classmate
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Pure Maths_ 3

Differential Equations

Exercise.1_ Solution (Revision)

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1 In a chemical reaction, a compound X is formed from two compounds Y and Z.
The masses in grams of X, Y and Z present at time t seconds after the start of the reaction are x, $10-x$ and $20-x$ respectively. At any time the rate of formation of X is proportional to the product of the masses of Y and Z present at the time. When $t=0$, $x=0$ and $\frac{dx}{dt}=2$

(a) Show that x and t satisfy the differential equation, $\frac{dy}{dx}=0.01(10-x)(20-x)$ ---[1]

(b) Solve this differential equation and obtain an expression for x in terms of t. --[9]

(c) State what happens to the value of x when t becomes large. --[1]

SP-20/03/Q10

Solution (a) $\frac{dx}{dt} \propto (10-x)(20-x) \Rightarrow \frac{dx}{dt} = k(10-x)(20-x)$ —— ①

$\frac{dx}{dt}=2$, when $x=0 \Rightarrow 2=k(10-0)(20-0) \Rightarrow k=0.01$

from ① $\frac{dx}{dt} = 0.01(10-x)(20-x)$ —— ②

(b) $\int \frac{1}{(10-x)(20-x)}dx = \int 0.01\,dt$ —— ③

$\Rightarrow \int \left(\frac{1}{10(10-x)} - \frac{1}{10(20-x)}\right)dx = \int 0.01\,dt$

$\Rightarrow -\frac{1}{10}\ln(10-x) + \frac{1}{10}\ln(20-x) = 0.01t + C$ —— ④

∵ Partial fractions: from ③

$\frac{1}{(10-x)(20-x)} = \frac{A}{(10-x)} + \frac{B}{(20-x)}$

$\frac{1}{20-x} = A + \frac{B(10-x)}{20-x}$

Put $x=10$, $A=\frac{1}{10}$

and $\frac{1}{10-x} = \frac{A(20-x)}{(10-x)} + B$

Put $x=20 \Rightarrow B=-\frac{1}{10}$

Now $t=0$, $x=0 \Rightarrow$ from ④

$-\frac{1}{10}\ln 10 + \frac{1}{10}\ln 20 = 0 + C$

$\Rightarrow C = \frac{1}{10}\ln\frac{20}{10} = \frac{1}{10}\ln 2$

∴ from ④ $-\frac{1}{10}\ln(10-x) + \frac{1}{10}\ln(20-x) = 0.01t + \frac{1}{10}\ln 2$

$\Rightarrow \frac{1}{10}\left[\ln\left(\frac{20-x}{10-x}\right)\right] = 0.01t + \frac{1}{10}\ln 2$

$\Rightarrow \ln\left(\frac{20-x}{10-x}\right) = 0.1t + \ln 2$

$\Rightarrow \frac{(20-x)}{2(10-x)} = e^{0.1t} \Rightarrow 20-x = (20-2x)e^{0.1t}$

$\Rightarrow x = \frac{20[e^{0.1t}-1]}{(2e^{0.1t}-1)}$ ✓

(c) $t \to \infty \Rightarrow x \to 10$ ✓

To Solve a diff. equation;
Using variable separable method: $\left\{ \begin{array}{l} \frac{dy}{dx} = f(y)\cdot g(x) \\ \Rightarrow \int \frac{1}{f(y)}dy = \int g(x)\,dx \end{array}\right.$

Example2: The variables x and y satisfy the differential equation; $\frac{dy}{dx} = \frac{1+4y^2}{e^x}$

It is given that $y=0$ when $x=1$.

(a) Solve the differential equation, obtaining a expression for y in terms of x. --[7]

(b) State what happens to the value of y as x tends to infinity. --[1]

[M-20/32/Q6]

Solution: $\frac{dy}{dx} = \frac{1+4y^2}{e^x}$

(a)

$\Rightarrow \int \frac{1}{1+4y^2}dy = \int e^{-x}dx$

$\frac{1}{4}\int \frac{1}{(\frac{1}{2})^2+y^2}dy = -e^{-x}+c$ $\left[as \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\cdot\tan^{-1}\frac{x}{a}\right]$

$\Rightarrow \frac{1}{4}\times\frac{1}{(\frac{1}{2})}\times\tan^{-1}\left(\frac{y}{\frac{1}{2}}\right) = -e^{-x}+c$

$\Rightarrow \frac{1}{2}\tan^{-1}2y = -e^{-x}+c$ —— (1)

Now $y=0$ for $x=1$

$\Rightarrow \frac{1}{2}\tan^{-1}0 = -e^{-1}+c$

$0 = -e^{-1}+c \Rightarrow c = e^{-1}$

from (1) $\frac{1}{2}\tan^{-1}2y = -e^{-x}+e^{-1}$

$\Rightarrow 2y = \tan(2e^{-1}-2e^{-x})$ ✓

or $y = \frac{1}{2}\tan(2e^{-1}-2e^{-x})$ —— (2)

is the req. Soln.

(b) When $x\to\infty$, $e^{-x}\to 0$

from (2) $y = \frac{1}{2}\tan(2e^{-1})$ ✓

3. The variables x and y satisfy the differential equation:

$$(1-\cos x)\frac{dy}{dx} = y\sin x.$$

It is given that $y=4$, when $x=\pi$ ---- [6]

(a) Solve the differential equation, obtaining an expression for y in terms of x.

(b) Sketch the graph of y against x for $0<x<2\pi$ ... [1]

M-21/32/Q4

Solution: $(1-\cos x)\frac{dy}{dx} = y\sin x$

Put $1-\cos x = u$

diff $\sin x\,dx = du$

$\Rightarrow \int \frac{1}{y}dy = \int \frac{\sin x}{(1-\cos x)}dx$

$\Rightarrow \ln y = \int \frac{1}{u}du$

$\Rightarrow \ln y = \ln u + C$

$\Rightarrow \ln y = \ln(1-\cos x) + C$ --- ①

$y=4$ when $x=\pi$ in ①

$\Rightarrow \ln 4 = \ln(1-\cos\pi) + C$

$\ln 4 = \ln 2 + C$

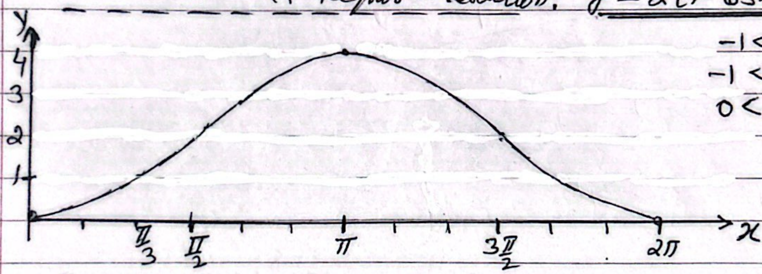
$\Rightarrow C = \ln 4 - \ln 2 = \ln \frac{4}{2} = \ln 2$ ✓

for ① $\ln y = \ln(1-\cos x) + \ln 2$

$\ln y = \ln 2(1-\cos x)$.

$\therefore$ Required Solution: $y = 2(1-\cos x)$ ✓

(b)



$-1 < \cos x < 1$

$-1 < -\cos x < 1$

$0 < 1-\cos x < 2$

Graph of $y = 2(1-\cos x)$ for $0<x<2\pi$

4. The variables x and y satisfy the differential equation:
$(x+1)(3x+1)\frac{dy}{dx} = y$; and it is given that $y=1$, when $x=1$.
Solve the differential equation and find the exact value of y when $x=3$, giving your answer in a simplified form. ---[9]

[M-22/32/Q9]

Solution: $(x+1)(3x+1)\frac{dy}{dx} = y$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{(x+1)(3x+1)}\,dx$

Partial fractions:

$\frac{1}{(x+1)(3x+1)} = \frac{a}{(x+1)} + \frac{b}{(3x+1)}$

Get $a = -\frac{1}{2}$ and $b = \frac{3}{2}$ ✓

$\ln y = \int\left(-\frac{1}{2}\cdot\frac{1}{(x+1)} + \frac{3}{2}\cdot\frac{1}{(3x+1)}\right)dx$

$\ln y = -\frac{1}{2}\ln(x+1) + \frac{\cancel{3}}{2}\frac{\ln(3x+1)}{\cancel{3}} + c$ — ①

Now Given $y=1$, $x=1$

$\Rightarrow \ln 1 = -\frac{1}{2}\ln 2 + \frac{1}{2}\ln 4 + c$

$\Rightarrow c = \frac{1}{2}[\ln 4 - \ln 2] = -\frac{1}{2}\ln 2$

from ① Solution is,

$\therefore \ln y = -\frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(3x+1) - \frac{1}{2}\ln 2$

$\ln y = \frac{1}{2}\cdot\ln\frac{(3x+1)}{2(x+1)}$

$\Rightarrow y = \sqrt{\frac{(3x+1)}{2(x+1)}}$ ✓ ✓

Now for $x=3$,

$y = \sqrt{\frac{10}{8}} = \frac{1}{2}\sqrt{5}$ ✓

5. The variables x and y satisfy the differential equation: $\frac{dy}{dx} = e^{3y} \cdot \sin^2 2x$.
It is given that $y=0$ when $x=0$.
Solve the differential equation and find the value of y when $x = \frac{1}{2}$ --- [7]

[M-23/32/Q9]

Solution: $\frac{dy}{dx} = e^{3y} \cdot \sin^2 x$

$\Rightarrow \int e^{-3y}\,dy = \int \sin^2 2x\,dx$

$\left[\sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$

$\Rightarrow \frac{e^{-3y}}{-3} = \int \frac{(1-\cos 4x)}{2}\,dx$

$\Rightarrow -\frac{1}{3}e^{-3y} = \frac{1}{2}\left(x - \frac{\sin 4x}{4}\right) + c$ --- ①

Given $y=0$ when $x=0$, put in ①

$-\frac{1}{3}e^{0} = \frac{1}{2}[0-0] + c \Rightarrow c = -\frac{1}{3}$

from ①

$-\frac{1}{3}e^{-3y} = \frac{1}{2}\left(x - \frac{\sin 4x}{4}\right) - \frac{1}{3}$ --- ②

for $x = \frac{1}{2}$ in ②

$-\frac{1}{3}e^{-3y} = \frac{1}{2}\left[\frac{1}{2} - \frac{\sin 2}{4}\right] - \frac{1}{3}$

$\Rightarrow e^{-3y} = -\frac{3}{2}\left[\frac{1}{2} - \frac{1}{4}\sin 2\right] + 1$

$\Rightarrow e^{-3y} = -\frac{3}{4} + 1 + \frac{3}{8}\sin 2$

$\Rightarrow -3y = \ln\left(\frac{1}{4} + \frac{3}{8}\sin 2\right) = -\frac{1}{3}\left(\frac{1}{4} + \frac{3}{8}\sin 2\right) = 0.175$ ✓

Example 6: A certain curve is such that its gradient at a point (x, y) is proportional to $\frac{y}{x\sqrt{x}}$. The curve passes through the points with coordinates $(1,1)$ and $(4,e)$.

(a) By setting up and solving a differential equation, find the equation of the curve, expressing y in terms of x. --[8]

(b) Describe what happens to y as x tends to infinity. --[1]

S-20/31/Q8

Solution:
(a) $\frac{dy}{dx} = k.\frac{y}{x\sqrt{x}}$

$\Rightarrow \int \frac{1}{y}\,dy = \int k\, x^{-3/2}\,dx$

$\Rightarrow \ln y = -2k \times \frac{1}{\sqrt{x}} + c$ —— ①

① passes through $(1,1)$

fr ① $\ln 1 = -2k + c \Rightarrow -2k + c = 0$ — ②

① also passes through $(4,e)$

$\ln e = -2k \times \frac{1}{\sqrt{4}} + c$

$\Rightarrow -k + c = 1$ —— ③

Solving ② & ③ $k = 1$ and $c = 2$.

fr ① $\ln y = -\frac{2}{\sqrt{x}} + 2$

or $y = e^{\left(-\frac{2}{\sqrt{x}} + 2\right)}$ ✓ $\left[y = e^{-\frac{2}{\sqrt{x}}} \cdot e^{2}\right]$

(b) as $x \to \infty$, $e^{\frac{-2}{\sqrt{x}}} \to 0$

$y \to e^{2}$

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Example 7: The variables x and y satisfy the differential equation: $\frac{dy}{dx} = \frac{y-1}{(x+1)(x+3)}$

It is given that $y=2$, when $x=0$

Solve the differential equation, obtain and expression for y in terms of x. [S-20/32/Q7] -- [9]

Solution: $\frac{dy}{dx} = \frac{y-1}{(x+1)(x+3)}$

$\Rightarrow \int \frac{1}{(y-1)} dy = \int \frac{1}{(x+1)(x+3)} dx$

$\Rightarrow \ln(y-1) = \int \left(\frac{1}{2(x+1)} - \frac{1}{2(x+3)}\right) dx$

$\Rightarrow \ln(y-1) = \frac{1}{2}[\ln(x+1) - \ln(x+3)] + \ln c$

$\Rightarrow \ln(y-1) = \frac{1}{2}\ln\left(\frac{x+1}{x+3}\right) + \ln c$

$\Rightarrow \ln(y-1) = \ln c \cdot \sqrt{\frac{x+1}{x+3}}$

$\Rightarrow y-1 = c \cdot \sqrt{\frac{x+1}{x+3}}$ ——— ②

Now $y=2$ for $x=0$

from ② $2-1 = c \cdot \sqrt{\frac{1}{3}} \Rightarrow c = \sqrt{3}$

from ①

$y-1 = \sqrt{3} \cdot \sqrt{\frac{x+1}{x+3}}$

or $y = 1 + \sqrt{\frac{3x+3}{x+3}}$ ✓

is the required solution.

Partial fraction:

$\frac{1}{(x+1)(x+3)} = \frac{a}{(x+1)} + \frac{b}{(x+3)}$ —— ①

$\frac{1}{x+3} = a + \frac{b(x+1)}{(x+3)}$

Put $x=-1$

$\frac{1}{2} = a$ ✓

Again multiply ① by $(x+3)$

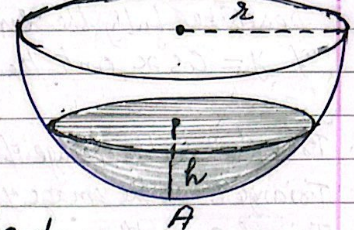
$\frac{1}{(x+1)} = \frac{a(x+3)}{(x+1)} + b$

Put $x=-3$

$-\frac{1}{2} = b$

$\therefore a = \frac{1}{2}, b = -\frac{1}{2}$

Example 8. A tank containing water is in the form of a hemisphere. The axis is vertical, the lowest point is A and the radius is r. The depth of water at time t is h. At time $t=0$ the tanks is full and the depth of water is r. A this instant a tap at A is opened and water begins to flow out at rate proportional to $\sqrt{h}$. The tank becomes empty at time $t=14$.



The volume of water in the tank is V, when the depth is h. It is given that $V=\frac{1}{3}\pi(3rh^2-h^3)$

(a) Show that h and r satisfy a differential equation of the form $\frac{dh}{dt}=-\frac{B}{2rh^{1/2}-h^{3/2}}$

where B is a positive contant. --[4]

(b) Solve the differential equation and an expression for t in terms of h and r. $\boxed{\text{S-20/33/Q 10}}$ --[8]

Solution: (a) Given $\frac{dV}{dt}\propto\sqrt{h}\Rightarrow\frac{dV}{dt}=-k\sqrt{h}$ —(1)

and $V=(\pi rh^2-\frac{1}{3}\pi h^3)$

$\Rightarrow\frac{dV}{dh}=2\pi rh-\pi h^2$ —(2) (r is constant)

$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$

$\Rightarrow -k\sqrt{h}=(2\pi rh-\pi h^2)\times\frac{dh}{dt}$

$\Rightarrow\frac{dh}{dt}=\frac{-k\sqrt{h}}{(2\pi rh-\pi h^2)}$

$=\frac{-k\sqrt{h}}{\pi\sqrt{h}(2rh^{1/2}-h^{3/2})}$

$=\frac{-B}{(2rh^{1/2}-h^{3/2})}$ ✓ —(3)

(b) from (3)

$\int(2rh^{1/2}-h^{3/2})dh=-\int B\,dt$

$\Rightarrow\frac{4}{3}rh^{3/2}-\frac{2}{5}h^{5/2}=-Bt+C$ —(4)

given $t=0$, $h=r$

for (4) $\Rightarrow C=\frac{14}{15}r^{5/2}$ ✓

and $t=14$, $h=0$ from (4)

$\Rightarrow 0=-14B+\frac{14}{15}r^{5/2}$

$\Rightarrow B=\frac{1}{15}r^{5/2}$ ✓

hence from (4) required Solution from (4)

$\frac{4}{3}rh^{3/2}-\frac{2}{5}h^{5/2}=-\frac{1}{15}r^{5/2}\cdot t+\frac{14}{15}r^{5/2}$

$\Rightarrow t=14-20\left(\frac{h}{r}\right)^{3/2}+6\left(\frac{h}{r}\right)^{5/2}$ ✓

9. The variables x and t satisfy the differential equation, $\frac{dx}{dt} = x^2(1+2x)$, and $x=1$, when $t=0$.
Using partial fractions, solve the differential equation, obtaining an expression for t in terms of x. --- [11]

[S-21|31|Q10]

Solution: $\frac{dx}{dt} = x^2(1+2x) \Rightarrow \int \frac{1}{x^2(1+2x)}\,dx = \int 1\,dt$ --- (1)

Consider $\frac{1}{x^2(1+2x)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{(1+2x)}$ --- (2)

multiply (2) by $x^2 \rightarrow \frac{1}{1+2x} = ax + b + \frac{c \cdot x^2}{(1+2x)}$ --- (3)

Put $x=0$ in (3) $\Rightarrow 1 = b$ ✓

Again multiply (2) by $(1+2x) \Rightarrow \frac{1}{x^2} = \frac{a(1+2x)}{x} + \frac{b(1+2x)}{x^2} + c$ --- (3)

Put $1+2x=0 \Rightarrow x = -\frac{1}{2}$ in (3) $\Rightarrow 4 = 0 + 0 + c \Rightarrow \underline{c=4}$ ✓

Put $x=1$ in (2) $\rightarrow \frac{1}{1\times 3} = \frac{a}{1} + \frac{1}{1} + \frac{4}{3} \Rightarrow a = -2$ ✓

$\therefore$ from (2) $\frac{1}{x^2(1+2x)} = \frac{-2}{x} + \frac{1}{x^2} + \frac{4}{(1+2x)}$ --- (4)

Now from (4) in (1) $\int \left(-\frac{2}{x} + \frac{1}{x^2} + \frac{4}{1+2x}\right)dx = \int 1\,dt$

$\Rightarrow -2\ln x - \frac{1}{x} + 2\ln(1+2x) + C = t$ --- (5)

Now $x=1$ for $t=0$ in (1)

$\Rightarrow 0 - 1 + 2\ln 3 + C = 0 \Rightarrow C = 1 - 2\ln 3$ --- (6)

from (6) in (5) $t = 2[\ln(1+2x) - \ln x] - \frac{1}{x} + 1 - 2\ln 3$

$t = 2[\ln(1+2x) - \ln x - \ln 3] - \frac{1}{x} + 1$

$t = 1 - \frac{1}{x} + 2\ln\left(\frac{1+2x}{3x}\right)$ ✓

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10. A curve is such that the gradient at a general point with coordinates (x, y) is proportional to $\frac{y}{\sqrt{x+1}}$, the curve passes through the points with coordinates $(0,1)$ and $(3, e)$.
By setting up and solving a differential equation, find the equation of the curve, expressing y in terms of x. --[7]
[S-21/32/Q7]

Solution: Given $\frac{dy}{dx} = k \cdot \frac{y}{\sqrt{x+1}}$ ---(1)

$\Rightarrow \int \frac{1}{y}\, dy = k \int \frac{1}{\sqrt{x+1}}\, dx$

$\Rightarrow \ln y = 2k\sqrt{x+1} + c$ --(2)

Curve (2) passes through $(0,1)$

$\Rightarrow \ln 1 = 2k + c \Rightarrow 2k + c = 0$ -(3)

Again (2) passes through $(3, e) \Rightarrow \ln e = 2k \times 2 + c = 1$

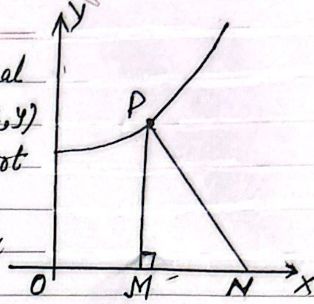
or $4k + c = 1$ --(4)

Solving (3) & (4) $\Rightarrow k = \frac{1}{2}$ and $c = -1$

$\therefore$ from (2) Required solution: $\ln y = 2 \times \frac{1}{2}\sqrt{x+1} - 1$

$\Rightarrow \ln y = (\sqrt{x+1} - 1) \Rightarrow y = e^{(\sqrt{x+1} - 1)}$ ✓

11. For the curve shown in the diagram, the normal to the curve at the point P with coordinates (x, y) meets the x-axis at N. The point M is the foot of the perp. from P to the x-axis.



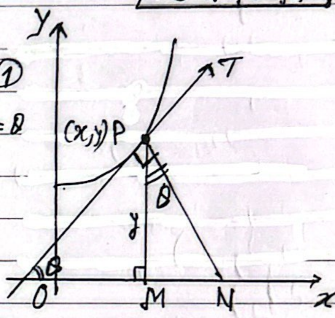
The curve is such that for all values of x in the interval $0 \leq x < \frac{1}{2}\pi$, the area of triangle PMN is equal to $\tan x$.

(a) (i) Show that $\frac{MN}{y} = \frac{dy}{dx}$ ---[1]

(ii) Hence show that x and y satisfy the differential equation $\frac{1}{2}y^2\frac{dy}{dx} = \tan x$ ---[2]

(b) Given that $y = 1$, when $x = 0$, solve the differential equation to find the equation of the curve, expressing y in terms of x. ---[6]

S-21/33/Q7



Solution: Let the tangent to curve at 'P' is inclined

(a) at angle angle θ with x-axis, $\rightarrow \frac{dy}{dx} = \tan\theta$ —— ①

Now normal to the curve at P is PN, $\rightarrow \angle MPN = \theta$

from ① $\frac{dy}{dx} = \tan\theta = \frac{MN}{y}$ — (In rt Δ MPN)

(i) $\Rightarrow \frac{MN}{y} = \frac{dy}{dx}$ —— ②

(ii) area of Δ PMN $= \frac{1}{2} \times MN \times PM$

or $\tan x = \frac{1}{2} \times y \cdot \frac{dy}{dx} \times y$ [from ② $MN = y\frac{dy}{dx}$

$\Rightarrow \frac{1}{2}y^2\frac{dy}{dx} = \tan x$ —— ③

(b) from ③ $\frac{1}{2}y^2\frac{dy}{dx} = \tan x \Rightarrow \int \frac{1}{2}y^2\,dy = \int \tan x\,dx$

$\Rightarrow \frac{1}{2} \cdot \frac{y^3}{3} = -\ln\cos x + c$ --- ④

Put in ④ $y = 1, x = 0 \Rightarrow \frac{1}{6} = 0 + c \Rightarrow c = \frac{1}{6}$

from ④ $\frac{1}{6}y^3 = -\ln\cos x + \frac{1}{6}$

$\Rightarrow y^3 = 1 - 6\ln\cos x$

$y = \sqrt[3]{1 - 6\ln\cos x}$ ✓

12. The variable x and y satisfy the differential equation,
$\frac{dy}{dx} = \frac{xy}{1+x^2}$ and $y = 2$ when $x = 0$
Solve the differential equation, obtaining a simplified expression for y in terms of x. ---[7]
[S-22/31/Q4]

Solution: $\frac{dy}{dx} = \frac{xy}{1+x^2}$

$\Rightarrow \int \frac{1}{y}\, dy = \int \frac{x}{1+x^2}\, dx$

$\left\{ \begin{array}{l} 1+x^2 = u \\ 2x\,dx = du \\ x\,dx = \frac{1}{2} du \end{array} \right.$

$\Rightarrow \int \frac{1}{y} dy = \frac{1}{2}\int \frac{1}{u} du$

$\ln y = \frac{1}{2} \ln u + \ln c$

$\Rightarrow \ln y = \ln(u^{1/2} \cdot c)$

$\Rightarrow y = c\, u^{1/2}$

$\Rightarrow y = c\sqrt{1+x^2}$ ---- (1)

$y = 2$ for $x = 0$ in (1)

$2 = c\sqrt{1+0} \Rightarrow c = 2$

$\therefore$ from (1) Req. Solution: $y = 2\sqrt{1+x^2}$

13. The variables x and y satisfy the differential equation:
$\frac{dy}{dx} = x e^{y-x}$ and $y = 0$ when $x = 0$
(a) Solve the differential equation, giving y in terms of x. ---[7]
(b) Find the value of y when $x = 1$, giving your answer in the form $a - \ln b$, where a and b are integers. ---[1]
[S-22/32/Q6]

Solution: $\frac{dy}{dx} = x e^{y-x} \Rightarrow \frac{dy}{dx} = x e^{y} \cdot e^{-x}$

$\Rightarrow \int e^{-y}\, dy = \int x \cdot e^{-x}\, dx$

$\Rightarrow \frac{e^{-y}}{-1} = x \cdot \int e^{-x} dx - \int\left(\frac{d}{dx}x \cdot \int e^{-x} dx\right) dx$

$\Rightarrow -e^{-y} = x\frac{e^{-x}}{-1} - \int 1 \times \frac{e^{-x}}{-1}\, dx$

$\Rightarrow -e^{-y} = -xe^{-x} + \frac{e^{-x}}{-1}$

$\Rightarrow e^{-y} = e^{-x}(x+1) + c$

$\ln e^{-y} = \ln(e^{-x}(x+1))$

$\Rightarrow -y = \ln((x+1)e^{-x}) - c$

$y = -\ln((x+1)e^{-x}) + c$ --(1)

for $x = 0$, $y = 0$

$\Rightarrow 0 = -\ln e^{0} + c \Rightarrow c = 0$

$\therefore$ from (1) The required Solution

$y = -\ln((x+1)e^{-x})$ — (2)

(b) for $x = 1$ in (2)

$y = -\ln(2e^{-1})$

$= -[\ln 2 + \ln e^{-1}]$

$= -(\ln 2 - 1)$

$y = 1 - \ln 2$

14. At time t days after the start of observations, the number of insects in a population is N. The variation in the number of insects is modelled by differential equation of the form:
$\frac{dN}{dt} = kN^{3/2} \cdot \cos 0.02t$, where k is a constant and N is a continuous variable. It is given that $t=0$, $N=100$

(a) Solve the differential equation, obtaining a relation between N, k and t. ---[5]

(b) Given also that $N=625$, when $t=50$, find the value of k. ---[2]

(c) Obtain an expression for N in terms of t, [S-22/33/Q8] [2]
and find the greatest value of N predicted by this model. ---[2]

Solution: (a)
$$\frac{dN}{dt} = kN^{3/2} \cdot \cos 0.02t$$
$$\Rightarrow \int N^{-3/2}\,dN = \int k \cos 0.02t\,dt$$
$$\Rightarrow \frac{N^{-1/2}}{-\frac{1}{2}} = k \cdot \frac{\sin 0.02t}{0.02}$$
$$\Rightarrow -\frac{2}{\sqrt{N}} = 50k \cdot \sin 0.02t + c \quad \text{—(1)}$$
for $t=0$, $N=100$ in (1)
$$-\frac{2}{10} = 50k \times 0 + c \Rightarrow c = -0.2$$
$\therefore$ from (1), Req. solution
$$-\frac{2}{\sqrt{N}} = 50k \sin 0.02t - 0.2 \quad \text{—(2)}$$

(b) Now $N=625$; $t=50$ in (2)
$$-\frac{2}{25} = 50k \cdot \sin 1 - 0.2$$
$$\Rightarrow 0.2 - 0.08 = 50 \times 0.8414k$$
$$\Rightarrow 42.073k = 0.12$$
$$\therefore k = \frac{0.12}{42.073} =$$
$$k = 0.00285$$

(c) from (2)
$$\frac{2}{\sqrt{N}} = 0.2 - 50 \times 0.00285 \sin 0.02t$$
$$N = 4(0.2 - 0.1426 \sin 0.02t)^{-2} \quad \text{—(3)}$$

Now for the greatest value of N: $\sin 0.02t = 1$
$$0.02t = \frac{\pi}{2}$$
$$t = 78.5$$
Put $\sin 0.02t = 1$ from (3)
$$N = 4[0.2 - 0.1426 \times 1]^{-2}$$
$$= 4 \times (0.574)^{-2}$$
$$= 4/0.00329$$
$$N = 1215$$

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15. The variables x and y satisfy the differential equation: $\cos 2x \frac{dy}{dx} = \frac{4\tan 2x}{\sin^2 3y}$
where $0 \le x < \frac{\pi}{4}$. It is given that $y = 0$ when $x = \frac{1}{6}\pi$.
Solve the differential equation to obtain the value of x when $y = \frac{1}{6}\pi$.
Give your answer correct to 3 decimal places. ---[8]

S-23/31/Q7

Solution: Given, $\cos 2x \cdot \frac{dy}{dx} = \frac{4\tan 2x}{\sin^2 3y}$

$\Rightarrow \int \sin^2 3y\, dy = \int 4\tan 2x \cdot \sec 2x\, dx$

$\Rightarrow \int \left(\frac{1-\cos 6y}{2}\right) dy = \int 4\tan 2x \sec 2x\, dx$

$\Rightarrow \frac{1}{2}\left(y - \frac{\sin 6y}{6}\right) = 4\frac{\sec 2x}{2} + c$

$\Rightarrow \frac{1}{2}y - \frac{1}{12}\sin 6y = 2\sec 2x + c$ ---(1)

Given $y = 0$ when $x = \frac{\pi}{6}$

for (1) $0 - 0 = 2 \times 2 + c \Rightarrow c = -4$

for (1) Req Solution: $\frac{1}{2}y - \frac{1}{12}\sin 6y = 2\sec 2x - 4$ ---(2)

Now $y = \frac{\pi}{6}$ in (2) $\quad x = ?$

$\frac{1}{2} \times \frac{\pi}{6} - \frac{1}{12}\sin \pi = 2\sec 2x - 4$

$\Rightarrow \frac{\pi}{12} = 2\sec 2x - 4 \quad [\sin \pi = 0]$

$\Rightarrow 2\sec 2x = \frac{\pi}{12} + 4 = 4.2618$

$\Rightarrow \sec 2x = 2.13$

$\Rightarrow \cos 2x = 1/2.13 = 0.46948$

$2x = \cos^{-1} 0.46948 = 1.082$

$\Rightarrow x = 0.541$

16(a) The variables x and y satisfy the differential equation $\frac{dy}{dx} = \frac{4+9y^2}{e^{2x+1}}$

It is given that $y=0$ when $x=1$

Solve the differential equation, obtain an expression for y in terms of x.

(b) State what happens to the value of y as x tends to infinity. [S-23|32|Q8] --[7]

Solution: $\frac{dy}{dx} = \frac{4+9y^2}{e^{2x+1}}$ ---①

$\Rightarrow \int \frac{1.dy}{4+9y^2} = \int \frac{1}{e^{2x+1}} dx$

Note: $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$\frac{1}{9}\int \frac{1}{(\frac{2}{3})^2 + y^2} dy = \int e^{-(2x+1)} dx$

$\Rightarrow \frac{1}{9}\times\frac{3}{2}\tan^{-1}\frac{y}{2/3} = \frac{e^{-(2x+1)}}{-2} + C$

$\Rightarrow \frac{1}{6}\tan^{-1}\frac{3y}{2} = -\frac{1}{2}e^{-(2x+1)} + C$ --②

Put $y=0$ when $x=1$ in ②

$\frac{1}{6}\tan^{-1}0 = -\frac{1}{2}e^{-3} + C$

$\Rightarrow C = \frac{1}{2}e^{-3}$ $(\tan^{-1}0 = 0)$

from ②

$\frac{1}{6}\tan^{-1}\left(\frac{3y}{2}\right) = -\frac{1}{2}e^{-(2x+1)} + \frac{1}{2}e^{-3}$

$= \frac{1}{2}[e^{-3} - e^{-(2x+1)}]$

$\Rightarrow \tan^{-1}\left(\frac{3y}{2}\right) = [3e^{-3} - 3e^{-(2x+1)}]$

$\Rightarrow y = \frac{2}{3}\tan[3e^{-3} - 3e^{-(2x+1)}]$ ---③

(b) when $x\to\infty$, $e^{-(2x+1)} \to 0$

Hence from ③ $y = \frac{2}{3}\tan(3e^{-3})$

17. The variables x and y satisfy the differential equation $\frac{dy}{dx} = \frac{y^2+4}{x(y+4)}$

for $x>0$, It is given that $x=4$ when $y=2\sqrt{3}$

Solve the differential equation to obtain the value of x when $y=2$. ---[8]

[S-23|33|Q8]

Solution: $\frac{dy}{dx} = \frac{y^2+4}{x(y+4)}$

$\Rightarrow \int \frac{y+4}{y^2+4} dy = \int \frac{1}{x} dx$

$\Rightarrow \frac{1}{2}\int \frac{2y}{(y^2+4)} dy + 4\int \frac{1}{2^2+y^2} dy = \ln x + c$

$\Rightarrow \frac{1}{2}\ln(y^2+4) + 4\times\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) = \ln x + c$ ----①

for $x=4$ when $y=2\sqrt{3}$ in ①

$\frac{1}{2}\ln(12+4) + 2\tan^{-1}\sqrt{3} = \ln 4 + C$

$\ln 4 + 2\times\frac{\pi}{3} = \ln 4 + C$

$\Rightarrow C = 2\pi/3$

Hence from ① $\frac{1}{2}\ln(y^2+4) + 2\tan^{-1}\left(\frac{y}{2}\right) = \ln x + \frac{2\pi}{3}$ ---②

when $y=2$, to find $x=?$

from ②

$\frac{1}{2}\ln 8 + 2\tan^{-1}1 = \ln x + \frac{2\pi}{3}$

$\ln\sqrt{8} + 2\times\frac{\pi}{4} = \ln x + \frac{2\pi}{3}$

$\Rightarrow \ln x = \ln\sqrt{8} - \frac{\pi}{6}$

$= \ln\sqrt{8} + \ln e^{-\pi/6}$

$\ln x = \ln 2\sqrt{2}\cdot e^{-\pi/6}$

$\Rightarrow x = 2\sqrt{2}e^{-\pi/6}$

18. The coordinates (x, y) of a general point of a curve satisfy the differential equation, $x\frac{dy}{dx} = (1-2x^2)y$, for $x > 0$.
It is given that $y = 1$ when $x = 1$.
Solve the differential equation, obtaining an expression for y in terms of x. ...[6]

W-20/31/Q8

Solution: $x\frac{dy}{dx} = (1-2x^2)y$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{(1-2x^2)}{x}\,dx$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \left(\frac{1}{x} - 2x\right)dx$$

$$\Rightarrow \ln y = \ln x - x^2 + c \quad \text{—— (1)}$$

Put $y = 1$ & $x = 1 \Rightarrow \ln 1 = \ln 1 - 1^2 + c \Rightarrow c = -1$

from (1) $\ln y = \ln x - x^2 + 1$

$$\Rightarrow \ln y - \ln x = 1 - x^2$$

$$\Rightarrow \ln \frac{y}{x} = 1 - x^2$$

$$\Rightarrow \frac{y}{x} = e^{1-x^2}$$

$$\Rightarrow y = x \cdot e^{1-x^2}$$

19. The variables x and t satisfy the differential equation;
$e^{3t}\frac{dx}{dt} = \cos^2 2x$, for $t \geq 0$. It is given that $x = 0$ when $t = 0$
(a) Solve the differential equation and obtain an expression for x in terms of t. --[7]
(b) State what happens to the value of x when t tends to infinity. --[1]

W-20/32/Q7

Solution (a) $e^{3t}\frac{dx}{dt} = \cos^2 2x$

$$\Rightarrow \int \frac{1}{\cos^2 2x}\,dx = \int \frac{1}{e^{3t}}\,dt$$

$$\Rightarrow \int \sec^2 2x\,dx = \int e^{-3t}\,dt$$

or $\frac{1}{2}\tan 2x = -\frac{1}{3}e^{-3t} + c$ —— (1)

for $x = 0$, $t = 0$ in (1)

$\frac{1}{2}\tan 0 = -\frac{1}{3}e^{0} + c \Rightarrow c = \frac{1}{3}$

$\therefore$ from (1)

$$\frac{1}{2}\tan 2x = -\frac{1}{3}e^{-3t} + \frac{1}{3}$$

$$\Rightarrow \tan 2x = -\frac{2}{3}e^{-3t} + \frac{2}{3}$$

$$\Rightarrow 2x = \tan^{-1}\left\{\frac{2}{3}\left(1 - e^{-3t}\right)\right\}$$

$$\Rightarrow x = \frac{1}{2}\tan^{-1}\left[\frac{2}{3}\left(1 - e^{-3t}\right)\right]$$

(b) when $t \to \infty$; $x \to \frac{1}{2}\tan^{-1}(2/3)$

20(a) Given that $y = \ln(\ln x)$, Show that $\frac{dy}{dx} = \frac{1}{x \ln x}$ ---[1]

The variables x and t satisfy the differential equation:
$x \ln x + t \frac{dx}{dt} = 0$; It is given that $x = e$ when $t = 2$

(b) Solve the differential equation obtaining an expression for x in terms of t, simplifying your answer. ---[7]

W-21/ 31/Q7

Solution: (a) $y = \ln(\ln x)$ ---- ①

$$\frac{dy}{dx} = \frac{1}{\ln x} \times \frac{1}{x} = \frac{1}{x \ln x}$$

$$\therefore \frac{dy}{d} = \frac{1}{x \ln x} \text{ ---- ②}$$

(b) $x \ln x + t \frac{dx}{dt} = 0$

$\Rightarrow t \frac{dx}{dt} = -x \ln x \quad \Rightarrow \int \frac{1}{x \ln x} dx = -\int \frac{1}{t} dt$

$\Rightarrow \ln(\ln x) = -\ln t + c$ --③ $\left\{ \begin{array}{l} \text{using Part (a)} \\ \frac{d}{dx} \ln(\ln x) = \frac{1}{x \cdot \ln x} \end{array} \right.$

Put $x = e$, for $t = 2$ in ③ $\Rightarrow$

$\ln(\ln e) = -\ln 2 + c$

$\Rightarrow \ln(1) = -\ln 2 + c \Rightarrow 0 = -\ln 2 + c$

$c = \ln 2$ —— ④

From ④ in ③ $\quad \ln(\ln x) = -\ln t + \ln 2$

$\Rightarrow \ln(\ln x) = \ln\left(\frac{2}{t}\right)$

$\Rightarrow \quad \ln x = \frac{2}{t}$

$\Rightarrow \quad \underline{x = e^{(2/t)}}$ ✓

21. The variables x and y satisfy the differential equation, $e^{2x} \cdot \frac{dy}{dx} = 4xy^2$. and it is given that $y=1$ when $x=0$,
Solve the differential equation, obtaining an expression for y in terms of x. [W-21/32/Q7] ---[7]

Solution: $e^{2x}\frac{dy}{dx} = 4xy^2 \Rightarrow \int \frac{1}{y^2}\,dy = \int \frac{4x}{e^{2x}}\,dx \Rightarrow -\frac{1}{y} = 4\int x \cdot e^{-2x}\,dx$

$\Rightarrow -\frac{1}{y} = 4\left[x \cdot \int e^{-2x}\,dx - \int\left(\frac{d}{dx}x \cdot \int e^{-2x}\,dx\right)dx\right.$

$\Rightarrow -\frac{1}{y} = 4\left[\frac{xe^{-2x}}{-2} - \int 1 \times \frac{e^{-2x}}{-2}\,dx\right]$

$\Rightarrow -\frac{1}{y} = 4\left[\frac{xe^{-2x}}{-2} + \frac{1}{2} \cdot \frac{e^{-2x}}{-2}\right]$

$\Rightarrow -\frac{1}{y} = -2xe^{-2x} - e^{-2x} + c$ —— ①

$y=1$ when $x=0$ in ① $\Rightarrow -1 = -1 + c \Rightarrow c = 0$

from ① $\frac{1}{y} = e^{-2x}(2x+1) \Rightarrow y = \frac{e^{2x}}{(2x+1)}$ ✓

22 A large plantation of area $20\ km^2$ is becoming infected with a plant disease. At time t years the area infected is $x\ km^2$ and the rate of increase of x is proportional to the ratio of the area infected to the area not yet infected. When $t=0$, $x=1$ and $\frac{dx}{dt}=1$.

(a) Show that x and t satisfy the differential equation: $\frac{dx}{dt}=\frac{19x}{20-x}$ --[2]

(b) Solve the diff. equation and show that when $t=1$, the value of x satisfy the equation: $x=e^{0.9+0.05x}$ --[5]

(c) Use an iterative formula based on the equation in part (b), with an initial value of 2, to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. --[3]

(d) Calculate the value of t at which the entire plantation becomes inflected. --[1]

W-21/33/Q10

Solution (a) $\frac{dx}{dt}\propto\frac{x}{(20-x)} \Rightarrow \frac{dx}{dt}=\frac{k.x}{(20-x)}$ --- (1)

$x=1,\ \frac{dx}{dt}=1 \Rightarrow 1=\frac{k\times 1}{19} \Rightarrow k=19$

$\therefore$ from (1) $\frac{dx}{dt}=\frac{19x}{(20-x)}$ — (2)

(b) $\frac{dx}{dt}=\frac{19x}{20-x} \Rightarrow \int\frac{20-x}{19x}\,dx=\int dt$

$\Rightarrow \frac{1}{19}\int\left(\frac{20}{x}-1\right)dx=t+c$

$\Rightarrow \frac{1}{19}(20\ln x-x)=t+c$ — (3)

when $x=1$, $t=0$

$\frac{1}{19}(-1)=c \Rightarrow c=-\frac{1}{19}$

for (3) $\frac{1}{19}(20\ln x-x)=t-\frac{1}{19}$

Now for $t=1$

$\Rightarrow \frac{1}{19}(20\ln x-x)=\frac{18}{19}$

$\Rightarrow 20\ln x-x=18$

$20\ln x=x+18$

$\ln x=0.9+0.05x$

$\Rightarrow x=e^{(0.9+0.05x)}$

(c) Using the Iterative formula:

$x_{n+1}=e^{(0.9+0.05x_n)}$

$x_0=2$, $x_1=2.7182$

$x_2=2.8176$

$x_3=2.8317$

$x_4=2.8337$

$x_5=2.8339$

$x_6=2.8340$

$x_7=2.8340$

$x_8=2.8340$

$\therefore x=2.83$ (2dp)

(d) Entire plantation is infected for $x=20$

from (3) $t=\frac{1}{19}(20\ln x-x)+\frac{1}{19}$

$=\frac{1}{19}[20\ln x-x+1]$ — (4)

Now for $x=20$ in (4)

$t=\frac{1}{19}[20.\ln 20-20+1]$

$=\frac{1}{19}[20.\ln 20-19]$

$t=2.15$

23. In a certain chemical reaction the amount, x grams, of a substance is increasing. The differential equation satisfied by x and t, the time in seconds since the reaction began, is

$$\frac{dx}{dt} = kxe^{-0.1t},$$ where k is a constant. It is given that $x = 20$ at the start of reaction.

(a) Solve the differential equation, obtaining a relation between x, t and k --- [5]

(b) Given that $x = 40$ when $t = 10$, find the value of k and find the value approached by x as t becomes large. --- [3]

W-22 | 31 | Q 8

<u>Solution:</u>
(a) $\frac{dx}{dt} = kxe^{-0.1t} \Rightarrow \int \frac{1}{x}dx = \int ke^{-0.1t}dt$

$\Rightarrow \ln x = \frac{ke^{-0.1t}}{-0.1} + c$

$\Rightarrow \ln x = -10ke^{-0.1t} + c$ --- ①

$\Rightarrow \ln 20 = -10ke^{0} + c$ [$x = 20$ when $t = 0$

$\Rightarrow c = 10k + \ln 20$]

from ① $\ln x = -10ke^{-0.1t} + (10k + \ln 20)$

$\Rightarrow \ln x = 10k(1 - e^{-0.1t}) + \ln 20$ --- ② ✓

(b) Now given $x = 40$, at $t = 10$

$\therefore$ from ② $\ln 40 = 10k(1 - e^{-1}) + \ln 20$

$\Rightarrow 10k = \frac{\ln 40 - \ln 20}{1 - e^{-1}} = \frac{\ln 2}{1 - 0.367879} = \frac{0.6931}{0.6321} = 1.0965$

Hence from ② $\ln x = 1.0965(1 - e^{-0.1t}) + \ln 20$

$\Rightarrow \ln \frac{x}{20} = 1.0965(1 - e^{-0.1t})$

$\Rightarrow \ln \frac{x}{20} = 1.0965(1 - 0)$ $\left\{ \begin{array}{l} \text{Now when } t \text{ is large} \\ \Rightarrow e^{-0.1t} \to 0 \end{array} \right.$

$\Rightarrow \frac{x}{20} = e^{1.0965}$

$\Rightarrow x = 20 \times 2.9936 = 59.8733$

$\therefore x = \underline{59.9}$ (3 s.f)

24. The variables x and θ satisfy the differential equation:

$$x \sin^2\theta \frac{dx}{d\theta} = \tan^2\theta - 2\cot\theta; \text{ for } 0<\theta<\tfrac{1}{2}\pi \text{ and } x>0$$

It is given that $x=2$ for $\theta=\frac{1}{4}\pi$

(a) Show that $\frac{d}{d\theta}(\cot^2\theta) = -\frac{2\cot\theta}{\sin^2\theta}$ --- [1]

(b) Solve the differential equation and find the of x when $\theta=\frac{1}{6}\pi$

[W-22/32/Q7] --- [7]

Solution: (a) $\frac{d}{d\theta}\cot^2\theta = 2\cot\theta \times \frac{d}{d\theta}\cot\theta$

$= 2\cot\theta \times (-\operatorname{cosec}^2\theta) = \frac{-2\cot\theta}{\sin^2\theta}$ ✓ --- (1)

(b) Solve; $x\sin^2\theta\frac{dx}{d\theta} = \tan^2\theta - 2\cot\theta$ for $0<\theta<\frac{\pi}{2}$, $x>0$

$\Rightarrow \int x\,dx = \int\left(\frac{\tan^2\theta}{\sin^2\theta} - \frac{2\cot\theta}{\sin^2\theta}\right)d\theta$

$\Rightarrow \frac{1}{2}x^2 = \int\left(\sec^2\theta - \frac{2\cot\theta}{\sin^2\theta}\right)dx$

$\Rightarrow \frac{1}{2}x^2 = \tan\theta + \cot^2\theta + c$ --- (2) [using part (a)]

now $x=2$, for $x=\frac{\pi}{4}$

for (2) $2 = 1+1+c \Rightarrow c=0$

Hence the required solution for (2)

$\frac{1}{2}x^2 = \tan\theta + \cot^2\theta$ --- (3)

also when $\theta=\frac{1}{6}\pi$ for (3) $\Rightarrow \frac{1}{2}x^2 = \tan\frac{\pi}{6} + \cot^2\frac{\pi}{6}$

$\Rightarrow \frac{1}{2}x^2 = \frac{1}{\sqrt{3}} + 3$

$\Rightarrow x^2 = 6 + \frac{2}{\sqrt{3}} = 6.1547$

$\Rightarrow x = \sqrt{6.1547} = 2.6748$

$x = 2.67$ ✓ $(x>0)$

25. A gardener is filling an ornamental pool with water, using a hose that delivers 30 litres of water per minute. Initially the pool is empty. At time t minutes after filling begins the volume of water in the pool is V litres. The pool has a small leak and loses water at a rate of 0.01V litres per minutes.
The differential equation satisfied by V and is: $\frac{dV}{dt} = a - bV$.

(a) Write down the values of the constants a and b. ---[1]

(b) Solve the differential equation and find the value of t when V=1000. ---[6]

(c) Obtain an expression for V in terms of t and hence state what happens to V as t becomes large. ---[2]

W-22/33/Q10.

Solution:

(a) Given $\frac{dV}{dt} = a - bV$ --- ①

also $\frac{dV}{dt} = 30 - 0.01V$ --- ②

Comparing ① & ② $\underline{a = 30}$ ✓ and $\underline{b = 0.01}$ ✓

(b) $\frac{dV}{dt} = 30 - 0.01V \Rightarrow \int \frac{1}{(30-0.01V)} dV = \int dt$

$\Rightarrow \frac{\ln(30-0.01V)}{-0.01} = t + c$

$\Rightarrow -100 \ln(30 - 0.01V) = t + c$ --- ②

Given $t = 0$ and $V = 0$

from ② $-100 \ln 30 = c$

$\therefore$ from ② $-100 \ln(30 - 0.01V) = t - 100 \ln 30$

$\Rightarrow \underline{t = 100 \ln 30 - 100 \ln(30 - 0.01V)}$ ✓ --- ③

Required solution

Now for V=1000 in ③

$\Rightarrow t = 100 \ln 30 - 100 \ln(30 - 0.01 \times 1000)$

$\Rightarrow t = 100(\ln 30 - \ln 20) = 100 \times \ln 3/2 = \underline{40.5}$ ✓

(c) from ③ $t = 100(\ln 30 - \ln(30 - 0.01V)) = 100 \ln \frac{30}{(30-0.01V)}$

$\Rightarrow \frac{t}{100} = \ln\left(\frac{30 \times 100}{3000 - V}\right) \Rightarrow \frac{3000}{3000-V} = e^{t/100}$

$\Rightarrow 3000 - V = 3000 . e^{-0.01t} \Rightarrow V = 3000(1 - e^{-0.01t})$ ✓ --- ④

Now when t is large then V approaches to $\underline{3000}$ ✓ $\left[\because e^{-0.01t} \to 0 \text{ as } t \to \infty\right.$