PURE MATHEMATICS -3

9709

(March, June and November series 2020 – 2023 With marking scheme)

DIFFERENTIAL EQUATION

EXERCISE -1

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1) SP-2020 _9709_3 Q10

In a chemical reaction, a compound X is formed from two compounds Y and Z.

The masses in grams of X, Y and Z present at time t seconds after the start of the reaction are x, $10 - x$ and $20 - x$ respectively. At any time the rate of formation of X is proportional to the product of the masses of Y and Z present at the time. When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 2$.

(a) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.01(10 - x)(20 - x).$$ [1]

(b) Solve this differential equation and obtain an expression for x in terms of t. [9]

(c) State what happens to the value of x when t becomes large. [1]

2) MARCH-2020 _9709_32 Q6

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{1 + 4y^2}{e^x}.$$

It is given that $y = 0$ when $x = 1$.

(a) Solve the differential equation, obtaining an expression for y in terms of x. [7]

(b) State what happens to the value of y as x tends to infinity. [1]

3) MARCH-2021 _9709_32 Q4

The variables x and y satisfy the differential equation

$$(1 - \cos x)\frac{dy}{dx} = y \sin x.$$

It is given that $y = 4$ when $x = \pi$.

(a) Solve the differential equation, obtaining an expression for y in terms of x. [6]

(b) Sketch the graph of y against x for $0 < x < 2\pi$. [1]

4) MARCH-2022 _9709_32 Q9

The variables x and y satisfy the differential equation

$$(x + 1)(3x + 1)\frac{dy}{dx} = y,$$

and it is given that $y = 1$ when $x = 1$.

Solve the differential equation and find the exact value of y when $x = 3$, giving your answer in a simplified form. [9]

5) MARCH-2023 _9709_32 Q9

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = e^{3y}\sin^2 2x.$$

It is given that $y = 0$ when $x = 0$.

Solve the differential equation and find the value of y when $x = \frac{1}{2}$. [7]

6) JUNE-2020 _9709_31 Q8

A certain curve is such that its gradient at a point (x, y) is proportional to $\frac{y}{x\sqrt{x}}$. The curve passes through the points with coordinates $(1, 1)$ and $(4, e)$.

(a) By setting up and solving a differential equation, find the equation of the curve, expressing y in terms of x. [8]

(b) Describe what happens to y as x tends to infinity. [1]

7) JUNE-2020 _9709_32 Q7

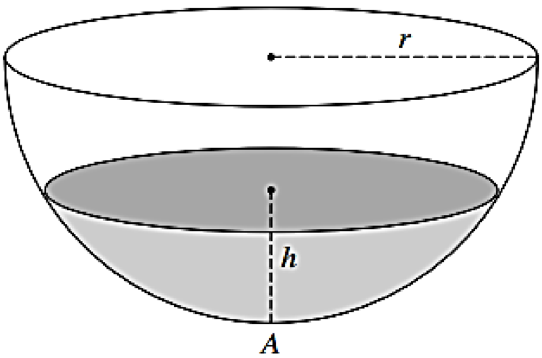
The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{y-1}{(x+1)(x+3)}.$$

It is given that $y = 2$ when $x = 0$.

Solve the differential equation, obtaining an expression for y in terms of x. [9]

8) JUNE-2020 _9709_33 Q10



A tank containing water is in the form of a hemisphere. The axis is vertical, the lowest point is A and the radius is r, as shown in the diagram. The depth of water at time t is h. At time $t = 0$ the tank is full and the depth of the water is r. At this instant a tap at A is opened and water begins to flow out at a rate proportional to $\sqrt{h}$. The tank becomes empty at time $t = 14$.

The volume of water in the tank is V when the depth is h. It is given that $V = \frac{1}{3}\pi(3rh^2 - h^3)$.

(a) Show that h and t satisfy a differential equation of the form

$$\frac{dh}{dt} = -\frac{B}{2rh^{\frac{1}{2}} - h^{\frac{3}{2}}},$$

where B is a positive constant. [4]

(b) Solve the differential equation and obtain an expression for t in terms of h and r. [8]

9) JUNE-2021 _9709_31 Q10

The variables x and t satisfy the differential equation $\frac{dx}{dt} = x^2(1+2x)$, and $x = 1$ when $t = 0$.

Using partial fractions, solve the differential equation, obtaining an expression for t in terms of x. [11]

10) JUNE-2021 _9709_32 Q7

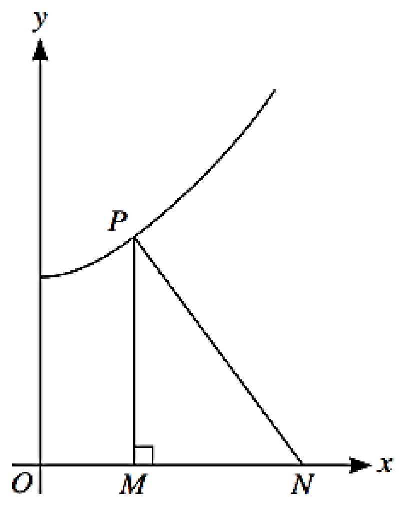
A curve is such that the gradient at a general point with coordinates (x, y) is proportional to $\frac{y}{\sqrt{x+1}}$. The curve passes through the points with coordinates $(0, 1)$ and $(3, e)$.

By setting up and solving a differential equation, find the equation of the curve, expressing y in terms of x. [7]

11) JUNE-2021 _9709_33 Q7

(ii) Hence show that x and y satisfy the differential equation $\frac{1}{2}y^2\frac{dy}{dx} = \tan x$. [2]

(b) Given that $y = 1$ when $x = 0$, solve this differential equation to find the equation of the curve, expressing y in terms of x. [6]



For the curve shown in the diagram, the normal to the curve at the point P with coordinates (x, y) meets the x-axis at N. The point M is the foot of the perpendicular from P to the x-axis.

The curve is such that for all values of x in the interval $0 \leqslant x < \frac{1}{2}\pi$, the area of triangle PMN is equal to $\tan x$.

(a) **(i)** Show that $\frac{MN}{y} = \frac{dy}{dx}$. [1]

12) JUNE-2022 _9709_31 Q4

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{xy}{1 + x^2},$$

and $y = 2$ when $x = 0$.

Solve the differential equation, obtaining a simplified expression for y in terms of x. [7]

13) JUNE-2022 _9709_32 Q6

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = xe^{y-x},$$

and $y = 0$ when $x = 0$.

(a) Solve the differential equation, obtaining an expression for y in terms of x. [7]

(b) Find the value of y when $x = 1$, giving your answer in the form $a - \ln b$, where a and b are integers. [1]

14) JUNE-2022 _9709_33 Q8

At time t days after the start of observations, the number of insects in a population is N. The variation in the number of insects is modelled by a differential equation of the form $\frac{dN}{dt} = kN^{\frac{3}{2}} \cos 0.02t$, where k is a constant and N is a continuous variable. It is given that when $t = 0$, $N = 100$.

(a) Solve the differential equation, obtaining a relation between N, k and t. [5]

(b) Given also that $N = 625$ when $t = 50$, find the value of k. [2]

(c) Obtain an expression for N in terms of t, and find the greatest value of N predicted by this model. [2]

15) JUNE-2023 _9709_31 Q7

The variables x and y satisfy the differential equation

$$\cos 2x \frac{dy}{dx} = \frac{4 \tan 2x}{\sin^2 3y},$$

where $0 \leqslant x < \frac{1}{4}\pi$. It is given that $y = 0$ when $x = \frac{1}{6}\pi$.

Solve the differential equation to obtain the value of x when $y = \frac{1}{6}\pi$. Give your answer correct to 3 decimal places. [8]

16) JUNE-2023 _9709_32 Q8

(a) The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{4 + 9y^2}{e^{2x+1}}.$$

It is given that $y = 0$ when $x = 1$.

Solve the differential equation, obtaining an expression for y in terms of x. [7]

(b) State what happens to the value of y as x tends to infinity. Give your answer in an exact form. [1]

17) JUNE-2023 _9709_33 Q8

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{y^2 + 4}{x(y + 4)}$$

for $x > 0$. It is given that $x = 4$ when $y = 2\sqrt{3}$.

Solve the differential equation to obtain the value of x when $y = 2$. [8]

18) OCT 2020_9709_31 Q8

The coordinates (x, y) of a general point of a curve satisfy the differential equation

$$x\frac{dy}{dx} = (1 - 2x^2)y,$$

for $x > 0$. It is given that $y = 1$ when $x = 1$.

Solve the differential equation, obtaining an expression for y in terms of x. [6]

19) OCT 2020_9709_32 Q7

The variables x and t satisfy the differential equation

$$e^{3t}\frac{dx}{dt} = \cos^2 2x,$$

for $t \geqslant 0$. It is given that $x = 0$ when $t = 0$.

(a) Solve the differential equation and obtain an expression for x in terms of t. [7]

(b) State what happens to the value of x when t tends to infinity. [1]

20) OCT 2021_9709_31 Q7

(a) Given that $y = \ln(\ln x)$, show that

$$\frac{dy}{dx} = \frac{1}{x\ln x}.$$ [1]

The variables x and t satisfy the differential equation

$$x \ln x + t\frac{dx}{dt} = 0.$$

It is given that $x = e$ when $t = 2$.

(b) Solve the differential equation obtaining an expression for x in terms of t, simplifying your answer. [7]

(c) Hence state what happens to the value of x as t tends to infinity. [1]

21) OCT 2021_9709_32 Q7

The variables x and y satisfy the differential equation

$$e^{2x}\frac{dy}{dx} = 4xy^2,$$

and it is given that $y = 1$ when $x = 0$.

Solve the differential equation, obtaining an expression for y in terms of x. [7]

22) OCT 2021_9709_33 Q10 (a,b)

A large plantation of area $20\,\text{km}^2$ is becoming infected with a plant disease. At time t years the area infected is $x\,\text{km}^2$ and the rate of increase of x is proportional to the ratio of the area infected to the area not yet infected.

When $t = 0$, $x = 1$ and $\frac{dx}{dt} = 1$.

(a) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = \frac{19x}{20 - x}.$$ [2]

(b) Solve the differential equation and show that when $t = 1$ the value of x satisfies the equation $x = e^{0.9+0.05x}$. [5]

(c) Use an iterative formula based on the equation in part **(b)**, with an initial value of 2, to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(d) Calculate the value of t at which the entire plantation becomes infected. [1]

23) OCT 2022-9709_31 Q8

In a certain chemical reaction the amount, x grams, of a substance is increasing. The differential equation satisfied by x and t, the time in seconds since the reaction began, is

$$\frac{dx}{dt} = kxe^{-0.1t},$$

where k is a positive constant. It is given that $x = 20$ at the start of the reaction.

(a) Solve the differential equation, obtaining a relation between x, t and k. [5]

24) OCT 2022_9709_32 Q7

The variables x and θ satisfy the differential equation

$$x \sin^2 \theta \frac{dx}{d\theta} = \tan^2 \theta - 2 \cot \theta,$$

for $0 < \theta < \frac{1}{2}\pi$ and $x > 0$. It is given that $x = 2$ when $\theta = \frac{1}{4}\pi$.

(a) Show that $\frac{d}{d\theta}(\cot^2 \theta) = -\frac{2 \cot \theta}{\sin^2 \theta}$.

(You may assume without proof that the derivative of $\cot \theta$ with respect to θ is $-\operatorname{cosec}^2 \theta$.) [1]

(b) Solve the differential equation and find the value of x when $\theta = \frac{1}{6}\pi$. [7]

25) OCT 2022_9709_33 Q10

A gardener is filling an ornamental pool with water, using a hose that delivers 30 litres of water per minute. Initially the pool is empty. At time t minutes after filling begins the volume of water in the pool is V litres. The pool has a small leak and loses water at a rate of $0.01V$ litres per minute.

The differential equation satisfied by V and t is of the form $\frac{dV}{dt} = a - bV$.

(a) Write down the values of the constants a and b. [1]

(b) Solve the differential equation and find the value of t when $V = 1000$. [6]

(c) Obtain an expression for V in terms of t and hence state what happens to V as t becomes large. [2]

MARKING SCHEME

1) SP-2020 _9709_3 Q10

(a)	State or imply $\frac{dx}{dt} = k(10-x)(20-x)$ and show $k = 0.01$	1	B1
(b)	Separate variables and attempt integration of at least one side	1	M1
	Carry out an attempt to find A and B such that $\frac{1}{(10-x)(20-x)} \equiv \frac{A}{10-x} + \frac{B}{20-x}$	1	M1
	Obtain $A = \frac{1}{10}$ and $B = -\frac{1}{10}$, or equivalent	1	A1
	Integrate and obtain $-\frac{1}{10}\ln(10-x) + \frac{1}{10}\ln(20-x)$, or equivalent	1	A1FT
	Integrate and obtain term $0.01t$, or equivalent	1	A1
	Evaluate a constant, or use limits $t = 0, x = 0$ in a solution containing terms of the form $a\ln(10-x)$, $b\ln(20-x)$ and ct	1	M1
	Obtain answer in any form, e.g. $-\frac{1}{10}\ln(10-x) + \frac{1}{10}\ln(20-x) = 0.01t + \frac{1}{10}\ln 2$	1	A1FT
	Use laws of logarithms correctly to remove logarithms	1	M1
	Rearrange and obtain $x = \frac{20(e^{0.1t}-1)}{2e^{0.1t}-1}$, or equivalent	1	A1
		9	
	State that x approaches 10	1	B1

2) MARCH-2020 _9709_32 Q6

a)	Separate variables correctly and attempt integration of at least one side	B1
	Obtain term of the form $a\tan^{-1}(2y)$	M1
	Obtain term $\frac{1}{2}\tan^{-1}(2y)$	A1
	Obtain term $-e^{-x}$	B1
	Use $x = 1$, $y = 0$ to evaluate a constant or as limits in a solution containing terms of the form $a\tan^{-1}(by)$ and $ce^{\pm x}$	M1
	Obtain correct answer in any form	A1
	Obtain final answer $y = \frac{1}{2}\tan\left(2e^{-1} - 2e^{-x}\right)$, or equivalent	A1
		7
b)	State that y approaches $\frac{1}{2}\tan\left(2e^{-1}\right)$, or equivalent	B1FT
		1

3) MARCH-2021 _9709_32 Q4

4(a)	Separate variables correctly and attempt integration of at least one side	M1
	Obtain term $\ln y$	A1
	Obtain term of the form $\pm\ln(1-\cos x)$	M1
	Obtain term $\ln(1-\cos x)$	A1
	Use $x=\pi, y=4$ to evaluate a constant, or as limits, in a solution containing terms of the form $a\ln y$ and $b\ln(1-\cos x)$	M1
	Obtain final answer $y=2(1-\cos x)$	A1
		6
(b)	Show a correct graph for $0<x<2\pi$ with the maximum at $x=\pi$	B1 FT
		1

4) MARCH-2022 _9709_32 Q9

Correctly separate variables and integrate at least one side	M1
Obtain term $\ln y$ from integral of $1/y$	B1
State or imply the form $\frac{A}{x+1}+\frac{B}{3x+1}$ and use a correct method to find a constant	M1
Obtain $A=-\frac{1}{2}$ and $B=\frac{3}{2}$	A1
Obtain terms $-\frac{1}{2}\ln(x+1)+\frac{1}{2}\ln(3x+1)$ or $-\frac{1}{2}\ln(2x+2)+\frac{1}{2}\ln(6x+2)$ or combination of these terms	A1 FT + A1 FT
Use $x=1$ and $y=1$ to evaluate a constant, or expression for a constant, (decimal equivalent of ln terms allowed) or as limits, in a solution containing terms $a\ln y$, $b\ln(x+1)$ and $c\ln(3x+1)$, where $abc\neq 0$	*M1
Obtain an expression for y or y^2 and substitute $x=3$	DM1
Obtain answer $y=\frac{1}{2}\sqrt{5}$ or $\sqrt{\frac{5}{4}}$ or $\sqrt{\frac{10}{8}}$	A1
	9

5) MARCH-2023 _9709_32 Q9

Separate variables correctly and obtain e^{-3y} and $\sin^2 2x$ on the opposite sides	**B1**
Obtain term $-\frac{1}{3}e^{-3y}$	**B1**
Use correct double angle formula for $\sin^2 2x = (1/2)[1 - \cos 4x]$	**M1**
Obtain terms $\frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]$ oe	**A1**
Use $x = 0$, $y = 0$ to evaluate a constant or as limits in a solution containing terms of the form ax and $b\sin 4x$ and $ce^{\pm 3y}$	**M1**
Obtain correct answer in any form e.g. $-\frac{1}{3}e^{-3y} = \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right] - \frac{1}{3}$	**A1**
Substitute $x = \frac{1}{2}$ and obtain $y = 0.175$ or $-\frac{1}{3}\ln\left(\frac{1}{4} + \frac{3}{8}\sin 2\right)$	**A1**
	7

6) JUNE-2020 _9709_31 Q8

a)	State $\frac{dy}{dx} = k\frac{y}{x\sqrt{x}}$, or equivalent	**B1**
	Separate variables correctly and attempt integration of at least one side	**M1**
	Obtain term $\ln y$, or equivalent	**A1**
	Obtain term $-2k\frac{1}{\sqrt{x}}$, or equivalent	**A1**
	Use given coordinates to find k or a constant of integration c in a solution containing terms of the form $a \ln y$ and $\frac{b}{\sqrt{x}}$, where $ab \neq 0$	**M1**
	Obtain $k = 1$ and $c = 2$	**A1 + A1**
	Obtain final answer $y = \exp\left(-\frac{2}{\sqrt{x}} + 2\right)$, or equivalent	**A1**
		8
(b)	State that y approaches e^2 (FT *their* c in part (a) of the correct form)	**B1FT**
		1

7) JUNE-2020 _9709_32 Q7

Separate variables correctly and integrate at least one side	B1
Obtain term $\ln(y-1)$	B1
Carry out a relevant method to determine A and B such that $\frac{1}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$	M1
Obtain $A = \frac{1}{2}$ and $B = -\frac{1}{2}$	A1
Integrate and obtain terms $\frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x+3)$ $\frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x+3)$, or equivalent (FT is on A and B)	A1 FT + A1 FT
Use $x = 0$, $y = 2$ to evaluate a constant, or as limits in a solution containing terms of the form $a\ln(y-1)$, $b\ln(x+1)$ and $c\ln(x+3)$, where $abc \neq 0$	M1
Obtain correct answer in any form	A1
Obtain final answer $y = 1 + \sqrt{\left(\frac{3x+3}{x+3}\right)}$, or equivalent	A1
	9

8) JUNE-2020 _9709_33 Q10

(a)	State or imply $\frac{dV}{dt} = -k\sqrt{h}$	B1
	State or imply $\frac{dV}{dh} = 2\pi rh - \pi h^2$, or equivalent	B1
	Use $\frac{dV}{dt} = \frac{dV}{dh}.\frac{dh}{dt}$	M1
	Obtain the given answer correctly	A1
		4
(b)	Separate variables and attempt integration of at least one side	M1
	Obtain terms $\frac{4}{3}rh^{\frac{3}{2}} - \frac{2}{5}h^{\frac{5}{2}}$ and $-Bt$	A3, 2, 1, 0
	Use $t = 0$, $h = r$ to find a constant of integration c	M1
	Use $t = 14$, $h = 0$ to find B	M1
	Obtain correct c and B, e.g. $c = \frac{14}{15}r^{\frac{5}{2}}$, $B = \frac{1}{15}r^{\frac{5}{2}}$	A1
	Obtain final answer $t = 14 - 20\left(\frac{h}{r}\right)^{\frac{3}{2}} + 6\left(\frac{h}{r}\right)^{\frac{5}{2}}$, or equivalent	A1
		8

9) JUNE-2021 _9709_31 Q10

State a suitable form of partial fractions for $\frac{1}{x^2(1+2x)}$	B1	e.g. $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{1+2x}$ or $\frac{Ax+B}{x^2}+\frac{C}{1+2x}$
Use a relevant method to determine a constant	M1	
Obtain one of $A=-2$, $B=1$ and $C=4$	A1	
Obtain a second value	A1	
Obtain the third value	A1	
Separate variables correctly and integrate at least one term	M1	
Obtain terms $-2\ln x-\frac{1}{x}+2\ln(1+2x)$ and t	B3 FT	The FT is on A, B and C. Withhold B1 for each error or omission.
Evaluate a constant, or use limits $x=1$, $t=0$ in a solution containing terms t, $a\ln x$ and $b\ln(1+2x)$, where $ab\neq 0$	M1	
Obtain a correct expression for t in any form, e.g. $t=-\frac{1}{x}+2\ln\left(\frac{1+2x}{3x}\right)+1$	A1	
	11	

10) JUNE-2021 _9709_32 Q7

State equation $\frac{dy}{dx}=k\frac{y}{\sqrt{x+1}}$	B1	OE. Must be a differential equation.
Separate variables correctly for *their* differential equation and integrate at least one side	*M1	$\int\frac{1}{y}dy=\int\frac{k}{\sqrt{x+1}}dx$
Obtain $\ln y$	A1	Allow M1A1A1 if they have assumed $k=1$ or are working with an incorrect value for k
Obtain $2[k]\sqrt{x+1}$	A1	
Use (0, 1) **and** (3, e) in an expression containing $\ln y$ and $\sqrt{x+1}$ and a constant of integration to determine k and/or a constant of integration c (or use (0, 1), (3, e) and (x, y) as limits on definite integrals)	DM1	If remove logs before finding the constant of integration then the constant must be of the correct form.
Obtain $k=\frac{1}{2}$ and $c=-1$	A1	OE. $(\ln y=\sqrt{x+1}-1)$ Their value of c will depend on where c is in their equation and whether they are working with $\frac{1}{k}\ln y$. The value of k must be consistent with what they integrated.
Obtain $y=\exp\left(\sqrt{x+1}-1\right)$	A1	NFWW, OE, ISW.
	7	

11) JUNE-2021 _9709_33 Q7

7(a)(i)	Justify the given statement $\frac{MN}{y} = \frac{dy}{dx}$	B1	
		1	
(a)(ii)	Express the area of PMN in terms of y and $\frac{dy}{dx}$ and equate to $\tan x$	M1	
	Obtain the given equation correctly	A1	
		2	
7(b)	Separate variables and integrate at least one side	M1	
	Obtain term $\frac{1}{6}y^3$	A1	
	Obtain term of the form $\pm \ln \cos x$	M1	
	Evaluate a constant or use $x = 0$ and $y = 1$ in a solution containing terms ay^3 and $\pm \ln \cos x$, or equivalent	M1	
	Obtain correct answer in any form, e.g. $\frac{1}{6}y^3 = -\ln \cos x + \frac{1}{6}$	A1	
	Obtain final answer $y = \sqrt[3]{(1-6\ln \cos x)}$	A1	OE
		6	

12) JUNE-2022 _9709_31 Q4

Separate variables correctly	B1	$\int \frac{x}{1+x^2}dx = \int \frac{1}{y}dy$ Accept without integral signs.
Obtain term $\ln y$	B1	
State term of the form $k\ln(1+x^2)$	M1	
State correct term $\frac{1}{2}\ln(1+x^2)$	A1	OE
Evaluate a constant, or use limits $x = 0, y = 2$ in a solution containing terms $a\ \ln y$ and $b\ln(1+x^2)$ where $ab \neq 0$	M1	If they remove logs first the constant must be of the correct form.
Obtain correct solution in any form	A1	e.g $\ln y + \ln\frac{1}{2} = \frac{1}{2}\ln(1+x^2)$
Simplify and obtain $y = 2\sqrt{1+x^2}$	A1	OE The question asks for simplification, so need to deal with $\exp(\ln(...))$.
	7	

13) JUNE-2022 _9709_32 Q6

a)	Correct separation of variables	B1	$\int e^{-y} dy = \int xe^{-x} dx$ Condone missing integral signs.
	Obtain term $-e^{-y}$	B1	
	Commence integration by parts and reach $\pm xe^{-x} \pm \int e^{-x} dx$	*M1	M0 if clearly using differentiation of a product.
	Complete integration and obtain $-xe^{-x} - e^{-x}$	A1	
	Use $x = 0$ and $y = 0$ to evaluate a constant or as limits in a solution containing or derived from terms ae^{-y}, bxe^{-x} and ce^{-x}, where $abc \neq 0$	DM1	Must see working for this. In a correct solution they should have $-e^{-y} + C = -xe^{-x} - e^{-x}$ or equivalent. If they take logarithms before finding the constant, the constant must be of the right form.
	Correct solution in any form Must follow from correct working	A1	e.g. $-e^{-y} = -xe^{-x} - e^{-x}$ A0 if constant of integration ignored or assumed to be zero.
	Obtain final answer $y = -\ln\left((x+1)e^{-x}\right)$ from correct working	A1	OE e.g. $y = x - \ln(x+1)$, $y = \ln\left(\frac{e^x}{x+1}\right)$. A0 if constant of integration ignored or assumed to be zero.
		7	
b)	Obtain answer $(y =) 1 - \ln 2$	B1	Must follow from at least 6 or7 obtained in part 6(a).
		1	

14) JUNE-2022 _9709_33 Q8

(a)	Separate variables correctly	B1	$\frac{dN}{N^{\frac{3}{2}}} = (k\cos 0.02t)\, dt$ Allow without integral signs.
	Obtain term $-\frac{2}{\sqrt{N}}$	B1	OE Ignore position of k.
	Obtain term $50\sin 0.02t$	B1	OE Ignore position of k.
	Use $t = 0$, $N = 100$ to evaluate a constant, or as limits, in a solution containing terms $\frac{a}{\sqrt{N}}$ and $b\sin 0.02t$, where $ab \neq 0$	M1	$\left[\text{e.g. } c = -0.2 \text{ or } c = \frac{-0.2}{k}\right]$
	Obtain correct solution in any form, e.g. $-\frac{2}{\sqrt{N}} = 50k\sin 0.02t - 0.2$	A1	OE ISW e.g. $N = \frac{1}{(25k\sin 0.02t - 0.1)^2}$ $-2N^{-\frac{1}{2}} = \frac{k}{0.02}\sin 0.02t - \frac{1}{5}$ $50k\sin 0.02t = -\frac{2}{\sqrt{N}} + \frac{1}{5}$ $\frac{1}{\sqrt{N}} = -\frac{1}{2}k(50\sin 0.02t) + \frac{1}{10}$ $50\sin\left(\frac{1}{50}t\right) = -\frac{2\sqrt{N}}{kN} + \frac{20}{100k}$
		5	
(b)	Use the substitution $N = 625$ and $t = 50$ in expression of appropriate form to evaluate k	M1	Expression must contain $a + bk\sin 0.02t$, $(\sqrt{N})^{\pm n}$, where $n = -1, 1, 3$ or 5 and a and b are constants $ab \neq 0$ or $(a + bk\sin 0.02t)^{\pm 2}$ and $(N)^{\pm n}$. Allow with k replaced by $\frac{1}{k}$, error due to $k(N^{-3/2})$ when separating variables in 8(a). If invert term by term when 3 terms shown then M0.
	Obtain $k = 0.00285[2148]$	A1	Must evaluate sin1. Degrees $k = 0.138$ M1 A0.
		2	

(c)	Rearrange and obtain $N = 4(0.2 - 0.142(607)\sin 0.02t)^{-2}$ Substitution for k required	M1	Anything of the form $N = c(d - ek\sin 0.02t)^{-2}$, where c, d and e are constants $cde \neq 0$ and value of k substituted. Allow with k replaced by $1/k$, error due to $k(N^{-3/2})$ when separating variables in 8(a). OE ISW e.g. $N = \left(-\frac{10}{0.7125\sin 0.02t - 1}\right)^2$ $N = \frac{1}{(-0.0713\sin 0.02t + 0.1)^2}$ $N = \frac{100}{\left(\left(\frac{0.6}{\sin 1}\right)\sin 0.02t - 1\right)^2}$ $N = \frac{1}{\left(\frac{3}{-50\sin 1} \times \sin 0.02t + \frac{1}{10}\right)^2}$ $N = \left(-\frac{0.06}{\sin 1}\sin 0.02t + 0.1\right)^{-2}$ $N = \left(\frac{800}{80 - 57\sin 0.02t}\right)^2$ Do not need to substitute for sin(0.02t) = 1, but must substitute for k.
	Accept answers between 1209 and 1215	A1	ISW Substitute $\sin 0.02t = 1$ or $t = 50\sin^{-1} 1$ or 78.5 or 25π. Answer with no working (rubric) 0/2. SC $N = \ldots$ not seen but correct numerical answer B1 1/2.
		2	

15) JUNE-2023 _9709_31 Q7

Correct separation of variables	B1	$\int \sin^2 3y\,dy = \int 4\sec 2x \tan 2x\,dx$ or equivalent. Condone missing integral signs or dx and dy.
Integrate to obtain $k\sec 2x$	M1	
Obtain $2\sec 2x$	A1	
Use double angle formula and integrate to obtain $py + q\sin 6y$	M1	Or two cycles of integration by parts.
Obtain $\frac{1}{2}y - \frac{1}{12}\sin 6y$	A1	
Use $y = 0, x = \frac{\pi}{6}$ in a solution containing terms $\lambda\sec 2x$ and $\mu\sin 6y$ to find the constant of integration	M1	
Obtain $\frac{1}{2}y - \frac{1}{12}\sin 6y = 2\sec 2x - 4$	A1	Or equivalent seen or implied by $\frac{\pi}{2}\left(-\frac{1}{12}\sin\pi\right) = 2\sec 2x - 4$.
Obtain $x = 0.541$	A1	From correct working (not by using the calculator to integrate).
	8	

16) JUNE-2023 _9709_32 Q8

(a)	Separate variables correctly	B1	$\int \frac{1}{4+9y^2}\mathrm{d}y = \int \mathrm{e}^{-(2x+1)}\mathrm{d}x$. Condone missing integral signs or dx and dy missing.
	Obtain term $-\frac{1}{2}\mathrm{e}^{-2x-1}$	B1	OE e.g. $-\frac{1}{2\mathrm{e}}\mathrm{e}^{-2x}$.
	Obtain term of the form $a\tan^{-1}\left(\frac{3y}{2}\right)$	M1	
	Obtain term $\frac{1}{6}\tan^{-1}\left(\frac{3y}{2}\right)$	A1	OE e.g. $\frac{1}{9}\times\frac{3}{2}\tan^{-1}\frac{3y}{2}$.
	Use $x=1, y=0$ to evaluate a constant or as limits in a solution containing or derived from terms of the form $a\tan^{-1}(by)$ and $c\mathrm{e}^{\pm(2x+1)}$	M1	If they rearrange before evaluating the constant, the constant must be of the correct form.
	Obtain correct answer in any form	A1	e.g. $\frac{1}{6}\tan^{-1}\frac{3y}{2} = \frac{1}{2}\mathrm{e}^{-3} - \frac{1}{2}\mathrm{e}^{-(2x+1)}$.
	Obtain final answer $y = \frac{2}{3}\tan\left(3\mathrm{e}^{-3} - 3\mathrm{e}^{-2x-1}\right)$	A1	OE Allow with $3\mathrm{e}^{-3} = 0.149....$
		7	
(b)	State that y approaches $\frac{2}{3}\tan\left(3\mathrm{e}^{-3}\right)$	B1 FT	Or exact equivalent. The FT is on correct work on a solution containing e^{-2x-1}. Condone $y = \ldots$ Accept correct answer stated with minimal wording. 0.10032… is not exact so B0.
		1	

17) JUNE-2023 _9709_33 Q8

Separate the variables correctly	B1	$\frac{y+4}{y^2+4}\mathrm{d}y = \frac{1}{x}\mathrm{d}x$.
Obtain $\ln x$	B1	
Split the fraction and integrate to obtain $p\ln\left(y^2+4\right)$ or $q\tan^{-1}\frac{y}{2}$ correctly	*M1	Only following subdivision into $\frac{y}{y^2+4}+\frac{4}{y^2+4}$. If no subdivision seen then both terms $p\ln\left(y^2+4\right)$ and $q\tan^{-1}\frac{y}{2}$ must be present.
Obtain $\frac{1}{2}\ln\left(y^2+4\right)$	A1	
Obtain $2\tan^{-1}\frac{y}{2}$	A1	
Use $\left(4, 2\sqrt{3}\right)$ in an expression containing at least 2 of $a\ln x, b\ln\left(y^2+4\right)$ and $c\tan^{-1}\frac{y}{2}$ to obtain constant of integration	DM1	Allow one sign or arithmetic error e.g. $\frac{2\pi}{3}$. May use $\left(4, 2\sqrt{3}\right)$ and $(x, 2)$ as limits to find x for the final 3 marks.
Correct solution (any form) e.g. $\frac{1}{2}\ln\left(y^2+4\right)+2\tan^{-1}\frac{y}{2} = \ln x + \frac{2\pi}{3}$ or $\frac{1}{2}\ln\left(y^2+4\right)+2\tan^{-1}\frac{y}{2} = \ln x + 2\tan^{-1}\sqrt{3} + \frac{1}{2}\ln 16 - \ln 4$	A1	However solution not asked for so allow $\frac{1}{2}\ln 8 + 2\tan^{-1}1 = \ln x + 2\tan^{-1}\sqrt{3} + \frac{1}{2}\ln 16 - \ln 4$.
Obtain $\sqrt{8}\mathrm{e}^{-\frac{1}{6}\pi}$ or 1.68 or more accurate or $2\sqrt{2}\mathrm{e}^{-\frac{1}{6}\pi}$ or $\frac{\sqrt{8}}{\mathrm{e}^{\frac{1}{6}\pi}}$ or $\mathrm{e}^{0.516}$	A1	ISW Must remove ln so $x = \mathrm{e}^{(\ln 2\sqrt{2} - \pi/6)}$ A0.

Alternative method for first *M1 A1 A1		
$p\left((y+4)\tan^{-1}\frac{y}{2}-\int\tan^{-1}\frac{y}{2}\,dy\right)$	*M1	Allow sign error.
$(y+4)\frac{1}{2}\tan^{-1}\frac{y}{2}-\frac{y}{2}\tan^{-1}\frac{y}{2}+\int\frac{y}{y^2+4}\,dy$	A1	
Obtain $2\tan^{-1}\frac{y}{2}+\frac{1}{2}\ln\left(y^2+4\right)$	A1	
	8	

18) OCT 2020_9709_31 Q8

Separate variables correctly and attempt integration of at least one side	B1	$\frac{1}{y}dy=\frac{1-2x^2}{x}dx$
Obtain term $\ln y$	B1	
Obtain terms $\ln x - x^2$	B1	
Use $x = 1$, $y = 1$ to evaluate a constant, or as limits, in a solution containing at least 2 terms of the form $a\ln y$, $b\ln x$ and cx^2	M1	The 2 terms of required form must be from correct working e.g. $\ln y=\ln x - x^2+1$
Obtain correct solution in any form	A1	
Rearrange and obtain $y = xe^{1-x^2}$	A1	OE
	6	

19) OCT 2020_9709_32 Q7

a)	Correct separation of variables	B1	$\int\sec^2 2x\,dx=\int e^{-3t}\,dt$ Needs correct structure
	Obtain term $-\frac{1}{3}e^{-3t}$	B1	
	Obtain term of the form $k\tan 2x$	M1	From correct working
	Obtain term $\frac{1}{2}\tan 2x$	A1	
	Use $x = 0$, $t = 0$ to evaluate a constant, or as limits in a solution containing terms of the form $a\tan 2x$ and be^{-3t}, where $ab \neq 0$	M1	
	Obtain correct solution in any form	A1	e.g. $\frac{1}{2}\tan 2x=-\frac{1}{3}e^{-3t}+\frac{1}{3}$
	Obtain final answer $x=\frac{1}{2}\tan^{-1}\left(\frac{2}{3}\left(1-e^{-3t}\right)\right)$	A1	
		7	
b)	State that x approaches $\frac{1}{2}\tan^{-1}\left(\frac{2}{3}\right)$	B1 FT	Correct value. Accept $x \to 0.294$ The FT is dependent on letting $e^{-3t}\to 0$ in a solution containing e^{-3t}.
		1	

20) OCT 2021_9709_31 Q7

(a)	Show sufficient working to justify the given answer	B1	
		1	
(b)	Correct separation of variables	B1	e.g. $-\int \frac{1}{t}\mathrm{d}t = \int \frac{1}{x\ln x}\mathrm{d}x$
	Obtain term $\ln(\ln x)$	B1	
	Obtain term $-\ln t$	B1	
	Evaluate a constant or use $x = \mathrm{e}$ and $t = 2$ as limits in an expression involving $\ln(\ln x)$	M1	
	Obtain correct solution in any form, e.g. $\ln(\ln x) = -\ln t + \ln 2$	A1	
	Use log laws to enable removal of logarithms	M1	
	Obtain answer $x = \mathrm{e}^{\frac{2}{t}}$, or simplified equivalent	A1	
		7	
(c)	State that x tends to 1 coming from $x = \mathrm{e}^{\frac{k}{t}}$	B1	
		1	

21) OCT 2021_9709_32 Q7

Separate variables correctly	B1	$\int \frac{1}{y^2}\mathrm{d}y = \int 4x\mathrm{e}^{-2x}\mathrm{d}x$
$\int \frac{1}{y^2}\mathrm{d}y = -\frac{1}{y}$	B1	OE
Commence the other integration and reach $ax\mathrm{e}^{-2x} + b\int \mathrm{e}^{-2x}\,\mathrm{d}x$	M1	
Obtain $-2x\mathrm{e}^{-2x} + 2\int \mathrm{e}^{-2x}\mathrm{d}x$ or $-\frac{1}{2}x\mathrm{e}^{-2x} + \frac{1}{2}\int \mathrm{e}^{-2x}\mathrm{d}x$	A1	SOI (might have taken out factor of 4)
Complete integration and obtain $-2x\mathrm{e}^{-2x} - \mathrm{e}^{-2x}$	A1	
Evaluate a constant or use $x = 0$ and $y = 1$ as limits in a solution containing terms of the form $\frac{p}{y}$, $qx\mathrm{e}^{-2x}$, $r\mathrm{e}^{-2x}$, or equivalent.	M1	
Obtain $y = \frac{\mathrm{e}^{2x}}{2x+1}$, or equivalent expression for y	A1	ISW
	7	

22) OCT 2021_9709_33 Q10 (a,b)

(a)	State or imply equation of the form $\frac{dx}{dt} = k\frac{x}{20-x}$	M1	
	Obtain $k = 19$	A1	AG
		2	
(b)	Separate variables and integrate at least one side	M1	
	Obtain terms $20\ln x - x$ and $19t$, or equivalent	A1 A1	
	Evaluate a constant or use $t = 0$ and $x = 1$ as limits in a solution containing terms $a\ln x$ and bt	M1	
	Substitute $t = 1$ and rearrange the equation in the given form	A1	AG
		5	

23) OCT 2022-9709_31 Q8

(a)	Separate variables correctly	B1	$\int \frac{1}{x}\,dx = \int ke^{-0.1t}\,dt$
	Obtain term $\ln x$	B1	
	Obtain term $-10ke^{-0.1t}$	B1	Not from $\int xe^{-0.1t}\,dt$
	Use $x = 20$, $t = 0$ to evaluate a constant or as limits in a solution containing terms $a\ln x, be^{-0.1t}$ where $ab \neq 0$	M1	
	Obtain $\ln x = 10k\left(1-e^{-0.1t}\right) + \ln 20$	A1	or equivalent ISW
		5	
(b)	Use $x = 40$, $t = 10$ to find k or $10k$	M1	Available for their function of the correct structure even if they found no constant in **(a)**.
	Obtain $10k = 1.09654$	A1	or equivalent e.g. $10k = \frac{\ln 2}{1-e^{-1}}$
	State that x tends to 59.9	A1	Need a number, not an expression for that value 3 sf or better 59.87595…..
		3	

24) OCT 2022_9709_32 Q7

a)	Show sufficient working to justify the given statement	B1	e.g. see $2\cot\theta \times -\operatorname{cosec}^2\theta$ in the working or express in terms of $\sin\theta$ and $\cos\theta$ and use quotient rule to obtain the given result. Solution must have θ present throughout and must reach the given answer.
		1	

(b)	Separate variables correctly Check for relevant working in (a)	B1	$\int x\,dx = \int \frac{\tan^2\theta}{\sin^2\theta} - \frac{2\cot\theta}{\sin^2\theta}\,d\theta$ Condone incorrect notation e.g. missing dx. Need either the integral sign or the dx, $d\theta$.
	Obtain term $\frac{1}{2}x^2$	B1	
	Obtain terms $\tan\theta + \cot^2\theta$	B1 + B1	Alternative: $\int \frac{2\cot\theta}{\sin^2\theta}\,d\theta = \int \frac{2\cos\theta}{\sin^3\theta}\,d\theta = -\frac{1}{\sin^2\theta}(+C)$
	Form an equation for the constant of integration, or use limits $x = 2$, $\theta = \frac{1}{4}\pi$, in a solution with **at least two correctly obtained terms** of the form ax^2, $b\tan\theta$ and $c\cot^2\theta$, where $abc \neq 0$	M1	Need to have 3 terms. Constant of correct form.
	State correct solution in any form, e.g. $\frac{1}{2}x^2 = \tan\theta + \cot^2\theta$	A1	or $\frac{1}{2}x^2 = \tan\theta + \operatorname{cosec}^2\theta - 1$ If everything else is correct, allow a correct final answer to imply this A1.
	Substitute $\theta = \frac{1}{6}\pi$ and obtain answer $x = 2.67$	A1	2.6748… $\sqrt{\frac{18+2\sqrt{3}}{3}}$ If see a correctly rounded value ISW.
		7	

25) OCT 2022_9709_33 Q10

(a)	$a = 30$ and $b = 0.01$	B1	
		1	
(b)	Separate variables and integrate one side	M1	
	Obtain terms $-100\ln(30-0.01V)$ and t, or equivalent	A1 FT + A1 FT	FT *their* a and b.
	Evaluate a constant, or use $t = 0$, $V = 0$ as limits, in a solution containing terms $c\ln(30-0.01V)$ and dt where $cd \neq 0$	M1	
	Obtain solution $100\ln 30 - 100\ln(30-0.01V) = t$, or equivalent	A1	
	Substitute $V = 1000$ and obtain answer $t = 40.5$	A1	
		6	
(c)	Obtain $V = 3000(1 - e^{-0.01t})$	B1	OE
	State that V approaches 3000	B1	
		2	