x}(\cos x - 3\sin x)$

$\frac{dy}{dx} = e^{2x}[7\cos x - \sin x]$ —— (2)

for Stationary point $e^{2x}(7\cos x - \sin x) = 0$

$\Rightarrow 7\cos x - \sin x = 0 \quad [\because e^{2x} \neq 0]$

$\tan x = 7 \Rightarrow x = 1.428899\text{ rad}$

$x = 1.43$ rad.

(b) diff (2) $\frac{d^2y}{dx^2} = 2e^{2x}(7\cos x - \sin x) + e^{2x}(-7\sin x - \cos x)$

$= e^{2x}[13\cos x - 9\sin x]$

$\left(\frac{d^2y}{dx^2}\right)_{x=1.43} = e^{2\times 1.43}[13\cos 1.43 - 9\sin 1.43]$

$= e^{2.86}[1.82 - 8.91] = -7.09 \times 17.46 < 0$

$\therefore$ There is a maximum at $x = 1.43$ radians.

Example 14(a) Find the quotient and remainder when $2x^3 - x^2 + 6x + 3$ is divided by $x^2 + 3$. --[3]

(b) using your answer to part (a), find $\int_1^3 \frac{2x^3 - x^2 + 6x + 3}{x^2 + 3}\,dx$ --[5] [S-20/31/Q5]

Solution: $x^2 + 3 \overline{)\, 2x^3 - x^2 + 6x + 3 \,}(2x - 1$

$-2x^3 \quad \pm 6x$

$-x^2 + 3$

$\mp x^2 \mp 3$

6

$\therefore$ Quotient $= (2x - 1)$,

Remainder $= 6$,

(b) $\int_1^3 \frac{2x^3 - x^2 + 6x + 3}{x^2 + 3}\,dx$

$= \int_1^3 \left((2x - 1) + \frac{6}{x^2 + (\sqrt{3})^2}\right) dx$

continued

(b) continued →

$\left[x^2 - x + 6 \times \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right]_1^3$

$= \left((9 - 3) + 2\sqrt{3}\tan^{-1}\sqrt{3}\right) - \left(0 - 2\sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}}\right)$

$= 6 + 2\sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$

$= 6 + 2\sqrt{3} \times \frac{\pi}{6}$

$= \left(6 + \frac{1}{\sqrt{3}}\pi\right)$

$\left[\because \int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

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Example 15: Let $f(x) = \frac{\cos x}{1+\sin x}$

(a) Show that $f'(x) < 0$ for all x in the interval $-\frac{1}{2}\pi < x < \frac{3}{2}\pi$ --[4]

(b) Find $\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} f(x)\,dx$. Give your answer in a simplified exact from. --[4]

[S-20|31|Q7]

Solution: $f(x) = \frac{\cos x}{1+\sin x}$

(a) diff. $f'(x) = \frac{-\sin x(1+\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$

$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$

$= \frac{-(1+\sin x)}{(1+\sin x)^2}$

$\therefore f'(x) = \frac{-1}{(1+\sin x)} < 0$

[as $0 < (1+\sin x) < 2 \quad -\frac{\pi}{2} < x < \frac{3\pi}{2}$]

(b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} f(x)\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)}\,dx$

$= \int_{3/2}^{2} \frac{1}{u}\,du$ $\quad$ $\left\{\begin{array}{l}\text{Put } 1+\sin x = u \\ \cos x\,dx = du\end{array}\right.$

$= [\ln u]_{3/2}^{2}$ $\quad$ $\left\{\begin{array}{l}x=\frac{\pi}{6},\ u = 3/2 \\ x=\frac{\pi}{2},\ u = 2\end{array}\right.$

$= \ln 2 - \ln 3/2$

$= \ln \frac{2}{3/2} = \ln\left(\frac{4}{3}\right)$ ✓

Example 16. Find the exact value of $\int_1^4 x^{3/2}\ln x\,dx$ --[5]

[S-20|32|Q3]

Solution:

Consider $\int x^{3/2}\ln x\,dx = \ln x \cdot \int x^{3/2}dx - \int\left(\left(\frac{d}{dx}\ln x\right)\cdot \int x^{3/2}dx\right)dx$

(using Integral by parts)

$= \ln x \cdot \frac{x^{5/2}}{5/2} - \int \frac{1}{x}\cdot\frac{x^{5/2}}{5/2}\,dx$

$= \frac{2}{5}x^{5/2}\ln x - \frac{2}{5}\int x^{3/2}\,dx$

$= \frac{2}{5}x^{5/2}\ln x - \frac{2}{5}\times\frac{x^{5/2}}{5/2}$

$\therefore \int_1^4 x^{3/2}\ln x\,dx = \left[\frac{2}{5}x^{5/2}\ln x - \frac{4}{25}x^{5/2}\right]_1^4$

$= \left(\frac{2}{5}\times 32\ln 4 - \frac{4}{25}\times 32\right) - \left(0 - \frac{4}{25}\right)$

$= \frac{64}{5}\ln 2^2 - \frac{128}{25} + \frac{4}{25}$

$= \left(\frac{128}{5}\ln 2 - \frac{124}{25}\right)$ ✓

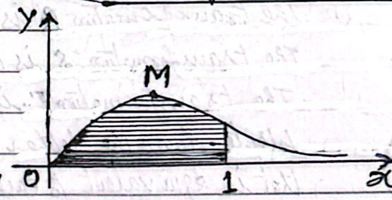
Example 17. A curve has equation $y = \cos x \sin 2x$
Find the x-coordinate of the stationary point in interval $0 < x < \frac{1}{2}\pi$, giving your answer correct to 3 s.f. [S-20/32/Q4] --[6]

Solution: $y = \cos x \sin 2x$

$\frac{dy}{dx} = \cos x \cdot 2\cos 2x + \sin 2x \cdot (-\sin x)$
$= 2\cos 2x \cos x - \sin 2x \cdot \sin x$
$= 2(2\cos^2 x - 1)\cos x - 2\sin x \cos x \cdot \sin x$
$= 2(2\cos^2 x - 1)\cos x - 2\cos x(1 - \cos^2 x)$
$= 4\cos^3 x - 2\cos x - 2\cos x + 2\cos^3 x$
$= 6\cos^3 x - 4\cos x$ $\quad 0 < x < \frac{\pi}{2}$
$= 2\cos x(3\cos^2 x - 2)$

for stationary point $\frac{dy}{dx} = 0 \Rightarrow 2\cos x(3\cos^2 x - 2) = 0$
$\Rightarrow \cos^2 x = 2/3$ $\quad [\cos x \neq 0$ as $0 < x < \frac{\pi}{2}]$
$\Rightarrow \cos x = \sqrt{2/3}$
$\Rightarrow x = \cos^{-1}\sqrt{2/3} = 0.615$
$\therefore x = 0.615$ ✓

Example 18: The diagram shows the curve $y = \frac{x}{1+3x^4}$, for $x \geq 0$, and its maximum point M.



(a) Find the x-coordinate of M, giving your answer correct to 3 d.p. --[4]
(b) Using substitution $u = \sqrt{3}x^2$, find by integration the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 1$. [S-20/32/Q6] --[5]

Solution: $y = \frac{x}{1+3x^4}$

(a) $\frac{dy}{dx} = \frac{(1+3x^4)\times 1 - x \times 12x^3}{1+3x^4}$
$= \left(\frac{1 - 9x^4}{1+3x^4}\right)$

for station point, $\frac{dy}{dx} = 0 \Rightarrow 1 - 9x^4 = 0$
$x = \frac{1}{\sqrt{3}}$
$x = \underline{0.577}$ ✓

(b) Area $= \int_0^1 \frac{x}{1+(\sqrt{3}x^2)^2}\,dx$ $\quad \left\{\begin{array}{l} u = \sqrt{3}x^2 \\ du = 2\sqrt{3}x\,dx \end{array}\right.$
$= \frac{1}{2\sqrt{3}} \int_0^{\sqrt{3}} \frac{1\,du}{1+u^2}$ $\quad x = 0 \to u = 0$, $x = 1 \to u = \sqrt{3}$
$= \frac{1}{2\sqrt{3}} \left[\tan^{-1} u\right]_0^{\sqrt{3}}$
$= \frac{1}{2\sqrt{3}} \times \left[\tan^{-1}\sqrt{3} - \tan^{-1} 0\right]$
$= \frac{1}{2\sqrt{3}}\left(\frac{\pi}{3} - 0\right) = \frac{\pi\sqrt{3}}{18}$ ✓

Area $= \frac{\sqrt{3}}{18}\pi$ ✓

Example 19: Find the exact value of $\int_0^1 (2-x)e^{-2x}dx$ [S-20/33/Q2] --[5]

Solution: Consider $\int (2-x)e^{-2x}dx = (2-x)\cdot\int e^{-2x}dx - \int\left(\frac{d}{dx}(2-x)\cdot\int e^{-2x}dx\right)dx$

$$= (2-x)\cdot\frac{e^{-2x}}{-2} - \int(-1)\cdot\frac{e^{-2x}}{-2}dx$$

$$= -\frac{1}{2}(2-x)e^{-2x} - \frac{1}{2}\int e^{-2x}dx$$

$$= -\frac{1}{2}(2-x)e^{-2x} - \frac{1}{2}\cdot\frac{e^{-2x}}{-2}$$

$$= e^{-2x}\left[-\frac{1}{2}(2-x)+\frac{1}{4}\right] = \frac{1}{4}e^{-2x}(2x-3)$$

$$\therefore \int_0^1 (2-x)e^{-2x}dx = \frac{1}{4}\left[e^{-2x}(2x-3)\right]_0^1 = \frac{1}{4}\left[e^{-2}(-1)-(-3)\right]$$

$$= \frac{1}{4}(3-e^{-2}) \checkmark$$

Example 20. The equation of a curve is $y = x\tan^{-1}(\frac{1}{2}x)$

(a) Find $\frac{dy}{dx}$ ---[3]

(b) The tangent to the curve at the point where $x=2$ meets y-axis at the point with coordinates $(0,p)$, find p. [S-20/33/Q4] ---[3]

Solution: $y = x\tan^{-1}(\frac{1}{2}x)$ —— ①

(a) $\frac{dy}{dx} = 1\times\tan^{-1}(\frac{1}{2}x) + x\times\frac{1}{1+(\frac{1}{2}x)^2}\times\frac{1}{2}$

$$= \tan^{-1}\left(\frac{x}{2}\right) + \frac{2x}{x^2+4} \checkmark \text{ —— ②}$$

(b) $\left(\frac{dy}{dx}\right)_{x=2} = \tan^{-1}1 + \frac{4}{4+4} = \frac{\pi}{4}+\frac{1}{2}$

from ① $x=2 \Rightarrow y = 2\tan^{-1}1 = 2\times\frac{\pi}{4} = \frac{\pi}{2}$

$\therefore$ Eqn of tangent to the curve at $(2,\frac{\pi}{2})$ is

$$y-\frac{\pi}{2} = \left(\frac{\pi}{4}+\frac{1}{2}\right)(x-2) \text{ —— ③}$$

Tangent intersects y-axis at $x=0$ put in ③

$$y-\frac{\pi}{2} = \left(\frac{\pi}{4}+\frac{1}{2}\right)(0-2)$$

$$y-\frac{\pi}{2} = -\frac{\pi}{2}-1 \Rightarrow y=-1$$

$\therefore$ Intersects y-axis at $(0,-1) \equiv (0,p) \Rightarrow p=-1 \checkmark$

Example 21. Let $f(x) = \frac{2}{(2x-1)(2x+1)}$

(a) Express $f(x)$ in partial fractions. --[2]

(b) Using your answer to part (a), show that

$$(f(x))^2 = \frac{1}{(2x-1)^2} - \frac{1}{(2x-1)} + \frac{1}{(2x+1)} + \frac{1}{(2x+1)^2} \quad \text{--[2]}$$

(c) Hence show that $\int_1^2 (f(x))^2\,dx = \frac{2}{5} + \frac{1}{2}\ln\left(\frac{5}{9}\right)$ --[5]

$\boxed{S\text{-}20/33/Q7}$

Solution:

(a) $$\frac{2}{(2x-1)(2x+1)} = \frac{a}{(2x-1)} + \frac{b}{(2x+1)} \quad \text{①}$$

$$\Rightarrow \frac{2}{(2x+1)} = a + \frac{b(2x-1)}{2x+1} \quad \left[\begin{array}{l}\text{Multiply by } (2x-1) \\ 2x-1=0,\ x=\tfrac{1}{2}\end{array}\right.$$

Put $x = \frac{1}{2} \Rightarrow 1 = a$ ✓

Again multiplying ① by $(2x+1) \Rightarrow \frac{2}{(2x-1)} = \frac{a(2x+1)}{(2x-1)} + b \quad \left[\begin{array}{l}2x+1=0 \\ x=-\tfrac{1}{2}\end{array}\right.$

Put $x = -\frac{1}{2} \Rightarrow -1 = b$ ✓

from ①
$$f(x) = \frac{2}{(2x-1)(2x+1)} = \frac{1}{(2x-1)} - \frac{1}{(2x+1)} \quad \text{②}$$

(b) from ② $(f(x))^2 = \left(\frac{1}{(2x-1)} - \frac{1}{2x+1}\right)^2$

$$= \frac{1}{(2x-1)^2} - 2\times\frac{1}{(2x-1)}\times\frac{1}{(2x+1)} + \frac{1}{(2x+1)^2}$$

$$= \frac{1}{(2x-1)^2} - \left(\frac{1}{(2x-1)} - \frac{1}{(2x+1)}\right) + \frac{1}{(2x+1)^2} \quad \left[\text{from ② in the middle Term}\right]$$

$$= \frac{1}{(2x-1)^2} - \frac{1}{(2x-1)} + \frac{1}{(2x+1)} + \frac{1}{(2x+1)^2} \quad \text{③}$$

(c) $$\int_1^2 (f(x))^2\,dx = \int_1^2 \left[\frac{1}{(2x-1)^2} - \frac{1}{(2x-1)} + \frac{1}{(2x+1)} + \frac{1}{(2x+1)^2}\right] dx$$

$$= \left[\frac{-1}{2(2x-1)} - \frac{1}{2}\ln(2x-1) + \frac{1}{2}\ln(2x+1) - \frac{1}{2(2x+1)}\right]_1^2$$

$$= \left(\frac{-1}{6} - \frac{1}{2}\ln 3 + \frac{1}{2}\ln 5 - \frac{1}{2\times 5}\right) - \left(-\frac{1}{2} - \frac{1}{2}\times 0 + \frac{1}{2}\ln 3 - \frac{1}{2\times 3}\right)$$

$$= -\frac{1}{6} - \frac{1}{10} + \frac{1}{2} + \frac{1}{6} + \frac{1}{2}(\ln 5 - \ln 3 - \ln 3)$$

$$= \frac{5-1}{10} + \frac{1}{2}(\ln 5 - 2\ln 3) = \frac{4}{10} + \frac{1}{2}(\ln 5 - \ln 3^2) = \frac{2}{5} + \frac{1}{2}\ln\frac{5}{9} \ ✓$$

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22. (a) Prove that $\frac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$ --- [2]

(b) Hence find the exact value of $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1-\cos 2\theta}{1+\cos 2\theta}\,d\theta$ --- [4]

[S-21/31/Q4]

Solution (a) $\frac{1-\cos 2\theta}{1+\cos 2\theta} = \frac{2\sin^2\theta}{2\cos^2\theta} = \tan^2\theta$

(b) $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1-\cos 2\theta}{1+\cos 2\theta}\,d\theta = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \tan^2\theta\,d\theta$ from part (a)

$= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} (\sec^2\theta - 1)\,d\theta$

$= \left[\tan\theta - \theta\right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} = \left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(\tan\frac{\pi}{6} - \frac{\pi}{6}\right)$

$= \left(\sqrt{3} - \frac{\pi}{3}\right) - \left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right) = \left(\frac{2}{3}\sqrt{3} - \frac{1}{6}\pi\right)$ ✓

---

23. The parametric equations of a curve are: $y = \frac{t}{2+3t}$; $x = \ln(2+3t)$

(a) Show that the gradient of the curve is always positive --- [5]

(b) Find the equation of the tangent to the curve at the point where it intersects the y-axis. --- [3]

[S-21/31/Q6]

Solution (a) $y = \frac{t}{2+3t}$; $x = \ln(2+3t)$

$\frac{dy}{dt} = \frac{(2+3t)\times\frac{d}{dt}t - t\cdot\frac{d}{dt}(2+3t)}{(2+3t)^2}$; $\frac{dx}{dt} = \frac{1}{(2+3t)}\times 3 = \frac{3}{(2+3t)}$ --- (2)

$= \frac{(2+3t)\times 1 - t\times 3}{(2+3t)^2}$

$= \frac{2}{(2+3t)^2}$ ---- (1)

Hence $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$

$= \frac{2}{(2+3t)^2} \Big/ \frac{3}{(2+3t)}$

$= \frac{2}{3(2+3t)}$

$\therefore$ Gradient of the curve $\frac{dy}{dx} = \frac{2}{3(2+3t)} > 0$ [$\because (2+3t) > 0$ for $\ln(2+3t)$ be defined.] is always positive.

(b) For the intersection with Y-axis, $x = 0$, $\Rightarrow \ln(2+3t) = 0$ $\Rightarrow 2+3t = 1$ $\Rightarrow t = -\frac{1}{3}$

$\therefore y = \frac{t}{2+3t} = \frac{-\frac{1}{3}}{1} = -\frac{1}{3}$

Point $(0, -\frac{1}{3})$

from (1) at $t = -\frac{1}{3}$; $\frac{dy}{dx} = \frac{2}{3\times 1^2}$ ($\because 2+3t = 1$)

$= 2/3$ ✓

$\therefore$ Equation of tangent at $(0, -\frac{1}{3})$ is $y + \frac{1}{3} = \frac{2}{3}(x - 0)$

$\Rightarrow 3y = 2x - 1$ ✓

24. The equation of a curve is $y = x^{-2/3} \cdot \ln x$ for $x > 0$. The curve has one stationary point.

(a) Find the exact coordinates of the stationary point. ---[5]

(b) Show that: $\int_1^8 y\,dx = 18\ln 2 - 9$ --[5]

[S-21/31/Q9]

Solution (a) $y = x^{-2/3} \cdot \ln x$ ----①

$\frac{dy}{dx} = -\frac{2}{3}x^{-5/3}\ln x + x^{-2/3} \cdot \frac{1}{x}$

$= x^{-5/3}\left[-\frac{2}{3}\ln x + 1\right]$ ---②

for stationary point, $\frac{dy}{dx} = 0$

from ②

$x^{-5/3}\left(1 - \frac{2}{3}\ln x\right) = 0$

$\Rightarrow 1 - \frac{2}{3}\ln x = 0$ [Given $x^{-5/3} \neq 0$, as $x > 0$]

$\Rightarrow x = e^{3/2}$, from ① $y = (e^{3/2})^{-2/3} \cdot \ln e^{3/2}$

$y = \frac{3}{2e}$ ✓

∴ Stationary point $\left(e^{3/2}, \frac{3}{2e}\right)$ ✓

(b) $\int_1^8 y\,dx = \int_1^8 x^{-2/3}\ln x\,dx$ ---③

consider

$\int x^{-2/3} \cdot \ln x\,dx$

$= \ln x \cdot \int x^{-2/3}dx - \int\left(\frac{d}{dx}\ln x \cdot \int x^{-2/3}dx\right)dx$

$= \ln x \cdot 3x^{1/3} - \int \frac{1}{x} \times 3x^{1/3}dx$

$= 3x^{1/3}\ln x - \int 3x^{-2/3}dx$

$= 3x^{1/3}\ln x - 9x^{1/3}$ --- ④

from ③ and ④

$\int_1^8 y\,dx = \left[3x^{1/3}\ln x - 9x^{1/3}\right]_1^8$

$= \left[(3 \times 2\ln 8 - 9 \times 2) - (0 - 9)\right]$

$= 6 \cdot \ln 2^3 - 9 = (18\ln 2 - 9)$ ✓

25. Using integration by parts, find the exact value of $\int_0^2 \tan^{-1}\left(\frac{1}{2}x\right)dx$ ---[5]

[S-21/32/Q4]

Solution: $\int_0^2 \tan^{-1}\left(\frac{1}{2}x\right)dx$ --- ①

consider $\int \tan^{-1}\left(\frac{1}{2}x\right)dx = \tan^{-1}\left(\frac{1}{2}x\right) \cdot \int 1\,dx - \int\left(\frac{d}{dx}\tan^{-1}\left(\frac{x}{2}\right) \cdot \int 1\,dx\right)dx$

$= x \cdot \tan^{-1}\left(\frac{1}{2}x\right) - \int x \cdot \frac{1}{\left(1 + \left(\frac{x}{2}\right)^2\right)} \times \frac{1}{2}dx = x\tan^{-1}\left(\frac{x}{2}\right) - \int \frac{2x}{(4+x^2)}dx$

$= x\tan^{-1}\left(\frac{1}{2}x\right) - \ln(4+x^2)$ ---② $\left[\because 4+x^2 = u \Rightarrow 2x\,dx = du\right\} \Rightarrow \int \frac{1}{u}du = \ln u$

from ① and ②

$\int_0^2 \tan^{-1}\left(\frac{1}{2}x\right)dx = \left[x\tan^{-1}\frac{x}{2} - \ln(4+x^2)\right]_0^2$

$= (2\tan^{-1}1 - \ln 8) - (0 - \ln 4)$

$= 2 \times \frac{\pi}{4} - (\ln 8 - \ln 4) = \frac{\pi}{2} - \ln 8/4$

$= \frac{1}{2}\pi - \ln 2$ ✓

26 (a) Prove that $\operatorname{Cosec} 2\theta - \operatorname{Cot} 2\theta \equiv \tan\theta$ --- [3]

(b) Hence show that $\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} (\operatorname{Cosec} 2\theta - \operatorname{Cot} 2\theta)\, d\theta = \frac{1}{2}\ln 2$ --- [4]

$\boxed{S\text{-}21/32/Q6}$

Solution (a) $\operatorname{Cosec} 2\theta - \operatorname{Cot} 2\theta$

$= \frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta}$

$= \frac{1 - \cos 2\theta}{\sin 2\theta}$

$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \tan\theta$ --- ①

(b) Consider $\int (\operatorname{Cosec} 2\theta - \operatorname{Cot} 2\theta)\, d\theta$

$= \int \tan\theta\, d\theta$ [using part (a) ①]

$= \ln\sec\theta$ --- ②

$\therefore \int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} (\operatorname{Cosec} 2\theta - \operatorname{Cot} 2\theta)\, d\theta = [\ln\sec\theta]_{\pi/4}^{\pi/3}$ from ②

$= \left(\ln\sec\frac{\pi}{3} - \ln\sec\frac{\pi}{4}\right)$

$= \ln\left[\frac{\sec\pi/3}{\sec\frac{\pi}{4}}\right] = \ln\left(\frac{2}{\sqrt{2}}\right) = \ln\sqrt{2}$

$= \frac{1}{2}\ln 2$ ✓

27 The equation of a curve is $y = e^{-5x}.\tan^2 x$ for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$

Find the x-coordinates of the stationary points of the curve. Give your answers correct to 3 decimal places where appropriate. --- [8]

$\boxed{S\text{-}21/32/Q8}$

<u>Solution</u>: $y = e^{-5x}.\tan^2 x$

$\frac{dy}{dx} = -5e^{-5x}.\tan^2 x + e^{-5x}.2\tan x.\sec^2 x$

$= e^{-5x}[-5\tan^2 x + 2\tan x(1+\tan^2 x)]$

$= e^{-5x}\tan x[2\tan^2 x - 5\tan x + 2]$

for stationary points $\frac{dy}{dx} = 0 \Rightarrow e^{-5x}.\tan x(\tan x - 2)(2\tan x - 1) = 0$

$\Rightarrow \tan x = 0,\ \tan x = \frac{1}{2},\ \tan x = 2$

$\Rightarrow x = 0;\ x = \tan^{-1}\frac{1}{2}$ or $x = \tan^{-1} 2$

$\underline{x = 0}$; $\underline{x = 0.468}$; $\underline{x = 1.107}$ ✓ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

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28. The parametric equations of a curve are, $y=(t-1)e^{-2t}$; $x=$ [illegible] where $t>-2$
(a) Express $\frac{dy}{dx}$ in terms of t, simplify your answer. ... [5]
(b) Find the exact Y-coordinate of the stationary point of the curve. --- [2]
[S-21/33/Q3]

Solution: $y=(t-1)e^{-2t}$ --- ①

(a) $\frac{dy}{dt}=e^{-2t}+(-2)(t-1)e^{-2t}$
$= e^{-2t}(3-2t)$ ---- ②

Now $x = t+\ln(t+2)$

$\frac{dx}{dt} = 1+\frac{1}{t+2} = \frac{(t+3)}{(t+2)}$ -- ③

$\therefore \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{e^{-2t}(3-2t)}{(t+3)/(t+2)}$

$= \frac{(3-2t)(t+2)}{(t+3)}.e^{-2t}$ -- ④

(b) for stationary point $\frac{dy}{dx}=0$

$\Rightarrow e^{-2t}.\frac{(3-2t)(t+2)}{(t+3)}=0$

$\Rightarrow (3-2t)=0$, $t+2=0$

$\Rightarrow t=3/2$; $t=-2^{\times}$ (Given $t>-2$)

from ①

$t=3/2$; $y=(\frac{3}{2}-1)e^{-2\times 3/2}$

$y=\frac{1}{2}e^{-3}$

Y-coord of stationary point

29. Let $f(x)=\frac{15-6x}{(1+2x)(4-x)}$
(a) Express in partial fractions. --[3]
(b) Hence find $\int_1^2 f(x)\,dx$, giving your answer in the form $\ln(a/b)$, where a and b are integers. [S-21/33/Q4] --[4]

Solution: $f(x)=\frac{15-6x}{(1+2x)(4-x)}=\frac{a}{(1+2x)}+\frac{b}{(4-x)}$ --- ①

multiply ① by $(1+2x)$

$\frac{15-6x}{(4-x)} = a+\frac{b(1+2x)}{(4-x)}$ --- ②

Put $(1+2x)=0 \Rightarrow x=-\frac{1}{2}$ in ②

$\frac{15+3}{4+\frac{1}{2}} = a \Rightarrow a=4$

Now multiply ① by $(4-x)$

$\frac{15-6x}{1+2x} = \frac{a(4-x)}{(1+2x)}+b$ --- ③

Put $4-x=0 \Rightarrow x=4$ in ③

$\frac{15-6\times 4}{1+2x} = b \Rightarrow b=-1$

Put $a=4$ & $b=-1$ in ① $\rightarrow$ $\frac{4}{(1+2x)}+\frac{-1}{(4-x)}$ -- ④ Partial Frac

(b) Consider
$\int_1^2 f(x)\,dx = \int_1^2\left(\frac{4}{1+2x}-\frac{1}{4-x}\right)dx$ (using Part (a))

$=\left[\frac{4}{2}\ln(1+2x)+\frac{\ln(4-x)}{-1}\right]_1^2$

$=\left[2\ln(1+2x)+\ln(4-x)\right]_1^2$

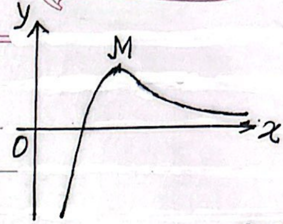
$=\left[\ln(1+2x)^2\cdot(4-x)\right]_1^2$

$=\ln(25)\times(2)-\ln(9)\times 3]$

$=\ln 50-\ln 27$

$=\ln\left(\frac{50}{27}\right)$

30. The diagram shows the curve $y = \frac{\ln x}{x^4}$ and its maximum point M.



(a) Find the exact coordinates of M. ---[4]

(b) By using integration by parts, show that for $a>1$ $\int_1^a \frac{\ln x}{x^4} < \frac{1}{9}$ ---[6]

S-21/33/Q8

Solution: $y = \frac{\ln x}{x^4}$ ---- (1)

$$\frac{dy}{dx} = \frac{x^4 \times \frac{1}{x} - \ln x \times 4x^3}{(x^4)^2}$$

$$= \frac{x^3[1-4\ln x]}{x^8} \text{ --- (2)}$$

for the point of Max, 'M' $\frac{dy}{dx} = 0$

$\Rightarrow \frac{1-4\ln x}{x^5} = 0$

$\Rightarrow 1 - 4\ln x = 0$

$\Rightarrow \ln x = \frac{1}{4} \Rightarrow x = e^{1/4}$

from (1) at $x = e^{1/4} \rightarrow y = \frac{\ln e^{\frac{1}{4}}}{(e^{\frac{1}{4}})^4} = \frac{1/4}{e} = \frac{1}{4e}$

$\therefore$ Point of Max, $M(\sqrt[4]{e}, \frac{1}{4e})$.

(b) Consider $\int \frac{\ln x}{x^4}\,dx$

$$= \ln x \cdot \int x^{-4}dx - \int\left(\frac{d}{dx}\ln x \cdot \int x^{-4}dx\right)dx$$

$$= \ln x \cdot \frac{x^{-3}}{-3} - \int \frac{1}{x} \cdot \frac{x^{-3}}{-3}\,dx$$

$$= -\frac{1}{3x^3}\ln x + \frac{1}{3}\int x^{-4}dx$$

$$= -\frac{1}{3x^3}\ln x + \frac{1}{3} \cdot \frac{x^{-3}}{-3}$$

$$= -\frac{1}{3x^3}\ln x - \frac{1}{9x^3} \text{ ---- (3)}$$

$$\therefore \int_1^a \frac{\ln x}{x^4}dx = \left[-\frac{1}{3x^3}\ln x - \frac{1}{9x^3}\right]_1^a$$

$$= -\frac{1}{3a^3}\ln a - \frac{1}{9a^3} + \frac{1}{9}$$

$$= \frac{1}{9} - \frac{1}{a^3}\left[\frac{\ln a}{3} + \frac{1}{9}\right] \quad \left[\begin{array}{l} a>1 \\ \ln a > 0\end{array}\right.$$

$$< \frac{1}{9} \quad \left[\because \frac{1}{a^3}\left(\frac{\ln a}{3} + \frac{1}{9}\right) > 0\right]$$