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Pure Maths-3

Exponential and Logarithmic Functions Notes

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Exponential Function

§ Exponential function:

$$f(x) = b^x \quad , \quad b \neq 1, b > 0, x \in R$$

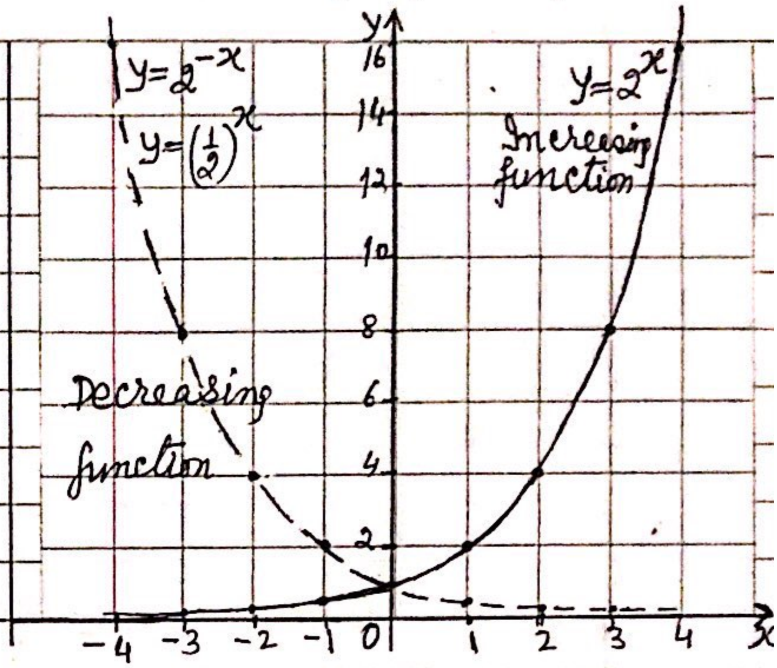
Case I. when $b > 1$

Example: $f(x) = 2^x$

Case II. $0 < b < 1$

Example: $f(x) = \left(\frac{1}{2}\right)^x$

or $= 2^{-x}$



§ Properties of Exponents:

(i) $b^x \cdot b^y = b^{x+y}$

(ii) $\frac{b^x}{b^y} = b^{x-y}$

(iii) $(b^x)^y = b^{x \cdot y}$

(iv) $b^0 = 1$

(v) $b^{-x} = \frac{1}{b^x}$

(vi) $b^{\frac{1}{n}} = \sqrt[n]{b}$

(vii) $b^{m/n} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$

$(\text{base})^{\text{Index}} = \text{Number}$

Example: $2^5 = 32$

$$\begin{cases} \text{base} = 2 \\ \text{Index} = 5 \\ \text{Value (Number)} = 32 \end{cases}$$

§ Special base: e (Natural base) $= 2.7182\cdots$

Note:

(i) $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

(ii) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \cdots$

Exponential and Logarithmic Functions

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§ **Logarithmic Function:**

$b \neq 1,\ b > 0,\ x > 0$

$\log_b x = n \Leftrightarrow b^n = x$

(i) $2^5 = 32 \Leftrightarrow \log_2 32 = 5$

(ii) $3^4 = 81 \Leftrightarrow \log_3 81 = 4$

(iii) $\log_{10} 100 = 2 \Leftrightarrow 10^2 = 100.$

§ **Laws of Logariths:**

$b > 0,\ b \neq 1$

1. $\log_b (p \cdot q) = \log_b p + \log_b q$
2. $\log_b \left(\frac{p}{q}\right) = \log_b p - \log_b q$
3. $\log_b (p)^q = q \cdot \log_b p$
4. $\log_b b = 1$
5. $\log_b 1 = 0$
6. $\log_b p = \dfrac{\log_a p}{\log_a b}$ (Change of base from b to a)
7. $\log_a b = \dfrac{1}{\log_b a}$
8. $\log_b \sqrt[n]{x} = \frac{1}{n} \log_b x$
9. (i) $\ln e^x = x$
   (ii) $e^{\ln x} = x$
10. $\ln e = 1.$
11. $\log_b x < 0$ for $0 < x < 1$ and $b > 1$

Log to natural base e.

$\ln x = \log_e x$ is called log to natural base e

$\log_{10} x = \dfrac{\ln x}{\ln 10} = 0.434 . \ln x$

Example 1: Using the substitution $u = 3^x$, solve the equation: $3^x + 3^{2x} = 3^{3x}$ --[5]

[SP-17/03/Q2]

Solution: $3^x + 3^{2x} = 3^{3x}$

$\Rightarrow 3^x + (3^x)^2 = (3^x)^3$

$\Rightarrow u + u^2 = u^3$ [$\because 3^x = u$]

$u^3 - u^2 - u = 0$

$u[u^2 - u - 1] = 0$

$u = 0^{\times}$, $u^2 - u - 1 = 0$

$\Rightarrow u = \frac{1+\sqrt{5}}{2}$; $u = \left(\frac{1-\sqrt{5}}{2}\right)^{\times} < 0$

$\Rightarrow u = 1.618$

$u = 1.618$

$\Rightarrow 3^x = 1.618$

$\Rightarrow \ln 3^x = \ln 1.618$

$x \cdot \ln 3 = \ln 1.618$

$x = \frac{\ln 1.618}{\ln 3}$

$\Rightarrow x = 0.43799$

$\Rightarrow \underline{x = 0.438}$ ✓

Example 2: $\log_{10}(x+9) = 2 + \log_{10} x$ ---[3]

[S-14/33/Q1]

Solution: $\log_{10}(x+9) = 2 + \log_{10} x \Rightarrow \log(x+9) - \log x = 2$

$\Rightarrow \log_{10} \frac{(x+9)}{x} = 2$

$\Rightarrow \frac{x+9}{x} = 10^2 = 100$

$\Rightarrow 100x = x+9 \Rightarrow x = \frac{1}{11}$ ✓

Example 3: Solve the equation: $2\ln(5 - e^{-2x}) = 1$ ---[4]

[S-14/32/Q2]

Solution: $2\ln(5 - e^{-2x}) = 1$

$\Rightarrow \ln(5 - e^{-2x}) = \frac{1}{2}$

$\Rightarrow 5 - e^{-2x} = e^{1/2}$

$\Rightarrow e^{-2x} = 5 - e^{1/2}$

$\Rightarrow e^{-2x} = 3.3513$

$\Rightarrow -2x = \ln(3.3513)$

[$\because e^x = y \Rightarrow x = \ln y$]

$\Rightarrow x = -\frac{1}{2} \times 1.2093$

$\Rightarrow x = -\underline{0.605}$ ✓

Example 4: Solve the equation: $\ln(x^4-4) = 4\ln x - \ln 4$ ---[4]
Giving your answer to 2 decimal places. [S-18/31/Q1]

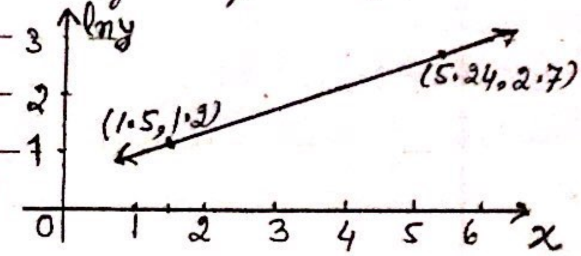
Solution: $\ln(x^4-4) = 4\ln x - \ln 4$
$\Rightarrow \ln(x^4-4) = \ln x^4 - \ln 4$
$\Rightarrow \ln(x^4-4) = \ln \frac{x^4}{4}$
$\Rightarrow x^4 - 4 = \frac{x^4}{4}$

$\frac{3}{4}x^4 = 4$
$\Rightarrow x^4 = \frac{16}{3}$
$x = \sqrt[4]{5.333}$
$= 1.519$
$\Rightarrow x = 1.52$ ✓

Example 5: The variables x and y satisfy the equation $y^n = Ax^3$, where n and A are constants. It is given that $y = 2.58$ when $x = 1.20$ and $y = 9.49$ when $x = 2.51$
(i) Explain why the graph of $\ln y$ against $\ln x$ is a straight line. ---[2]
(ii) Find the values of n and A, giving your answer correct to 2 decimal places. --[4]
[M-18/32/Q4]

Solution: Given $y^n = Ax^3$
(i) $\Rightarrow \ln y^n = \ln Ax^3$
$\Rightarrow n \cdot \ln y = \ln A + 3\ln x$ ---- (1)
It is a linear equation in $\ln y$ and $\ln x$; hence the graph of $\ln y$ against $\ln x$ will be a straight line.
(ii) Now $y = 2.58$ when $x = 1.20$,
$\therefore$ from (1) $n \times \ln 2.58 = \ln A + 3 \times \ln 1.20$
$\Rightarrow 0.9477\, n = \ln A + 0.5469$ -- (2)
and $y = 9.49$ when $x = 2.51$
$\therefore$ from (1) $n \ln 9.49 = \ln A + 3\ln 2.51$
$\Rightarrow 2.25n = \ln A + 2.76$ -- (3)
Subtract (1) from (2)
$1.3022\, n = 2.2131 \Rightarrow n = \frac{2.2131}{1.3022} = 1.699 = 1.70 \Rightarrow \underline{n = 1.70}$ ✓
Now put $n = 1.70$ in (2)
$\Rightarrow \ln A = 0.9477 \times 1.70 - 0.5469 = 1.0641$
$\Rightarrow A = e^{1.0641} = 2.898 \Rightarrow \underline{A = 2.90}$ ✓

Example 6: The variables x and y satisfy the equation $y^2 = e^{kx}$, where A and k are constants.
The graph of $\ln y$ against x is a straight line passing through the points $(1.5, 1.2)$ and $(5.24, 2.7)$ as shown in the diagram.



Find the values of A and k correct to two decimal places. --[5]

[S-20/32/Q2]

Solution: $y^2 = Ae^{kx}$

$\ln y^2 = \ln Ae^{kx}$

$\Rightarrow 2\ln y = \ln A + kx$ --- ①

Given line ① passes through $(1.5, 1.2)$ $(x, \ln y)$

$\therefore$ from ① $2 \times 1.2 = \ln A + k \times 1.5$

$\Rightarrow \ln A + 1.5k = 2.4$ --- ②

Also $(5.24, 2.7)$ lies on ①

$2 \times 2.7 = \ln A + k \times 5.24$

$\Rightarrow \ln A + 5.24k = 5.4$ --- ③

Subtract ② from ③

$3.74k = 3$

$\Rightarrow k = \frac{3}{3.74} = 0.80$ ✓

Put $k = 0.80$ in ②

$\ln A + 1.5 \times 0.8 = 2.4$

$\Rightarrow \ln A = 2.4 - 1.2 = 1.2$

$\Rightarrow A = e^{1.2} = 3.32$ ✓

$\therefore$ $\underline{A = 3.32}$ ✓ and $\underline{k = 0.80}$ ✓

Example 7 (i) Show that the equation $\log_{10}(x-4) = 2 - \log_{10} x$ can be written as a quadratic in x. --[3]

(ii) Hence solve the equation $\log_{10}(x-4) = 2 - \log_{10} x$, giving your answer correct to 3 s.f. --[2]

[M-19/32/Q1]

Solution: $\log_{10}(x-4) = 2 - \log_{10} x$

(i) $\Rightarrow \log_{10}(x-4) = \log_{10} 100 - \log_{10} x$

$\Rightarrow \log_{10}(x-4) = \log_{10} \frac{100}{x}$

$\Rightarrow x - 4 = \frac{100}{x}$

$\Rightarrow x^2 - 4x = 100$

$\Rightarrow x^2 - 4x - 100 = 0$ ✓

(ii) $x^2 - 4x - 100 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Using Quadratic formula
$b^2 - 4ac$
$= 16 + 400$
$= 416$

$= \frac{4 \pm \sqrt{416}}{2}$

$= 2 \pm 10.2$

$= 12.2$ ✓ or -8.2 ✗ $\because \log_{10}(-8.2)$ is not defined.

$\therefore$ $\underline{x = 12.2}$ ✓

Example 8: The variables x and y satisfy the relation $3^y = 4^{2-x}$,

(i) By taking logarithms, show that the graph of y against x is a straight line. State the exact value of the gradient of this line. --- [3]

(ii) Calculate the exact x-coordinate of the point of intersection of this line with line with equation $y = 2x$, simplify your answer. -- [2]

[S-16/33/Q2]

Solution: $3^y = 4^{2-x}$

$\Rightarrow \ln 3^y = \ln 4^{2-x}$

$\Rightarrow y.\ln 3 = (2-x)\ln 4$

$\Rightarrow y = -\frac{\ln 4}{\ln 3}\cdot x + 2\frac{\ln 4}{\ln 3}$ -- ①

which is of the form $y = mx + c$

∴ Eqn ① represent a straight line in y and x;

Gradient of line ① is $-\frac{\ln 4}{\ln 3}$ ✓

Now given another line $y = 2x$ —— ②

Put $y = 2x$ in ①

$2x = -\frac{\ln 4}{\ln 3}.x + 2\frac{\ln 4}{\ln 3}$

$\Rightarrow x\left[2 + \frac{\ln 4}{\ln 3}\right] = 2\frac{\ln 4}{\ln 3}$

$\Rightarrow x\left[\frac{2\ln 3 + 2\ln 2}{\ln 3}\right] = \frac{2\ln 4}{\ln 3}$

$\Rightarrow \frac{2\ln 6}{\ln 3}x = 2\frac{\ln 4}{\ln 3} \Rightarrow x = \frac{\ln 4}{\ln 6}$ ✓

Example 9: Solve: $\log_x 36 + \log_x 4 = 2$

$\Rightarrow \log_x 36 \times 4 = 2 \Rightarrow 144 = x^2 \Rightarrow x = \pm 12$

$\therefore x = 12$ ✓, -12 ✗ (∵ log does not exists for -ve base)

$\therefore x = 12$ ✓

Example 10: Solve: $0.8^x < 0.3$

Solution: $0.8^x < 0.3$

$\ln 0.8^x < \ln 0.3$

$\Rightarrow x \cdot \ln 0.8 < \ln 0.3$

(※) $\Rightarrow x > \frac{\ln 0.3}{\ln 0.8}$ [∵ $\ln 0.8 < 0$]

$x > \frac{-1.2039}{-0.2231} = 5.395$

$\therefore x > 5.4$ ✓

Example 11: Solve the equation: $2\log_2 x = 3 + \log_2(x+1)$ ---[5]
giving your answer correct to 3 s.f.
[W-17/32/Q2]

Solution: $2\log_2 x = 3 + \log_2(x+1)$

$\Rightarrow \log_2 x^2 - \log_2(x+1) = 3$

$\Rightarrow \log_2 \frac{x^2}{(x+1)} = 3 \Rightarrow \frac{x^2}{x+1} = 2^3$

$\Rightarrow x^2 = 8(x+1)$

$x^2 - 8x - 8 = 0$

$x = \frac{+8 \pm \sqrt{96}}{2}$

$b^2 - 4ac = 64 - 4\times1(-8) = 96$

$= \frac{+8 \pm 2\sqrt{24}}{2} = +4 \pm \sqrt{24}$

$x = 4 \pm 4.90$

$x = \underline{8.90}$ ✓ ; $x = -0.90$ ✗ [$\because \log_2 x$ is not defined for $x < 0$]

Example 12: Solve the equation: $\frac{e^x + e^{-x}}{e^x + 1} = 4$ ---[5]
giving your answer to 3 decimal places.
[W-18/32/Q4]

Solution: $\frac{e^x + e^{-x}}{e^x + 1} = 4$ Let $e^x = u$, $e^{-x} = \frac{1}{u}$

$\Rightarrow \frac{u + \frac{1}{u}}{u+1} = 4$

$\Rightarrow u^2 + 1 = 4u(u+1)$

$\Rightarrow 3u^2 + 4u - 1 = 0$

$u = \frac{-4 \pm \sqrt{28}}{6}$

$= \frac{-4 \pm 5.2915}{6}$ ⇗

$u = \frac{1.2915}{6}$ or $\frac{-9.2915}{6}$

$\Rightarrow e^x = 0.2152$ or $e^x = -1.55$ ✗

$\Rightarrow x = \ln 0.2152$ [$\because e^x > 0$]

$x = \underline{-1.536}$ ✓