21.04.21

(P_3)

Pure Maths_3

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Differentiation Notes.

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Differentiation

"Revision from P_1"

§ Differentiation Formulae:

(i) $\frac{d}{dx} x^n = n x^{n-1}$ ✓ or $\begin{cases} y = x^n \\ \frac{dy}{dx} = n x^{n-1} \end{cases}$ or $\begin{cases} f(x) = x^n \\ f'(x) = n \cdot x^{n-1} \end{cases}$

(Power function)

(ii) $y = a f(x)$

$\frac{dy}{dx} = a \cdot f'(x)$

(iii) $\frac{dc}{dx} = 0$ (c is a constant)

(iv) $\frac{d}{dx} ax = a$ $\left\{ \frac{d}{dx} x = 1 \right.$

(v) $\frac{d}{dx} a x^n = a n x^{n-1}$

(vi) $\frac{d}{dx} \sqrt{x} = \frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ ✓

(vii) $\frac{d}{dx} \frac{1}{x} = \frac{d}{dx} x^{-1} = -1 x^{-2} = \frac{-1}{x^2}$ ✓

(viii) $y = a \cdot f(x) + b \cdot g(x) - c \cdot h(x)$

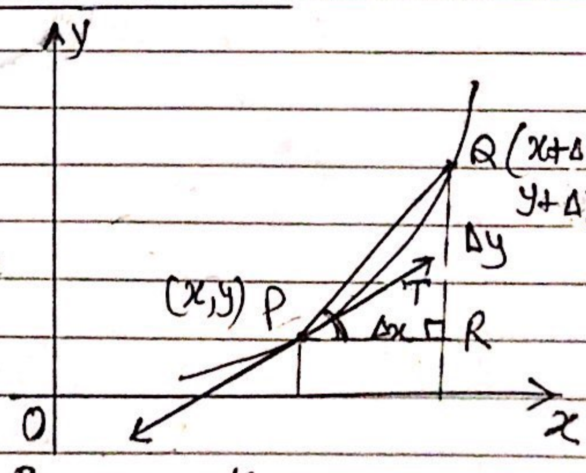
$\frac{dy}{dx} = a f'(x) + b g'(x) - c h'(x)$

Example: $y = x^5 - 3x^4 + 7x^2 + 8$

$\frac{dy}{dx} = 5x^4 - 3 \times 4x^3 + 7 \times 2x + 0$

$= 5x^4 - 12x^3 + 14x$ ✓

§ Geometric interpretation of $\frac{dy}{dx}$.



Given a function $y = f(x)$

Then $\frac{dy}{dx}$ or $f'(x)$ denotes the gradient (or slope) of the tangent to the curve $y = f(x)$ at any point $P(x, y)$ on the curve.

$f'(x) = \frac{dy}{dx} = \lim_{x \to 0} \frac{\Delta y}{\Delta x}$ = Gradient of the tangent to the curve at any point (x, y).

as $\Delta x \to 0$, $\Delta y \to 0$ and the chord $PQ \to$ tangent PT.

"PT is the tangent at P."

Differentiation

'Revision'

Example 1. Differentiate with respect to x (or w.r.t x)

(i) $\frac{1}{x^3}$ (ii) $x^2(1+x)$ (iii) $\frac{1+x}{x^2}$ (iv) $\frac{x^2+5x}{3\sqrt{x}}$

(i) $\frac{d}{dx}\frac{1}{x^3} = \frac{d}{dx}x^{-3} = -3x^{-3-1} = -3x^{-4} = \frac{-3}{x^4}$ ✓

Since $\frac{d}{dx}x^n = n \cdot x^{n-1}$

(ii) $\frac{d}{dx} x^2(1+x) = \frac{d}{dx}(x^2+x^3) = 2x+3x^2$ ✓

(iii) $\frac{d}{dx}\frac{1+x}{x^2} = \frac{d}{dx}\left(\frac{1}{x^2}+\frac{x}{x^2}\right) = \frac{d}{dx}\left(x^{-2}+x^{-1}\right) = -2x^{-3}-1x^{-2}$

$= \frac{-2}{x^3} - \frac{1}{x^2}$ ✓

(iv) $\frac{d}{dx}\left(\frac{x^2+5x}{3\sqrt{x}}\right) = \frac{1}{3}\frac{d}{dx}\left(\frac{x^2}{x^{1/2}}+\frac{5x}{x^{1/2}}\right) = \frac{1}{3}\frac{d}{dx}\left(x^{3/2}+5x^{1/2}\right)$

$= \frac{1}{3}\left[\frac{3}{2}x^{1/2} + 5\times\frac{1}{2}x^{-\frac{1}{2}}\right]$

$= \frac{1}{3}\left[\frac{3}{2}\sqrt{x} + \frac{5}{2\sqrt{x}}\right]$ ✓

Example 2. Find the value of $\frac{dy}{dx}$ at the given points.

(a) $y = x^4$ at $x=2$

$\frac{dy}{dx} = 4x^3$

$\left(\frac{dy}{dx}\right)_{x=2} = 4\times 2^3 = 32$ ✓

(b) $y = 5x^3 - 3x^2 + 7x + 6$ at $x=5$

$\frac{dy}{dx} = 5\times 3x^2 - 3\times 2x + 7\times 1 + 0$

$= 15x^2 - 6x + 7$

$\left(\frac{dy}{dx}\right)_{x=5} = 15\times 5^2 - 6\times 5 + 7 = 375 - 30 + 7 = 352$ ✓

Example 6. Given $f(x) = 4\sqrt{x}$ find $f'(9)$

$f'(x) = 4\times\frac{1}{2\sqrt{x}} = \frac{2}{\sqrt{x}}$

$\therefore f'(9) = \frac{2}{\sqrt{9}} = \frac{2}{3}$ ✓

$f(x) = 4\sqrt{x} = 4x^{\frac{1}{2}}$

$f'(x) = 4\times\frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{2}{\sqrt{x}}$

§ Derivative of Composite functions (Using Chain Rule)

Given $y = f(u)$ and $u = g(x)$

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ ✓

§ $y = (ax+b)^n$ to find $\frac{dy}{dx}$.

$y = u^n$ Let $u = ax+b$

$\frac{dy}{du} = nu^{n-1}$ and $\frac{du}{dx} = a$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (Chain rule)

$= nu^{n-1} \cdot a$

$\therefore \frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$ ✓ —— (1)

Example 3: $f(x) = (3-5x)^8$, find the value of $f'(0)$.

Solution: $f(x) = (3-5x)^8 \Rightarrow f'(x) = -5 \times 8(3-5x)^7$

$\therefore f'(0) = -40 \times 3^7 = -87480$ ✓

Example 4: $y = \sqrt{2x^3+4x^2+1}$; find $\frac{dy}{dx}$.

Solution: $y = \sqrt{u}$ and $u = 2x^3+4x^2+1$

$\frac{dy}{du} = \frac{1}{2\sqrt{u}}$ $\frac{du}{dx} = 6x^2+8x$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{1}{2\sqrt{u}} \times (6x^2+8x) = \frac{\cancel{2}(3x^2+4x)}{\cancel{2}\sqrt{2x^3+4x^2+1}}$

$\therefore \frac{dy}{dx} = \frac{3x^2+4x}{\sqrt{(2x^3+4x^2+1)}}$

Differentiation

§ Differentiation of Product of two functions:

$y = u \cdot v$ $\quad u = f(x)$, $\quad v = g(x)$

$$\boxed{\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}}$$

Example 5: Given $y = x^3\sqrt{2x-1}$ find $\frac{dy}{dx}$.

Solution: $y = x^3 \cdot \sqrt{2x-1}$

$$\Rightarrow \frac{dy}{dx} = x^3 \cdot \frac{d}{dx}(2x-1)^{\frac{1}{2}} + \sqrt{2x-1} \cdot \frac{d}{dx} x^3$$

$$= x^3 \times \frac{1}{2}(2x-1)^{-\frac{1}{2}} \times 2 + \sqrt{2x-1} \times 3x^2$$

$$= \frac{x^3}{\sqrt{2x-1}} + 3x^2 \cdot \sqrt{2x-1}$$

$$= \frac{x^3 + 3x^2(2x-1)}{\sqrt{2x-1}} = \frac{7x^3 - 3x^2}{\sqrt{2x-1}}$$

$$\frac{dy}{dx} = \frac{x^2(7x-3)}{\sqrt{2x-1}} \checkmark$$

Example 6: Find the equation of the tangent to the curve, $y = (2-x)^3(x+1)^4$ at the point where $x = 1$.

Solution: $y = (2-x)^3 \cdot (x+1)^4$ ---(1)

$$\frac{dy}{dx} = (2-x)^3 \cdot \frac{d}{dx}(x+1)^4 + (x+1)^4 \cdot \frac{d}{dx}(2-x)^3$$

$$= (2-x)^3 \cdot 4(x+1)^3 + (x+1)^4 \cdot 3(2-x)^2 \times (-1)$$

$$= (x+1)^3 \cdot (2-x)^2 [4(2-x) - 3(x+1)]$$

$$= (x+1)^3 (2-x)^2 \cdot (5-7x)$$

$$\left(\frac{dy}{dx}\right)_{x=1} = 2^3 \times 1^3 \times (-2) = -16 \checkmark$$

from (1) at $x = 1$, $y = 1^3 \times 2^4 = 16$.

$\therefore$ Equation of tangent to the curve (1) at the point $(1, 16)$

$$y - y_1 = \left(\frac{dy}{dx}\right)_{x=x_1}(x - x_1)$$

$$\Rightarrow y - 16 = -16(x-1)$$

$$\Rightarrow \underline{y + 16x = 32} \checkmark$$

§ Differentiation of Quotient of two funtions:

Given $y = \frac{u}{v}$ $\quad u = f(x)$ and $v = g(x)$

$$\frac{dy}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

Example 7: Given $y = \frac{x^2-3}{2x-1}$, find $\frac{dy}{dx}$.

Solution: $y = \frac{x^2-3}{(2x-1)}$

$$\frac{dy}{dx} = \frac{(2x-1) \cdot \frac{d}{dx}(x^2-3) - (x^2-3)\frac{d}{dx}(2x-1)}{(2x-1)^2}$$

$$= \frac{(2x-1) \times 2x - (x^2-3) \times 2}{(2x-1)^2}$$

$$= \frac{4x^2 - 2x - 2x^2 + 6}{(2x-1)^2} = \frac{2x^2 - 2x + 6}{(2x-1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{2(x^2 - x + 3)}{(2x-1)^2}$$

Example 8: $y = \frac{5x^4}{(x^2-1)^2}$ find $\frac{dy}{dx}$.

Solution: $y = \frac{5x^4}{(x^2-1)^2} \Rightarrow \frac{dy}{dx} = \frac{(x^2-1)^2 \cdot \frac{d}{dx} 5x^4 - 5x^4 \cdot \frac{d}{dx}(x^2-1)^2}{[(x^2-1)^2]^2}$

$$= \frac{(x^2-1)^2 \times 20x^3 - 5x^4 \times 2(x^2-1) \times 2x}{(x^2-1)^4}$$

$$= \frac{(x^2-1)[20x^3(x^2-1) - 20x^5]}{(x^2-1)^4}$$

$$= \frac{\cancel{20x^5} - 20x^3 - \cancel{20x^5}}{(x^2-1)^3}$$

$$= -\frac{20x^3}{(x^2-1)^3}$$

§ The derivative of e^x:

1. Given $y = e^x$ $\quad\left[\lim_{x\to 0} \frac{e^x - 1}{x} = 1\right]$

$$\frac{dy}{dx} = e^x \checkmark$$

2. $y = e^u$; $u = f(x)$

$$\frac{dy}{dx} = e^u \cdot \frac{du}{dx} \checkmark$$

3. $\frac{d}{dx} e^{ax+b} = a \cdot e^{ax+b} \checkmark$

Example 9: Given $y = e^{x^2}$; find $\frac{dy}{dx}$.

Solution: $y = e^{x^2} \Rightarrow \frac{dy}{dx} = e^{x^2} \times \frac{d}{dx} x^2$

$$= e^{x^2} \cdot 2x \Rightarrow \underline{\frac{dy}{dx} = 2x \cdot e^{x^2}}$$

Example 10: Given the equation of the curve $y = \frac{e^{\frac{1}{2}x}}{x}$ for $x > 0$ and its minimum point M, find the x-coordinate of M. ---[4]

[S-17 | 33 | Q7(i)]

Solution: $y = \frac{e^{\frac{1}{2}x}}{x}$ ---(i) $x > 0$

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx} e^{\frac{1}{2}x} - e^{\frac{1}{2}x} \cdot \frac{d}{dx} x}{x^2} = \frac{x \cdot \frac{1}{2} e^{\frac{1}{2}x} - e^{\frac{1}{2}x} \times 1}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\frac{1}{2}x}\left(\frac{1}{2}x - 1\right)}{x^2} \quad \text{---(ii)}$$

for any stationary point $\frac{dy}{dx} = 0 \Rightarrow \frac{e^{\frac{1}{2}x}\left(\frac{1}{2}x - 1\right)}{x^2} = 0$

$$\left\{\begin{array}{l} \Rightarrow \frac{1}{2}x - 1 = 0 \quad [\because e^{\frac{1}{2}x} \neq 0] \\ \Rightarrow x = 2 \checkmark \end{array}\right.$$

$\therefore$ x-coordinate of M $= \underline{2} \checkmark$

Example 11: Given the equation of curve, $y=(x+1)e^{-\frac{1}{3}x}$ and its maximum point M. Find the x-coordinate of M. ---[4]

[S-18/32/Q8(i)]

Solution: $y=(x+1)e^{-\frac{1}{3}x}$

$$\frac{dy}{dx}=(x+1)\frac{d}{dx}e^{-\frac{1}{3}x}+e^{-\frac{1}{3}x}\frac{d}{dx}(x+1)$$

$$=(x+1)\cdot\left(-\frac{1}{3}\right)e^{-\frac{1}{3}x}+e^{-\frac{1}{3}x}\times 1$$

$$=e^{-\frac{1}{3}x}\left[-\frac{1}{3}x-\frac{1}{3}+1\right]$$

$$=e^{-\frac{1}{3}x}\left[-\frac{1}{3}x+\frac{2}{3}\right] \quad ---(1)$$

for any stationary point $\frac{dy}{dx}=0$

$\therefore$ from (1) $e^{-\frac{1}{3}x}\left(-\frac{1}{3}x+\frac{2}{3}\right)=0$

$\Rightarrow -\frac{1}{3}x+\frac{2}{3}=0 \quad (\because e^{-\frac{1}{3}x}\neq 0)$

$\therefore$ x-coord. of M = $\Rightarrow$ $x=2$ ✓

Example 12: The curve with equation $y=\frac{e^{2x}}{4+e^{3x}}$ has one stationary point. Find the exact values of the coordinates of this point.

[S-15/33/Q4] ---[6]

Solution: $y=\frac{e^{2x}}{4+e^{3x}}$ ---(1) $\Rightarrow \frac{dy}{dx}=\frac{(4+e^{3x})\cdot\frac{d}{dx}e^{2x}-e^{2x}\cdot\frac{d}{dx}(4+e^{3x})}{(4+e^{3x})^2}$

$$\Rightarrow \frac{dy}{dx}=\frac{(4+e^{3x})\cdot 2e^{2x}-e^{2x}\cdot 3e^{3x}}{(4+e^{3x})^2}$$

$$=\frac{8e^{2x}+2e^{3x}\cdot e^{2x}-3e^{3x}\cdot e^{2x}}{(4+e^{3x})^2}$$

$$=\frac{e^{2x}[8-e^{3x}]}{(4+e^{3x})^2} \quad ---(11)$$

For any stationary point, $\frac{dy}{dx}=0$ from (11)

$$\Rightarrow \frac{e^{2x}[8-e^{3x}]}{(4+e^{3x})^2}=0$$

$$\Rightarrow 8-e^{3x}=0 \Rightarrow e^{3x}=8$$

$$\Rightarrow (e^{x})^3=2^3$$

$\Rightarrow e^{x}=2$

$\Rightarrow x=\ln 2$ ✓

from (1)

$$y=\frac{2^2}{4+2^3} \quad \left[e^{x}=2\right.$$

$$=\frac{4}{12}=\frac{1}{3}$$

$\therefore$ Coordinates of stationary point $\left(\ln 2, \frac{1}{3}\right)$ ✓

§ Natural Logarithm

$$y = \ln x = \log_e x \qquad \left[\begin{array}{l} y = \log_e x \\ \Leftrightarrow e^y = x \end{array}\right.$$

(i) $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$ ✓ $\qquad x > 0$

(ii) $y = \ln(ax+b) \Rightarrow \frac{dy}{dx} = \frac{a}{(ax+b)}$ ✓

Example 13: The curve with equation $y = \frac{(\ln x)^2}{x}$ has two stationary points. Find the exact values of the coordinates of these points. ---[6]

S-16/32/Q4

Solution: $y = \frac{(\ln x)^2}{x}$ ---(i) $\Rightarrow \frac{dy}{dx} = \frac{x\frac{d}{dx}(\ln x)^2 - (\ln x)^2 \cdot \frac{d}{dx}x}{x^2}$

$$= \frac{x \times 2\ln x \times \frac{1}{x} - (\ln x)^2 \times 1}{x^2}$$

$$= \frac{\ln x[2 - \ln x]}{x^2} \quad \text{---(ii)}$$

for stationary point;

$$\frac{dy}{dx} = 0 \Rightarrow \frac{\ln x[2 - \ln x]}{x^2} = 0 \quad \text{from (ii)}$$

$$\Rightarrow \ln x = 0 \quad \text{or} \quad \ln x = 2 \quad [\log_e x = 2]$$

$$\Rightarrow x = 1 \quad ; \quad x = e^2$$

from (1) $\left.\begin{array}{l} x = 1 \\ y = 0 \end{array}\right\}$ and $\left\{\begin{array}{l} x = e^2 \\ y = \frac{(\ln e^2)^2}{e^2} = \frac{(2\times1)^2}{e^2} = 4e^{-2} \end{array}\right.$ ✓

∴ Stationary points are;

$(1, 0)$ and $(e^2, 4e^{-2})$ ✓

Example 14: Find the exact coordinates of the point on the curve $y = \frac{x}{1+\ln x}$ at which the gradient of the tangent is equal to $\frac{1}{4}$.
[S-19|32|Q4] -- [7]

Solution: $y = \frac{x}{1+\ln x}$ ---(i) $\Rightarrow$ $\frac{dy}{dx} = \frac{(1+\ln x)\frac{d}{dx}x - x\frac{d}{dx}(1+\ln x)}{(1+\ln x)^2}$

$$= \frac{(1+\ln x) - x \times \frac{1}{x}}{(1+\ln x)^2}$$

$$\frac{dy}{dx} = \frac{\ln x}{(1+\ln x)^2} \quad \text{---(ii)}$$

$\therefore$ the gradient of the tangent to the curve (i)

$$\frac{dy}{dx} = \frac{\ln x}{(1+\ln x)^2} = \frac{1}{4} \text{ (given)}$$

$$\Rightarrow (1+\ln x)^2 = 4\ln x \Rightarrow 1 + (\ln x)^2 + 2\ln x = 4\ln x$$

$$\Rightarrow (\ln x)^2 - 2\ln x + 1 = 0$$

$$\Rightarrow (\ln x - 1)^2 = 0$$

$$\Rightarrow \ln x - 1 = 0$$

$$\Rightarrow \ln x = 1 = \ln e$$

$$\Rightarrow x = e$$

from (1) at $x = e$, $y = \frac{e}{1+\ln e} = \frac{1}{2}e$

$\therefore$ The required point $(e, \frac{1}{2}e)$ ✓

§ Differentiation of Trigonometric functions:

1. $\frac{d}{dx} \sin x = \cos x$
2. $\frac{d}{dx} \cos x = -\sin x$
3. $\frac{d}{dx} \tan x = \sec^2 x$
4. $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$
5. $\frac{d}{dx} \sec x = \sec x \cdot \tan x$
6. $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$
7. $\frac{d}{dx} \sin(ax+b) = a \cdot \cos(ax+b)$
8. (i) $\frac{d}{dx} \ln \sin x = \cot x$

   (ii) $\frac{d}{dx} \ln \sec x = \tan x$

Example 15: Find the derivative of the following functions:
(a) $\ln \sin x$ (b) $\ln \sec x$ (c) $\sin^3 x$ (d) $\tan(4x+5)$
(e) $\cos^3 x^5$

Solution (a) $y = \ln \sin x$

$$\frac{dy}{dx} = \frac{1}{\sin x} \times \frac{d}{dx} \sin x$$

$$= \frac{1}{\sin x} \times \cos x = \underline{\cot x}$$

(b) $y = \ln \sec x$

$$= \frac{1}{\sec x} \times \frac{d}{dx} \sec x$$

$$= \frac{1}{\sec x} \times \sec x \cdot \tan x = \underline{\tan x}$$

(c) $y = \sin^3 x$

$$y = (\sin x)^3$$

$$= 3 \sin^2 x \times \frac{d}{dx} \sin x$$

$$= \underline{3 \sin^2 x \cdot \cos x}$$

(d) $y = \tan(4x+5)$

$$\frac{dy}{dx} = \sec^2(4x+5) \times 4$$

$$= 4 \sec^2(4x+5)$$

(e) $y = \cos^3 x^5$

or $y = (\cos x^5)^3$

$$\frac{dy}{dx} = 3 \cos^2 x^5 \times \frac{d}{dx} \cos x^5$$

$$= 3 \cos^2 x^5 \times (-\sin x^5) \times \frac{d}{dx} x^5$$

$$= -3 \cos^2 x^5 \times \sin x^5 \times 5x^4$$

$$= \underline{-15 x^4 \cdot \cos^2 x^5 \cdot \sin x^5}$$

Example 16: The equation of a curve is $y = e^{-2x}\tan x$, for $0 \le x \le \frac{\pi}{2}$

(i) Obtain an expression for $\frac{dy}{dx}$ and show that it can be written in the form $e^{-2x}(a + b\tan x)^2$, where a and b are constants. ---[5]

(ii) Explain why the gradient of the curve is never negative. ---[1]

(iii) Find the value of x for which the gradient is least. ---[1]

W-15/31/Q5

Solution: $y = e^{-2x} \cdot \tan x$

(i) $\frac{dy}{dx} = e^{-2x} \cdot \frac{d}{dx}\tan x + \tan x \cdot \frac{d}{dx} e^{-2x}$

$= e^{-2x} \cdot \sec^2 x + \tan x \cdot (-2)e^{-2x}$

$= e^{-2x}(\sec^2 x - 2\tan x)$

$= e^{-2x}(1 + \tan^2 x - 2\tan x$

$= \underline{e^{-2x}(1 - \tan x)^2}$ ✓

(ii) $\frac{dy}{dx} = e^{-2x}(1 - \tan x)^2 \geq 0$ $\left\{ \because e^{-2x} > 0 \text{ and } (1-\tan x)^2 \geq 0 \right.$ being square of a real number

(iii) The least value of gradient can be zero.

$\Rightarrow e^{-2x}(1 - \tan x)^2 = 0$

$\Rightarrow (1 - \tan x)^2 = 0 \quad (\because e^{-2x} \neq 0)$

$\Rightarrow 1 - \tan x = 0$

$\Rightarrow \tan x = 1$

$\Rightarrow \underline{x = \frac{\pi}{4}}$ ✓ for $0 \le x \le \frac{\pi}{2}$

Example 17: By differentiating $\frac{1}{\sin x}$ Show that if $y = \operatorname{cosec} x$ then $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$ ---[3]

[S-16/32/Q8(i)]

Solution: $y = \frac{1}{\sin x} \Rightarrow \frac{dy}{dx} = \frac{\sin x \times \frac{d}{dx}1 - 1 \times \frac{d}{dx}\sin x}{\sin^2 x}$

$$= \frac{\sin x \times 0 - \cos x}{\sin^2 x}$$

$$= -\frac{\cos x}{\sin x \times \sin x} = -\frac{1}{\sin x} \times \frac{\cos x}{\sin x}$$

$$\Rightarrow \frac{dy}{dx} = -\operatorname{cosec} x \cot x$$

$$\Rightarrow \frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x \left[\because \frac{1}{\sin x} = \operatorname{cosec} x\right]$$

Example 18: The equation of a curve is: $y = 3\cos 2x + 7\sin x + 2$. Find the x-coordinate of the stationary points in the interval $0 \le x \le \pi$. Give each answer correct to 3 significant figures. ---[7]

[S-15/31/Q4]

Solution: $y = 3\cos 2x + 7\sin x + 2$

$$\frac{dy}{dx} = 3 \times (-2)\sin 2x + 7\cos x$$

$$= -6\sin 2x + 7\cos x$$

$$= -6 \times 2\sin x\cos x + 7\cos x$$

$$= \cos x[7 - 12\sin x] \quad \text{---(i)}$$

for stationary points $\frac{dy}{dx} = 0$

from (i)

$$\cos x = 0 \text{ or } 7 - 12\sin x = 0$$

$$x = \frac{\pi}{2} \text{ or } \sin x = \frac{7}{12} = 0.5833 \qquad 0 \le x \le \pi$$

$$x = \sin^{-1} 0.5833$$

$$x = 0.623 \text{ or } \pi - 0.623$$

$\therefore$ $x = 1.57$; 0.623; 2.52 radians

Example 19: The equation of a curve is, $y = 5\sin^2 x \cos^3 x$, for $0 < x < \frac{1}{2}\pi$. Find the x-coordinate of the point of maximum of the curve. ---[5]

[W-18/31/Q7(i)]

Solution: $y = 5\sin^2 x \cdot \cos^3 x \qquad 0 < x < \frac{\pi}{2}$

$\frac{dy}{dx} = 5\left[\sin^2 x \cdot \frac{d}{dx}\cos^3 x + \cos^3 x \cdot \frac{d}{dx}\sin^2 x\right]$

$= 5[\sin^2 x \times 3\cos^2 x \cdot (-\sin x) + \cos^3 x \times 2\sin x \cos x]$

$= 5\sin x \cdot \cos^2 x[-3\sin^2 x + 2\cos^2 x] = 0$ ----(i)

For the point of maximum: $\frac{dy}{dx} = 0$

from (i) $5\sin x \cdot \cos^2 x[-3\sin^2 x + 2\cos^2 x] = 0 \qquad 0 < x < \pi/2$

$\Rightarrow -3\sin^2 x + 2\cos^2 x = 0 \qquad \left[\begin{array}{l}\sin x \neq 0 \\ \cos x \neq\end{array} \text{ in } 0 < x < \frac{\pi}{2}\right]$

$\Rightarrow \tan^2 x = \frac{2}{3}$

$\tan x = +\sqrt{\frac{2}{3}}$ or $\tan x = -\sqrt{\frac{2}{3}}$ ✗ $\qquad 0 < x < \frac{\pi}{2}$

$\therefore x = \tan^{-1}\sqrt{\frac{2}{3}} = \tan^{-1} 0.8165 = \underline{0.685 \text{ rad}}$ ✓

Example 20: A curve has equation, $y = \cos x \cdot \sin 2x$

Find the x-coordinate of the stationary point in the interval $0 < x < \frac{1}{2}\pi$, giving your answer correct to 3 s.f. ----[6]

[S-20/32/Q4]

Solution: $y = \cos x \cdot \sin 2x$

$\frac{dy}{dx} = \cos x \cdot 2\cos 2x + \sin 2x \cdot (-\sin x)$

$= 2\cos 2x \cdot \cos x - \sin 2x \cdot \sin x$

$= 2(2\cos^2 x - 1)\cos x - 2\sin x \cos x \cdot \sin x$

$= 2(2\cos^2 x - 1)\cos x - 2\cos x(1 - \cos^2 x)$

$= 4\cos^3 x - 2\cos x - 2\cos x + 2\cos^3 x$

$= 6\cos^3 x - 4\cos x$

$= 2\cos x(3\cos^2 x - 2)$

For stationary points $\frac{dy}{dx} = 0$ ↗

$\Rightarrow 2\cos x(3\cos^2 x - 2) = 0$

$\Rightarrow \cos^2 x = \frac{2}{3} \left[\because \cos x \neq 0 \text{ as } 0 < x < \frac{\pi}{2}\right]$

$\Rightarrow \cos x = \sqrt{\frac{2}{3}}, \ -\sqrt{\frac{2}{3}}$ ✗

$x = \cos^{-1}\sqrt{\frac{2}{3}}$ ✓

$x = \underline{0.615}$ radians

§ Differentiation of Implicit functions:

(The functions of the type $y = f(x)$ are called explicit functions; $y = x^2 + e^x - \sin x$, $y = x^5 - 3x^3 + \tan x$)

But many times y cannot be expressed as a function of x. For example: (i) $x^2 + 2xy + y^2 + 7 = 0$

(ii) $x \sin y + y \cos x = 0$ etc

are called implicit functions, and to differentiate we say find $\frac{dy}{dx}$, we use chain rule.

$$\left\{ \begin{aligned} &\frac{d}{dx} y^3 = 3y^2 \times \frac{dy}{dx} \\ &\frac{d}{dx} \sin y = \cos y \cdot \frac{dy}{dx} \\ &\frac{d}{dx}(xy) = x \cdot \frac{dy}{dx} + y \end{aligned} \right.$$

Example 21: The equation of a curve is, $x^3 + 3xy^2 - y^3 = 5$

Show that; $\frac{dy}{dx} = \frac{x^2 + y^2}{y^2 - 2xy}$ ---[4]

[M-20/32/Q7(a)]

Solution:

$$x^3 + 3xy^2 - y^3 = 5$$

differentiating w.r.t x

$$3x^2 + 3\left(1 \cdot y^2 + x \cdot 2y\frac{dy}{dx}\right) - 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + 3y^2 + 6xy\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(6xy - 3y^2) = -3(x^2 + y^2)$$

$$\frac{dy}{dx} = \frac{-3(x^2 + y^2)}{-3(y^2 - 2xy)}$$

or $\frac{dy}{dx} = \frac{x^2 + y^2}{y^2 - 2xy}$ ✓

Example 22: The equation of a curve is $xy(x-6y)=9a^3$, where a is non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the x-axis, find the coordinates of this point. [W-16/31/Q4] --[7]

Solution: Given curve is; $xy(x-6y)=9a^3$ ---(i)

or $x^2y-6xy^2=9a^3$

differentiating with respect to x,

$$x^2\cdot\frac{dy}{dx}+y\times 2x-6\left(x\cdot 2y\frac{dy}{dx}+y^2\times 1\right)=0$$

or $$x^2\cdot\frac{dy}{dx}+2xy-12xy\frac{dy}{dx}-6y^2=0$$

or $$\frac{dy}{dx}\cdot(x^2-12xy)=6y^2-2xy$$

$$\therefore\quad \frac{dy}{dx}=\frac{6y^2-2xy}{x^2-12xy}\quad\text{---(ii)}$$

as the tangent to the curve is parallel to x-axis,

$\Rightarrow\quad \frac{dy}{dx}=0$ or $\frac{6y^2-2xy}{x^2-12xy}=0$ (from (ii))

$$\Rightarrow 6y^2-2xy=0$$

$$\Rightarrow 2y(3y-x)=0$$

$\Rightarrow\quad x=3y$ ✓ ...(iii) ; $y=0$ ✗ (from (i) $a\neq 0$)

from (i) $3y\times y(3y-6y)=9a^3$ [Put $x=3y$ in (i)]

$\Rightarrow 3y^2\times(-3y)=9a^3$

$\Rightarrow\quad y^3=-a^3\Rightarrow y=-a$ ✓

from (iii) $x=3y=3(-a)$
$=-3a$ ✓

$\therefore$ Required point $\underline{(-3a,-a)}$ ✓

Example 23: A curve has equation, $\sin y \ln x = x - 2\sin y$; $-\frac{1}{2}\pi \le x \le \frac{1}{2}\pi$

(i) Find $\frac{dy}{dx}$ in terms of x and y. -- [5]

(ii) Hence find the x-coordinate of the point on the curve at which the tangent is parallel to x-axis. [M-16/32/Q6] --[3]

Solution: $\sin y \ln x = x - 2\sin y$

(i) rearranging. $2\sin y + \sin y . \ln x = x$

$$\sin y(2+\ln x) = x$$

or $$\sin y = \frac{x}{2+\ln x} \text{ --- (i)}$$

diff. w.r.t x,

$$\cos y \frac{dy}{dx} = \frac{(2+\ln x)\cdot\frac{d}{dx}x - x\cdot\frac{d}{dx}(2+\ln x)}{(2+\ln x)^2}$$

$$= \frac{(2+\ln x)\times 1 - x\times\frac{1}{x}}{(2+\ln x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{(1+\ln x)}{(2+\ln x)^2} \text{ --- (ii)}$$

(ii) Given, tangent to the curve is parallel to x-axis,

$$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow \frac{1+\ln x}{(2+\ln x)^2} = 0 \quad (\text{from (ii)})$$

$$\Rightarrow 1+\ln x = 0$$

$$\Rightarrow \ln x = -1 \quad (\log_e x = -1)$$

$$\Rightarrow x = e^{-1} = \frac{1}{e}$$

Example 24: The variables x and y satisfy the relation, $\sin y = \tan x$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$, Show that $\frac{dy}{dx} = \frac{1}{\cos x\sqrt{(\cos 2x)}}$ [M-19/32/Q5] ---[5]

Solution: $\sin y = \tan x$ ---(i)

diff. w.r.t x

$$\cos y \frac{dy}{dx} = \sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{\cos y}$$

$$\frac{dy}{dx} = \frac{\sec^2 x}{\sqrt{1-\tan^2 x}} \quad \left[\because \cos y = \sqrt{1-\sin^2 y}\right. \text{ from (i)} = \sqrt{1-\tan^2 x}$$

$$= \frac{1}{\cos^2 x\sqrt{\frac{\cos^2 x - \sin^2 x}{\cos x}}}$$

$$= \frac{1}{\cos x\sqrt{(\cos 2x)}}$$

§ Differentiation of Parametric functions:

When x and y are both expressed as a function of third variable 't' (called parameter).

$$y = f(t) \quad \text{and} \quad x = g(t)$$

These two functions are called parametric equations.

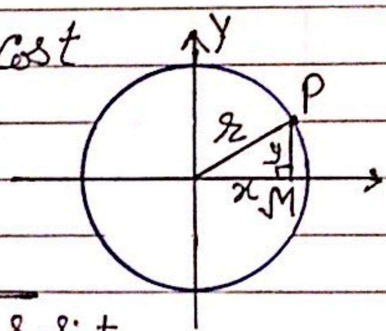
Now $y = f(t)$

find $\frac{dy}{dt} = f'(t)$ and $\frac{dx}{dt} = g'(t)$

Then $$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}}$$ ✓

Example: The parametric equations of a circle, Centre O, and radius 'r' are:

$$y = r\sin t, \quad x = r\cos t$$



Find $\frac{dy}{dx}$ in terms of t.

$y = r\sin t$

$\frac{dy}{dt} = r\cos t$ ---(i)

$x = r\cos t$

$\frac{dx}{dt} = -r\sin t$ ---(ii)

$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{r\cos t}{-r\sin t} \longrightarrow \left(-\frac{x}{y}\right)$

$= -\cot t$ ✓

$y = r\sin t$

$x = r\cos t$

Square and add.

$x^2 + y^2 = r^2$

Diff. w.r.t x

$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x}{y}$ ✓

$= -\frac{r\cos t}{r\sin t}$

$= -\cot t$ ✓

Example 25: The parametric equations of a curve are:

$x = 3 - \cos 2\theta$, $y = 2\theta + \sin 2\theta$, for $0 < \theta < \frac{1}{2}\pi$

Show that $\frac{dy}{dx} = \cot\theta$ [W-20/31/Q3] ---[5]

Solution: $y = 2\theta + \sin 2\theta$ and $x = 3 - \cos 2\theta$

$\frac{dy}{d\theta} = 2 + 2\cos 2\theta$; $\frac{dx}{d\theta} = +2\sin 2\theta$

Now $\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{dx} = \frac{2 + 2\cos 2\theta}{2\sin 2\theta}$

$= \frac{\cancel{2}(1 + \cos 2\theta)}{\cancel{2}\sin 2\theta} = \frac{2\cos^2\theta}{2\sin\theta\cos\theta} = \frac{\cos\theta}{\sin\theta}$

$\therefore \frac{dy}{dx} = \cot\theta$ ✓

Example 26: The parametric equations of a curve are:

$x = 2t + \sin 2t$, $y = \ln(1 - \cos 2t)$ ---[5]

Show that $\frac{dy}{dx} = \operatorname{cosec} 2t$. [W-19/31/Q3]

Solution: $y = \ln(1 - \cos 2t)$; $x = 2t + \sin 2t$

$\Rightarrow \frac{dy}{dt} = \frac{1}{(1 - \cos 2t)} \times (+2\sin 2t)$

$= \frac{2 \times 2\sin t\cos t}{2\sin^2 t}$

$= \frac{2\cos t}{\sin t}$ ----(i)

$\frac{dx}{dt} = 2 + 2\cos 2t$

$= 2(1 + \cos 2t)$

$= 2 \times 2\cos^2 t$

$= 4\cos^2 t$ ---(ii)

Now $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{\frac{2\cos t}{\sin t}}{4\cos^2 t} = \frac{2\cos t}{\sin t} \times \frac{1}{4\cos^2 t}$

$= \frac{1}{2\sin t\cos t}$

$= \frac{1}{\sin 2t} = \operatorname{cosec} 2t$

$\therefore \frac{dy}{dx} = \operatorname{cosec} 2t$ ✓

Example 27: The parametric equations of a curve are,

$x = \ln\cos\theta \quad ; \quad y = 3\theta - \tan\theta \quad ; \quad 0 \leq \theta \leq \frac{\pi}{2}$

(i) Express $\frac{dy}{dx}$ in terms of $\tan\theta$ ---[5]

(ii) Find the exact y-coordinate of the point on the curve at which the gradient of the normal is equal to 1. ---[3]

[S-17/31/Q4]

Solution:

(i)

$$y = 3\theta - \tan\theta \quad ; \quad x = \ln\cos\theta$$

$$\frac{dy}{d\theta} = 3 - \sec^2\theta \quad ; \quad \frac{dx}{d\theta} = \frac{1}{\cos\theta} \times -\sin\theta$$

$$= 3 - (1+\tan^2\theta) \quad ; \quad = -\tan\theta$$

$$= 2 - \tan^2\theta \quad ;$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

$$= \frac{2-\tan^2\theta}{-\tan\theta} = \frac{\cancel{-}(\tan^2\theta - 2)}{\cancel{-}\tan\theta}$$

$$\therefore \frac{dy}{dx} = \frac{\tan^2\theta - 2}{\tan\theta} \text{ ----- (i)}$$

(ii) Gradient of the normal $= -\dfrac{1}{\frac{dy}{dx}} = 1$ (given)

$$\Rightarrow \frac{dy}{dx} = -1$$

$$\Rightarrow \frac{\tan^2\theta - 2}{\tan\theta} = -1 \quad \text{(from (i))}$$

$$\Rightarrow \tan^2\theta + \tan\theta - 2 = 0$$

$$(\tan\theta + 2)(\tan\theta - 1) = 0$$

$$\Rightarrow \tan\theta = 1 \checkmark \text{ or } \tan\theta = -2^{\times} \left(\because 0 \leq \theta \leq \frac{\pi}{2}\right)$$

$$\text{or } \theta = \tan^{-1} = \frac{\pi}{4} \checkmark$$

Now y-coordinat at $\theta = \frac{\pi}{4}$ is

$$y = 3 \times \frac{\pi}{4} - \tan\frac{\pi}{4}$$

$$= \left(\frac{3\pi}{4} - 1\right) \checkmark$$