PURE MATHEMATICS -3
9709
(March, June and November series 2020 – 2023 With marking scheme)
LOGARITHEMIC AND EXPONENTIAL FUNCTIONS
EXERCISE -1

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1) SP-2020 _9709_3 Q1

Find the set of values of x for which $3(2^{3x+1}) < 8$. Give your answer in a simplified exact form. [3]

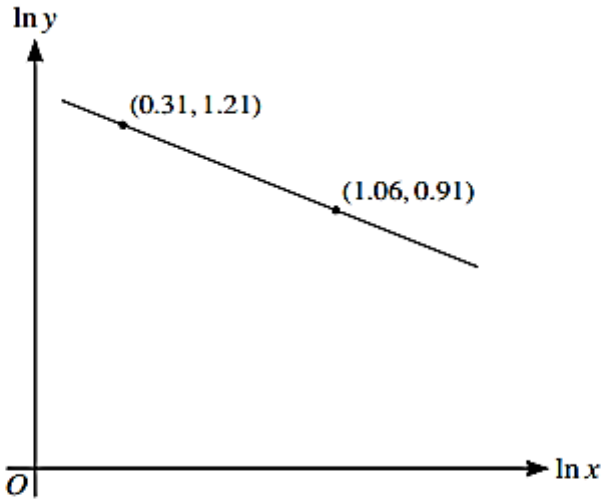
2) MARCH 2020_9709_32 Q2

Solve the equation $\ln 3 + \ln(2x + 5) = 2\ln(x + 2)$. Give your answer in a simplified exact form. [4]

3) MARCH 2021_9709_32 Q1

Solve the equation $\ln(x^3 - 3) = 3\ln x - \ln 3$. Give your answer correct to 3 significant figures. [3]

4) MARCH 2022_9709_32 Q3



The variables x and y satisfy the equation $x^n y^2 = C$, where n and C are constants. The graph of $\ln y$ against $\ln x$ is a straight line passing through the points (0.31, 1.21) and (1.06, 0.91), as shown in the diagram.

Find the value of n and find the value of C correct to 2 decimal places. [5]

5) MARCH 2023_9709_32 Q1

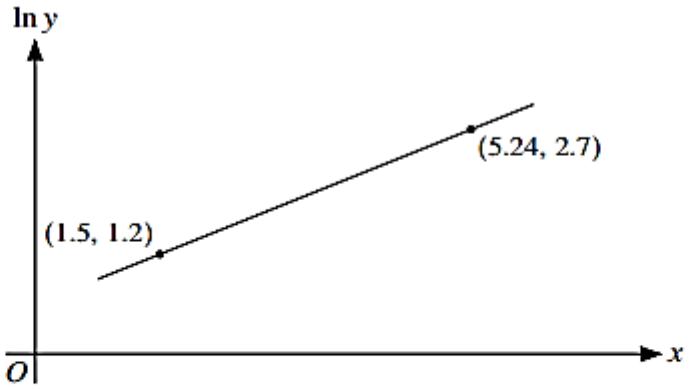
It is given that $x = \ln(2y - 3) - \ln(y + 4)$.

Express y in terms of x. [3]

6) JUNE 2020_9709_31 Q1

Find the set of values of x for which $2(3^{1-2x}) < 5^x$. Give your answer in a simplified exact form. [4]

7) JUNE 2020_9709_32 Q2



The variables x and y satisfy the equation $y^2 = Ae^{kx}$, where A and k are constants. The graph of $\ln y$ against x is a straight line passing through the points (1.5, 1.2) and (5.24, 2.7) as shown in the diagram.

Find the values of A and k correct to 2 decimal places. [5]

8) JUNE 2020_9709_33 Q3

(a) Show that the equation

$$\ln(1 + e^{-x}) + 2x = 0$$

can be expressed as a quadratic equation in e^x. [2]

(b) Hence solve the equation $\ln(1 + e^{-x}) + 2x = 0$, giving your answer correct to 3 decimal places. [4]

9) JUNE 2021_9709_31 Q2

Find the real root of the equation $\dfrac{2e^x + e^{-x}}{2 + e^x} = 3$, giving your answer correct to 3 decimal places. Your working should show clearly that the equation has only one real root. [5]

10) JUNE 2021_9709_32 Q3

The variables x and y satisfy the equation $x = A(3^{-y})$, where A is a constant.

(a) Explain why the graph of y against $\ln x$ is a straight line and state the exact value of the gradient of the line. [3]

It is given that the line intersects the y-axis at the point where $y = 1.3$.

(b) Calculate the value of A, giving your answer correct to 2 decimal places. [2]

11) JUNE 2021_9709_33 Q2

Solve the equation $4^x = 3 + 4^{-x}$. Give your answer correct to 3 decimal places. [5]

12) JUNE 2022_9709_31 Q1

Solve the equation $2(3^{2x-1}) = 4^{x+1}$, giving your answer correct to 2 decimal places. [4]

13) JUNE 2022_9709_32 Q1

Solve the equation $\ln(e^{2x} + 3) = 2x + \ln 3$, giving your answer correct to 3 decimal places. [4]

14) JUNE 2022_9709_33 Q3

(a) Show that the equation $\log_3(2x+1) = 1 + 2\log_3(x-1)$ can be written as a quadratic equation in x. [3]

(b) Hence solve the equation $\log_3(4y+1) = 1 + 2\log_3(2y-1)$, giving your answer correct to 2 decimal places. [2]

15) JUNE 2023_9709_31 Q1

Solve the equation

$$3e^{2x} - 4e^{-2x} = 5.$$

Give the answer correct to 3 decimal places. [3]

16) JUNE 2023_9709_32 Q2

Solve the equation $\ln(2x^2 - 3) = 2\ln x - \ln 2$, giving your answer in an exact form. [3]

17) JUNE 2023_9709_33 Q1

Solve the equation $\ln(x+5) = 5 + \ln x$. Give your answer correct to 3 decimal places. [4]

18) OCT 2020_9709_31 Q4

Solve the equation

$$\log_{10}(2x+1) = 2\log_{10}(x+1) - 1.$$

Give your answers correct to 3 decimal places. [6]

19) OCT 2020_9709_32 Q1

Solve the equation

$$\ln(1 + e^{-3x}) = 2.$$

Give the answer correct to 3 decimal places. [3]

20) OCT 2020_9709_32 Q3

The variables x and y satisfy the relation $2^y = 3^{1-2x}$.

(a) By taking logarithms, show that the graph of y against x is a straight line. State the exact value of the gradient of this line. [3]

(b) Find the exact x-coordinate of the point of intersection of this line with the line $y = 3x$. Give your answer in the form $\dfrac{\ln a}{\ln b}$, where a and b are integers. [2]

21) OCT 2021_9709_31 Q1

Solve the equation $4|5^x - 1| = 5^x$, giving your answers correct to 3 decimal places. [4]

22) OCT 2021_9709_32 Q1

Find the value of x for which $3(2^{1-x}) = 7^x$. Give your answer in the form $\dfrac{\ln a}{\ln b}$, where a and b are integers. [4]

23) OCT 2021_9709_33 Q3

Solve the equation $4^{x-2} = 4^x - 4^2$, giving your answer correct to 3 decimal places. [4]

24) OCT 2022_9709_31 Q3

Solve the equation $2^{3x-1} = 5(3^{-x})$. Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers. [4]

25) OCT 2022_9709_32 Q1

Solve the equation $2^{3x-1} = 5(3^{1-x})$. Give your answer in the form $\frac{\ln a}{\ln b}$ where a and b are integers. [4]

26) OCT 2022_9709_33 Q1

Solve the equation $\ln(2x-1) = 2\ln(x+1) - \ln x$. Give your answer correct to 3 decimal places. [4]

MARKING SCHEME

1) SP-2020 _9709_3 Q1

Use law of the logarithm of a product, quotient or power	1	**M1**
Obtain a correct linear inequality in any form, e.g. $\ln 3 + (3x + 1)\ln 2 < \ln 8$	1	**A1**
Obtain final answer $x < \frac{\ln \frac{4}{3}}{\ln 8}$, or equivalent	1	**A1**
	3	

4

2) MARCH 2020_9709_32 Q2

Use law of logarithm of a power and sum and remove logarithms	**M1**	
Obtain a correct equation in any form, e.g. $3(2x+5)=(x+2)^2$	**A1**	
Use correct method to solve a 3-term quadratic, obtaining at least one root	**M1**	
Obtain final answer $x = 1+2\sqrt{3}$ or $1 + \sqrt{12}$ only	**A1**	
	4	

3) MARCH 2021_9709_32 Q1

Use law of the logarithm of a product or power	**M1**
Obtain a correct equation free of logarithms, e.g. $3(x^3-3)=x^3$	**A1**
Obtain $x = 1.65$	**A1**
	3

4) MARCH 2022_9709_32 Q3

State or imply $n \ln x + 2 \ln y = \ln C$	**B1**	
Substitute values of $\ln y$ and $\ln x$, or equate gradient of line to $\pm\frac{1}{2}n$, but not $\pm n$, and solve for n	**M1**	Using lnx and lny values
Obtain $n = 0.8[0]$ or $0.8[00]$ or $\frac{4}{5}$	**A1**	
Solve for C	**M1**	Using lnx and lny values in equation of correct form, that is ln$C$ not C. Allow $C = e^{2.668}$.
Obtain $C = 14.41$	**A1**	Must be 2 d.p.
Alternative method for question 3		
Obtain two correct equations in n and C by substituting x and y values in the given equation	**B1**	$(2.886)^n \times (2.484)^2 = C$ and $(1.363)^n \times (3.353)^2 = C$
Solve for n	**M1**	Using x and y values
Obtain $n = 0.8[0]$ or $0.8[00]$ or 4/5	**A1**	$\left(\frac{2.886}{1.363}\right)^n \times \left(\frac{2.484}{3.353}\right)^2 = 1$ leading to $n = 0.7995$
Solve for C	**M1**	Using x and y values
Obtain $C = 14.41$	**A1**	Must be 2 d.p.
	5	

5) MARCH 2023_9709_32 Q1

Use law of the logarithm of a quotient or express x as $\ln e^x$	**M1**	$x = \ln[(2y-3)/(y+4)]$ or $\ln e^x = \ln(2y-3) - \ln(y+4)$].
Remove logarithms and obtain a correct equation e.g. $e^x = \frac{2y-3}{y+4}$	**A1**	
Obtain answer $y = \frac{3+4e^x}{2-e^x}$	**A1**	OE ISW
	3	

6) JUNE 2020_9709_31 Q1

Use law of the logarithm of a product or power	**M1**
Obtain a correct linear inequality in any form, e.g. $\ln 2 + (1-2x)\ln 3 < x \ln 5$	**A1**
Solve for x	**M1**
Obtain $x > \frac{\ln 6}{\ln 45}$	**A1**
	4

7) JUNE 2020_9709_32 Q2

State or imply $2\ln y = \ln A + kx$	**B1**
Substitute values of $\ln y$ and x, or equate gradient of line to k, and solve for k	**M1**
Obtain $k = 0.80$	**A1**
Solve for $\ln A$	**M1**
Obtain $A = 3.31$	**A1**
Alternative method for question 2	
Obtain two correct equations in y and x by substituting y- and x- values in the given equation	**B1**
Solve for k	**M1**
Obtain $k = 0.80$	**A1**
Solve for A	**M1**
Obtain $A = 3.31$	**A1**
	5

8) JUNE 2020_9709_33 Q3

a)	Remove logarithms correctly and state $1 + e^{-x} = e^{-2x}$, or equivalent	**B1**
	Show equation is $u^2 + u - 1 = 0$, where $u = e^x$, or equivalent	**B1**
		2
b)	Solve a 3-term quadratic for u	**M1**
	Obtain root $\frac{1}{2}\left(-1+\sqrt{5}\right)$, or decimal in [0.61, 0.62]	**A1**
	Use correct method for finding x from a positive root	**M1**
	Obtain answer $x = -0.481$ only	**A1**
		4

9) JUNE 2021_9709_31 Q2

Reduce to a 3-term quadratic $u^2+6u-1=0$ OE	**B1**	Allow '= 0' implied
Solve a 3-term quadratic for u	**M1**	
Obtain root $\sqrt{10}-3$	**A1**	
Obtain answer $x=-1.818$ only	**A1**	The question asks for 3 d.p.
Reject $-\sqrt{10}-3$ correctly	**B1**	e.g. by stating that $e^x>0$ or $\ln\left(-10-\sqrt{3}\right)$ is impossible Not "math error".
Alternative method for Question 2		
Rearrange to obtain a correct iterative formula	**B1**	e.g. $x_{n+1}=-\ln\left(6+e^{x_n}\right)$
Use the iterative process at least twice	**M1**	
Obtain answer $x=-1.818$	**A1**	
Show sufficient iterations to at least 4 d.p. to justify $x=-1.818$	**A1**	1, –2.165..., –1.811..., –1.819..., –1.818..., –1.818...
Clear explanation of why there is only one real root	**B1**	
	5	

10) JUNE 2021_9709_32 Q3

(a)	State or imply $\ln x=\ln A-y\ln 3$	**B1**	$\left(y=-\frac{1}{\ln 3}\ln x+\frac{\ln A}{\ln 3}\right)$
	State that the graph of y against $\ln x$ has an equation that is *linear* in y and $\ln x$, or has an equation of the standard form '$y=mx+c$' and is thus a straight line	**B1**	Must be a correct statement. Accept if the 2 equations are written side by side with no comment. An equation with $y\ln 3$ should be compared with the form $py+q\ln x=c$.
	State that the gradient is $-\frac{1}{\ln 3}$	**B1**	OE. Exact answer required. ISW after a correct statement.
(b)	Substitute $\ln x=0$, $y=1.3$ and use correct method to solve for A	**M1**	$(\ln A=1.3\ln 3)$ Follow *their* equation in y and $\ln x$. Must be substituting $\ln x=0$, not $x=0$. $\ln 0$ 'used' in the solution scores M0A0.
	Obtain answer $A=4.17$ only	**A1**	Must be 2 d.p. as specified in question

11) JUNE 2021_9709_33 Q2

State or imply $u^2-3u-1=0$, or equivalent in 4^x	**B1**
Solve for u or 4^x	**M1**
Obtain root $\frac{1}{2}\left(3+\sqrt{13}\right)$, or decimal in [3.30, 3.31]	**A1**
Use correct method for finding x from a positive root	**M1**
Obtain answer $x=0.862$ and no other	**A1**
	5

12) JUNE 2022_9709_31 Q1

Use law of the logarithm of a product or a quotient or a power	*M1	
Obtain a correct linear equation in any form	A1	e.g. $\ln 2+(2x-1)\ln 3=(x+1)\ln 4$ or $\log_2 2+(2x-1)\log_2 3=(2x+2)\log_2 2$
Solve for x	DM1	Allow for unsimplified expression $x=\ldots$ Allow M1 M1 for $x=1.45$ from $6^{2x-1}=4^{x+1}$.
Obtain answer $x = 2.21$	A1	The question asks for 2 dp.
Alternative method for question 1		
Correct use of indices to obtain $2.25^x = 6$ or $1.5^{2x} = 6$	M1 A1	
Correct use of logarithms to solve for x	M1	Allow solution of $2.25^x = 6$ by trial and improvement as far as 2.2…
Obtain answer $x = 2.21$	A1	Need to see an intermediate step / sequence of iterations.

13) JUNE 2022_9709_32 Q1

Use law of the logarithm of a product, power or quotient or a law of indices (on an expression that is relevant to the question)	M1	e.g. $\ln(e^{2x}+3)-\ln 3=\ln\left(\frac{e^{2x}+3}{3}\right)$ or $e^{(2x+\ln 3)}=e^{2x}e^{\ln 3}$
State a correct equation without logs (in any form)	A1	e.g. $3+e^{2x}=3e^{2x}$
Carry out correct method to solve an equation of the form $e^{2x}=a$, where $a>0$, or for solving $e^x=b$ $(b>0)$ if they have already taken the square root	M1	Allow for $x=\frac{1}{2}\ln\frac{3}{2}$. M1 can be implied by correct answer.
Obtain answer $x = 0.203$	A1	CAO. The question requires 3 d.p. Answer only with no working shown is 0/4.
	4	

14) JUNE 2022_9709_33 Q3

(a)	Use law of logarithm of a power	M1	$\log_3(2x+1)=1+\log_3(x-1)^2$
	Use $\log_3 3=1$	B1	$\log_3(2x+1)=\log_3 3+2\log_3(x-1)$ $\left[\log 3\left(\frac{2x+1}{(x-1)^2}\right)=\log_3 3 \quad \text{or} \quad \left(\frac{2x+1}{(x-1)^2}\right)=3\right]$ **SC** For candidates scoring M0 B0 due to combining logs before dealing with coefficient 2, and confusing coefficients, allow $\log_3(\ldots)=c$ leading to $(\ldots)=3^c$ **B1**.
	Obtain $3x^2-8x+2=0$ or $1.5x^2-4x+1=0$	A1	OE 3 terms only and = 0 required.
(b)	Solve 3-term quadratic equation from part **3(a)** or restart to find y	M1	$y=\frac{4\pm\sqrt{10}}{6}$ or $y=1.1937\ldots$ or $y=0.1396\ldots$ $(x=2.3874$ or $x=0.2792)$ May solve for x but must find $y=\frac{x}{2}$ to gain M1.
	Obtain answer 1.19	A1	CAO. 2 dp required.

15) JUNE 2023_9709_31 Q1

$3\left(e^{2x}\right)^2-5\left(e^{2x}\right)-4=0$	**B1**	OE Form 3 term quadratic in e^{2x}.
$e^{2x}=\frac{5\pm\sqrt{73}}{6}, \quad x=\frac{1}{2}\ln\left(\frac{5+\sqrt{73}}{6}\right)$	**M1**	Use correct method to solve for x.
$x=0.407$	**A1**	Only

16) JUNE 2023_9709_32 Q2

Use law of the logarithm of a power, quotient or product	**M1**	Must be used correctly on a **correct** term. e.g. M1 for $2\ln x=\ln x^2$ but M0 for $2\ln x-\ln 2=2\ln\frac{x}{2}$. M0 for $\ln(2x^2-3)=\ln 2x^2-\ln 3$ $=\ln 2+2\ln x-\ln 3$.
Remove logarithms and obtain a correct equation in x	**A1**	e.g. $2x^2-3=\frac{x^2}{2}$.
Obtain final answer $x=\sqrt{2}$ only	**A1**	If $x=-\sqrt{2}$ is mentioned, it must be rejected.

17) JUNE 2023_9709_33 Q1

Use exponentials or law for the logarithm of a product, quotient or power	**M1***	$e^{\ln(5+x)}=e^{5+\ln x}$ insufficient. Need e.g. $\ln\left(\frac{x+5}{x}\right)=5$ or $\ln(x+5)=\ln\left(e^5\right)+\ln x$ or $\ln(x+5)=\ln(e^5x)$ or $x+5=e^{5+\ln x}$ or $x+5=e^5e^{\ln x}$ and others.
Correctly remove logarithms	**DM1**	
Obtain a correct equation in x	**A1**	e.g. $\frac{x+5}{x}=e^5$ (or 148.4…) or $x+5=xe^5$.
Obtain 0.034	**A1**	CAO Final answer must be 3d.p.

18) OCT 2020_9709_31 Q4

State or imply $\log_{10}10=1$	**B1**	$\log_{10}10^{-1}=-1$
Use law of the logarithm of a power, product or quotient	**M1**	
Obtain a correct equation in any form, free of logs	**A1**	e.g. $(2x+1)/(x+1)^2=10^{-1}$ or $10(2x+1)/(x+1)^2=10^0$ or 1 or $x^2+2x+1=20x+10$
Reduce to $x^2-18x-9=0$, or equivalent	**A1**	
Solve a 3-term quadratic	**M1**	
Obtain final answers $x=18.487$ and $x=-0.487$	**A1**	Must be 3 d.p. Do not allow rejection.

19) OCT 2020_9709_32 Q1

State that $1+e^{-3x}=e^2$	**B1**	With no errors seen to that point
Use correct method to solve an equation of the form $e^{-3x}=a$, where $a>0$, for x or equivalent	**M1**	$(e^{-3x}=6.389...)$ Evidence of method must be seen.
Obtain answer $x=-0.618$ only	**A1**	Must be 3 decimal places
Alternative method for question 1		
State that $1+e^{-3x}=e^2$	**B1**	
Rearrange to obtain an expression for e^x and solve an equation of the form $e^x=a$, where $a>0$, or equivalent	**M1**	$e^x=\sqrt[3]{\frac{1}{e^2-1}}$
Obtain answer $x=-0.618$ only	**A1**	Must be 3 decimal places
	3	

20) OCT 2020_9709_32 Q3

(a)	State or imply $y\log 2=\log 3-2x\log 3$	**B1**	Accept $y\ln 2=(1-2x)\ln 3$
	State that the graph of y against x has an equation which is linear in x and y, or is of the form $ay=bx+c$	**B1**	Correct equation. Need a clear statement/comparison with matching linear form.
	Clear indication that the gradient is $-\frac{2\ln 3}{\ln 2}$	**B1**	Must be exact. Any equivalent e.g. $-\frac{2\log_k 3}{\log_k 2}$, $\log_2\frac{1}{9}$
		3	
(b)	Substitute $y=3x$ in an equation involving logarithms and solve for x	**M1**	
	Obtain answer $x=\frac{\ln 3}{\ln 72}$	**A1**	Allow M1A1 for the correct answer following decimals

21) OCT 2021_9709_31 Q1

State or imply non-modular equation $4^2(5^x-1)^2=(5^x)^2$ or pair of equations $4(5^x-1)=\pm 5^x$	**M1**
Obtain $5^x=\frac{4}{3}$ and $5^x=\frac{4}{5}$ (or $5^{x+1}=4$)	**A1**
Use correct method for solving an equation of the form $5^x=a$, or $5^{x+1}=b$ where $a>0$, or $b>0$	**M1**
Obtain answers $x=0.179$ and $x=-0.139$	**A1**
Alternative method for question 1	
Obtain $5^x=\frac{4}{3}$ by solving an equation	**B1**
Obtain $5^x=\frac{4}{5}$ (or $5^{x+1}=4$) by solving an equation	**B1**
Use correct method for solving an equation of the form $5^x=a$, or $5^{x+1}=b$ where $a>0$, or $b>0$	**M1**
Obtain answers $x=0.179$ and $x=-0.139$	**A1**
	4

22) OCT 2021_9709_32 Q1

Use law of the logarithm of a product, a quotient or power	*M1	e.g. $\ln\left(7^x\right)=x\ln 7$
Obtain a correct linear equation in any form	A1	e.g. $\ln 3+(1-x)\ln 2=x\ln 7$
Solve a linear equation for x	DM1	
Obtain answer $x=\frac{\ln 6}{\ln 14}$	A1	Maximum 3 out of 4 available if final answer not in required form e.g. 0.67… ISW once correct answer seen.
Alternative method for Question 1		
$2^{1-x}=2\times 2^{-x}$	*M1	OE
$6=2^x7^x\left[=14^x\right]$	A1	
Use law of the logarithm of a power to solve for x	DM1	Must be a linear power. Allow $x=\ln_{14}(6)$.
Obtain answer $x=\frac{\ln 6}{\ln 14}$	A1	ISW once correct answer seen.

23) OCT 2021_9709_33 Q3

Use laws of indices correctly and solve for 4^x	M1
Obtain correct solution in any form, e.g. $4^x=\frac{256}{15}$	A1
Use a correct method for solving an equation of the form $4^x=a$, where $a>0$	M1
Obtain answer 2.047	A1
	4

24) OCT 2022_9709_31 Q3

Use law of logarithm of a product or power	M1
Obtain a correct linear equation in any form, e.g. $(3x-1)\ln 2=\ln 5-x\ln 3$	A1
Solve for x	M1
Obtain answer $x=\frac{\ln 10}{\ln 24}$	A1
Alternative method for question 3	
Use laws of indices to split at least one exponential term	M1
Obtain $24^x=10$	A1
Solve for x	M1
Obtain answer $x=\frac{\ln 10}{\ln 24}$	A1

25) OCT 2022_9709_32 Q1

Use law of the logarithm of a power or product	**M1**
Obtain a correct linear equation in any form, e.g. $(3x-1)\ln 2 = \ln 5 + (1-x)\ln 3$	**A1**
Solve for x	**M1**
Obtain **final** answer $x = \dfrac{\ln 30}{\ln 24}$	**A1**
Alternative method for question 1	
Use laws of indices to split at least one exponential term	**M1**
Obtain $24^x = 30$	**A1**
Solve for x	**M1**
Obtain **final** answer $x = \dfrac{\ln 30}{\ln 24}$	**A1**
	4

26) OCT 2022_9709_33 Q1

Use law for the logarithm of a product, quotient or power	**M1**
Remove logarithms and state a correct equation, e.g. $x(2x-1)=(x+1)^2$	**A1**
Solve a 3-term quadratic obtaining at least one root	**M1**
Obtain answer 3.303 only	**A1**
	4