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Pure Maths-3

Integration
Notes-1

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Integration

§ Integration of exponential function:

1 (i) $\int e^{x}\,dx = e^{x} + c$ $\qquad \left(\frac{d}{dx}e^{x} = e^{x}\right)$

(ii) $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}$ $\qquad \left(\frac{d}{dx}e^{ax+b} = ae^{ax+b}\right)$

Example 1 (i) Find $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + c$ ✓

(ii) Find $\int 2e^{8x-3} = \frac{2}{8}e^{8x-3} = \frac{1}{4}e^{8x-3} + c$ ✓

2. Evaluate: $\int_0^1 (e^{x} + e^{2x})^2\,dx$

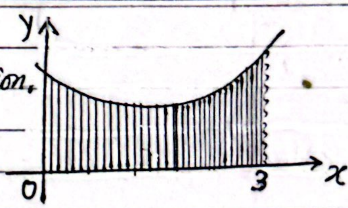
$= \int_0^1 [(e^{x})^2 + 2e^{x}.e^{2x} + (e^{2x})^2]\,dx = \int_0^1 (e^{2x} + 2e^{3x} + e^{4x})\,dx$

$= \left[\frac{e^{2x}}{2} + \frac{2e^{3x}}{3} + \frac{1}{4}e^{4x}\right]_0^1 = \left(\frac{1}{2}e^{2} + \frac{2}{3}e^{3} + \frac{1}{4}e^{4}\right) - \left(\frac{1}{2}e^{0} - \frac{2}{3}e^{0} - \frac{1}{4}e^{0}\right)$

$= \frac{1}{2}e^{2} + \frac{2}{3}e^{3} + \frac{1}{4}e^{4} - \frac{1}{2} - \frac{2}{3} - \frac{1}{4} = \frac{1}{12}(6e^{2} + 8e^{3} + 3e^{4} - 6 - 8 - 3)$

$= \frac{1}{12}(6e^{2} + 8e^{3} + 3e^{4} - 17)$ ✓

3 Given $y = 2e^{\frac{1}{2}x} - 2x + 3$
Find the exact area of the shaded region.



Solution:
Area of the shaded region $= \int_0^3 y\,dx$

$= \int_0^3 (2e^{\frac{1}{2}x} - 2x + 3)\,dx$

$= \left[\frac{2e^{\frac{1}{2}x}}{\frac{1}{2}} - \frac{\cancel{2}x^2}{\cancel{2}} + 3x\right]_0^3$

$= (4e^{3/2} - 9 + 9) - (4e^{0} - 0 + 0)$

$= 4e^{3/2} - 4$

$= 4(e^{3/2} - 1)$ ✓

§ Integration of Logarithmic function:

2 (i) $\int \frac{1}{x}\,dx = \ln x + c$, $x>0$ $\left(\frac{d}{dx}\ln x = \frac{1}{x};\ x>0\right)$

(ii) $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b)$, $ax+b>0$ $\left(\frac{d}{dx}\ln(ax+b) = \frac{a}{ax+b};\ ax+b>0\right)$

as $\ln x$ is defined for $x>0$

3 (i) $\int \frac{1}{x}\,dx = \ln|x| + c$; $x<0$

(ii) $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|$; $ax+b<0$.

Example 4: (i) Find $\int \frac{6}{2x-5}\,dx$

$= \frac{6}{2}\ln|2x-5| + c$

$= 3\ln|2x-5| + c$

(ii) Find $\int \frac{5}{2-3x}\,dx$

$= \frac{5}{-3}\ln|2-3x| + c$

$= -\frac{5}{3}\ln|2-3x| + c$

5. Evaluate: $\int_4^{10}\left(2+\frac{5}{3x-2}\right)dx$

$= \left[2x + \frac{5}{3}\ln|3x-2|\right]_4^{10}$

$= \left(20+\frac{5}{3}\ln 28\right) - \left(8+\frac{5}{3}\ln 10\right)$

$= 20-8+\frac{5}{3}(\ln 28 - \ln 10)$

$= 12 + \frac{5}{3}\ln\left(\frac{28}{10}\right) = 12 + \ln\left(\frac{14}{5}\right)$ ✓

6. A curve is such that: $\frac{dy}{dx} = 2x + \frac{3}{x+e}$
Given that the curve passes through the point (e, e^2),
Find the equation of the curve.

Solution: $\frac{dy}{dx} = 2x + \frac{3}{x+e}$

$\Rightarrow y = \int\left(2x+\frac{3}{x+e}\right)dx$

$\Rightarrow y = x^2 + 3\ln|x+e| + c$ -- (1)

given that curve passes through (e, e^2)

from (1): $e^2 = e^2 + 3\ln 2e + c \Rightarrow c = -3\ln 2e$

hence from (1) equation of the curve is: $y = x^2 + 3\ln|x+e) - 3\ln 2e$.

7. Find the exact value of $\int_4^{14}\left(2+\frac{6}{3x-2}\right)dx$, giving your answer in the form $\ln(ae^b)$, where a and b are integers. ...[5]

[S-16/21/Q7]

Solution: $\int_4^{14}\left(2+\frac{6}{(3x-2)}\right)dx$

$= \left[2x+\frac{6}{3}\ln|3x-2|\right]_4^{14}$

$= (28+2\ln 40)-(8+2\ln 10)$

$= 20+2(\ln 40-\ln 10)$

$= 20+2\ln\left(\frac{40}{10}\right)$

$= 20+2\ln 4$

$= \ln e^{20}+\ln 4^2$ $(\ln e^x = x)$

$= \ln(16\cdot e^{20})$ ✓

8. Find $\int 4e^x(3+e^{2x})dx$ ---[3]

[S-11/21/Q6]

Solution: $\int 4e^x(3+e^{2x})dx = \int(12e^x+4e^{3x})dx$

$= 12e^x+\frac{4}{3}e^{3x}+c$ ✓

9. (a) Find the value of the constant A such that: $\frac{6x}{(3x-2)} = 2+\frac{A}{3x-2}$

(b) Hence show that $\int_2^6 \frac{6x}{(3x-2)}dx = 8+\frac{8}{3}\ln 2$.

[S-09/P2/Q8]

Solution (a) $\frac{6x}{3x-2} = 2+\frac{A}{3x-2}$ $\Rightarrow A=4$

$$\begin{array}{r|l} 3x-2 \;\big)\; 6x & 2 \\ -\;6x \mp 4 & \\ \hline 4 & \end{array}$$

(b) $\int_2^6 \frac{6x}{(3x-2)}dx = \int_2^6\left(2+\frac{4}{3x-2}\right)dx$ (from Part (a))

$= \left[2x+\frac{4}{3}\ln|3x-2|\right]_2^6 = \left(12+\frac{4}{3}\ln 16\right)-\left(4+\frac{4}{3}\ln 4\right)$

$= 8+\frac{4}{3}(\ln 16-\ln 4) = 8+\frac{4}{3}\ln 4 = 8+\frac{4}{3}\times 2\ln 2 = 8+\frac{8}{3}\ln 2$ ✓

10. Find the exact value of the constant k for which: $\int_1^k \frac{1}{2x-1}dx = 1$

[W-07/03/Q1]

Solution: $\int_1^k \frac{1}{(2x-1)}dx = 1$

$\Rightarrow \left[\frac{1}{2}\ln|2x-1|\right]_1^k = 1$

$\Rightarrow (\ln|2k-1|-\ln 1) = 2$

$\Rightarrow \ln(2k-1)-0 = 2$

$\Rightarrow 2k-1 = e^2 \Rightarrow k = \frac{1}{2}(e^2+1)$ ✓

§ Integration of trigonometric functions:

4. (i) $\int \cos x\,dx = \sin x + C$ $\quad \left(\frac{d}{dx}\sin x = \cos x\right)$

(ii) $\int \sin x\,dx = -\cos x + C$ $\quad \left(\frac{d}{dx}\cos x = -\sin x\right)$

(iii) $\int \sec^2 x\,dx = \tan x + C$ $\quad \left(\frac{d}{dx}\tan x = \sec^2 x\right)$

(iv) $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$ $\quad \left(\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x\right)$

(v) $\int \sec x \tan x\,dx = \sec x + C$ $\quad \left(\frac{d}{dx}\sec x = \sec x \cdot \tan x\right)$

(vi) $\int \operatorname{cosec} x \cdot \cot x\,dx = -\operatorname{cosec} x + C$ $\quad \left(\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x\right)$

5. (i) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$ $\quad \left(\frac{d}{dx}\sin(ax+b) = a \cdot \cos(ax+b)\right)$

(ii) $\int \sin(ax+b)\,dx = \frac{-1}{a}\cos(ax+b) + C$ $\quad \left(\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)\right)$

(iii) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$ $\quad \left(\frac{d}{dx}\tan(ax+b) = a\sec^2(ax+b)\right)$

6. (i) $\int \tan x\,dx = -\ln\cos x + C$ (or $\ln \sec x$) $\quad \left(\frac{d}{dx}\ln\sec x = \tan x\right)$

(ii) $\int \cot x\,dx = \ln\sin x + C$ $\quad \left(\frac{d}{dx}\ln\sin x = \cot x\right)$

Application of Trigonometric identities in integration of trig functions:

1. (i) $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ $\quad \because \cos 2x = 1 - 2\sin^2 x \Rightarrow 2\sin^2 x = 1 - \cos 2x$

(ii) $\sin^2 \frac{x}{2} = \frac{1}{2}(1-\cos x)$ $\quad \because \cos x = 1 - 2\sin^2 \frac{x}{2} \Rightarrow 2\sin^2\frac{x}{2} = 1 - \cos x$

2. (i) $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ $\quad \cos 2x = 2\cos^2 x - 1 \Rightarrow 2\cos^2 x = 1 + \cos 2x$

(ii) $\cos^2 \frac{x}{2} = \frac{1}{2}(1+\cos x)$ $\quad \cos x = 2\cos^2\frac{x}{2} - 1 \Rightarrow 2\cos^2\frac{x}{2} = 1 + \cos x$

3. (i) $\tan^2 x = \sec^2 x - 1$ $\quad 1 + \tan^2 x = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

(ii) $\cot^2 x = \operatorname{cosec}^2 x - 1$ $\quad 1 + \cot^2 x = \operatorname{cosec}^2 x \Rightarrow \cot^2 x = \operatorname{cosec}^2 x - 1$

4. (i) $\sin^4 x = (\sin^2 x)^2 = \left[\frac{1}{2}(1-\cos 2x)\right]^2 = \frac{1}{4}\left[1 - 2\cos 2x + \cos^2 2x\right]$
$= \frac{1}{4}\left[1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right]$

(ii) $\cos^4 x \rightarrow$ similer steps.

5. (i) $\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$ $\quad (\sin 3x = 3\sin x - 4\sin^3 x)$

(ii) $\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$ $\quad (\cos 3x = 4\cos^3 x - 3\cos x)$

6. $\sin x \cdot \cos x = \frac{1}{2}\sin 2x$ $\quad (\sin 2x = 2\sin x\cos x)$

7. (i) $a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$ $\quad$ Put $a = R\cos\alpha$, $b = R\sin\alpha$

(ii) $a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$ $\quad R = \sqrt{a^2+b^2}$; $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

11. (i) Find $\int(\sin x - 2\cos x)\,dx$

$= -\cos x - 2\sin x + C$ ✓

(ii) Find $\int(3\sec^2 3x - \sin 2x)\,dx$

$= 3\,\frac{\tan 3x}{3} + \frac{\cos 2x}{2} + C$

$= \tan 3x + \frac{1}{2}\cos 2x + C$ ✓

(iii) Find the exact value of: $\int_0^{\frac{2\pi}{3}} \sin 2x\,dx$

$= \left[-\frac{\cos 2x}{2}\right]_0^{2\pi/3} = -\frac{1}{2}\left[\cos\frac{4\pi}{3} - \cos 0\right]$

$= -\frac{1}{2}\left[-\frac{1}{2} - 1\right] = -\frac{1}{2}\times -\frac{3}{2} = \frac{3}{4}$ ✓

(iv) Evaluate: $\int_{\pi/8}^{\pi/6} \sec^2 2x\,dx$

$= \frac{1}{2}\left[\tan 2x\right]_{\pi/8}^{\pi/6} = \frac{1}{2}\left[\tan\frac{\pi}{3} - \tan\frac{\pi}{4}\right] = \frac{1}{2}(\sqrt{3}-1)$ ✓

12. (i) Prove the identity. $(\cos x + 3\sin x)^2 = 5 - 4\cos 2x + 3\sin 2x$

(ii) Find the exact value of: $\int_0^{\frac{\pi}{4}} (\cos x + 3\sin x)^2\,dx$

W-07/02/Q7

Solution (i) $(\cos x + 3\sin x)^2$

$= \cos^2 x + 9\sin^2 x + 6\sin x\cos x$

$= (\cos^2 x + \sin^2 x) + 4\times 2\sin^2 x + 3\times 2\sin x\cos x$

$= 1 + 4(1-\cos 2x) + 3\sin 2x$

$= 5 - 4\cos 2x + 3\sin 2x$ ✓

(ii) $\int_0^{\pi/4} (\cos x + 3\sin x)^2\,dx$ (using Part (a)

$= \int_0^{\frac{\pi}{4}} (5 - 4\cos 2x + 3\sin 2x)\,dx$

$= \left[5x - \frac{4\sin 2x}{2} - \frac{3}{2}\cos 2x\right]_0^{\pi/4}$

$= \left(\frac{5\pi}{4} - 2\sin\frac{\pi}{2} - \frac{3}{2}\cos\frac{\pi}{2}\right) - \left(0 - 0 - \frac{3}{2}\times 1\right)$

$= \frac{5\pi}{4} - 2\times 1 - 0 + 3/2 = \frac{1}{4}(5\pi - 2)$ ✓

13. (i) By expanding $\sin(2x+x)$ and using double angle formula, show that: $\sin 3x = 3\sin x - 4\sin^3 x$

(ii) Hence show that: $\int_0^{\pi/3} \sin^3 x = \frac{5}{24}$

S-05/02/Q7

Solution (i) $\sin 3x = \sin(2x + x)$

$= \sin 2x\cdot\cos x + \cos 2x\cdot\sin x$

$= 2\sin x\cos x\cdot\cos x + \sin x(1 - 2\sin^2 x)$

$= 2\sin x\cdot\cos^2 x + \sin x(1 - 2\sin^2 x)$

$= 2\sin x(1 - \sin^2 x) + \sin x(1 - 2\sin^2 x)$

$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$. ✓ ---(1)

(ii) from (1)

$\sin 3x = 3\sin x - 4\sin^3 x$

$\Rightarrow 4\sin^3 x = 3\sin x - \sin 3x$

$\Rightarrow \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$ --(2)

(ii) $\int_0^{\pi/3} \sin^3 x\,dx$

$= \int_0^{\pi/3} \frac{1}{4}(3\sin x - \sin 3x)\,dx$

$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right]_0^{\pi/3}$

$= \frac{1}{4}\left[\left(-3\cos\frac{\pi}{3} + \frac{1}{3}\cos\pi\right) - \left(-3\cos 0 + \frac{1}{3}\cos 0\right)\right]$

$= \frac{1}{4}\left[\left(-\frac{3}{2} + \frac{1}{3}(-1)\right) - \left(-3\times 1 + \frac{1}{3}\right)\right]$

$= \frac{1}{4}\left[-3/2 - 1/3 + 3 - 1/3\right]$

$= \frac{1}{4}\times\frac{5}{6} = \frac{5}{24}$ ✓

14. Find $\int \cos^4 x\,dx$

Solution: $\int \cos^4 x\,dx$ ------ (1)

$\left[\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)\right.$ for $\theta = 2x$ $\left.\cos^2 2x = \frac{1}{2}(1+\cos 4x)\right]$

Consider $\cos^4 x = (\cos^2 x)^2 = \left[\frac{1}{2}(1+\cos 2x)\right]^2$

$$= \frac{1}{4}\left[\cos^2 2x + 2\cos 2x + 1\right]$$
$$= \frac{1}{4}\left[\frac{1}{2}(1+\cos 4x) + 2\cos 2x + 1\right]$$
$$= \frac{1}{4}\left[\frac{1}{2} + \frac{\cos 4x}{2} + 2\cos 2x + 1\right]$$
$$= \frac{1}{4}\left[\frac{1}{2}\cos 4x + 2\cos 2x + \frac{3}{2}\right]$$
$$= \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8} \quad \text{--- (2)}$$

Hence. $\int \cos^4 x\,dx = \int\left(\frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}\right)dx$ from (2)

$$= \frac{1}{8}\times\frac{1}{4}\sin 4x + \frac{1}{2}\times\frac{1}{2}\sin 2x + \frac{3}{8}x + C$$
$$= \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3}{8}x + C$$

15. (i) Using the expansion of $\cos(3x+x)$ and $\cos(3x-x)$, Show that: $\frac{1}{2}(\cos 4x + \cos 2x) = \cos 3x\cdot\cos x$ --- [3]

(ii) Hence show that: $\int_{-\frac{1}{6}\pi}^{\frac{1}{6}\pi} \cos 3x\cdot\cos x\,dx = \frac{3}{8}\sqrt{3}$. --- [3]

M-18/32/Q3

Solution (i) $\cos(3x+x) = \cos 3x\cos x - \sin 3x\sin x$ --- (1)

$\cos(3x-x) = \cos 3x\cos x + \sin 3x\sin x$ --- (2)

add (1) and (2) $\Rightarrow \cos(3x+x) + \cos(3x-x) = 2\cos 3x\cdot\cos x$

$\Rightarrow \frac{1}{2}(\cos 4x + \cos 2x) = \cos 3x\cos x$ --- (3)

(ii) $\int_{-\frac{1}{6}\pi}^{\frac{1}{6}\pi} \cos 3x\cdot\cos x\,dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(\cos 4x + \cos 2x)\,dx$ (from (3))

$$= \frac{1}{2}\left[\frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right]_{-\pi/6}^{\pi/6}$$
$$= \frac{1}{8}\left[\left(\sin\frac{2\pi}{3} + 2\sin\frac{\pi}{3}\right) - \left(\sin\left(-\frac{2\pi}{3}\right) + 2\sin\left(-\frac{\pi}{3}\right)\right)\right]$$
$$= \frac{1}{8}\left[\left(\frac{\sqrt{3}}{2} + 2\cdot\frac{\sqrt{3}}{2}\right) - \left(-\frac{\sqrt{3}}{2} - 2\frac{\sqrt{3}}{2}\right)\right] = \frac{1}{8}(3\sqrt{3}) = \frac{3\sqrt{3}}{8}$$

16.(i) Express $\cos\theta + 2\sin\theta$ in the form $R\cos(\theta-\alpha)$, where $R>0$, and $0<\alpha<\frac{1}{2}\pi$, Give the exact value of R and α. ---[3]

(ii) Hence Show that: $\int_0^{\pi/4} \frac{15}{(\cos\theta+2\sin\theta)^2}\,d\theta = 5$ --[5]

S-18/33/Q7

Solution (i) $\cos\theta + 2\sin\theta$

$= R\cos\alpha\cdot\cos\theta + R\sin\alpha\cdot\sin\theta$

$= R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$= R\cos(\theta-\alpha)$: $R=\sqrt{5}$ and $\tan\alpha = 2$. ---(1)

Put $R\cos\alpha = 1 \Rightarrow R^2 = 1^2+2^2$
$R\sin\alpha = 2 \Rightarrow R=\sqrt{5}$
and $\tan\alpha = \frac{2}{1} \Rightarrow \alpha = \tan^{-1}2$

(ii) $\int_0^{\pi/4} \frac{15}{(\cos\theta+2\sin\theta)^2}\,d\theta$

$= 15\int_0^{\pi/4} \frac{1}{(\sqrt{5}\cos(\theta-\alpha))^2}\,d\theta$

[from (1) $\cos\theta + 2\sin\theta = \sqrt{5}\cos(\theta-\alpha)$

$= 15\int_0^{\pi/4} \frac{1}{5}\cdot\frac{1}{\cos^2(\theta-\alpha)}\,d\theta = 3\int_0^{\pi/4}\sec^2(\theta-\alpha)\,d\theta$

$= 3\left[\tan(\theta-\alpha)\right]_0^{\pi/4} = 3\left[\tan\left(\frac{\pi}{4}-\alpha\right) - \tan(-\alpha)\right]$

$= 3\left[\frac{\tan\frac{\pi}{4} - \tan\alpha}{1+\tan\frac{\pi}{4}\cdot\tan\alpha} + \tan\alpha\right] = 3\left[\frac{1-2}{1+1\times 2} + 2\right]$ ($\because \tan\alpha = 2$)

$= 3\left[-\frac{1}{3}+2\right] = 3\times\frac{5}{3} = 5$ ✓

17(i) Show that $12\sin^2 x\cos^2 x = \frac{3}{2}(1-\cos 4x)$

(ii) Hence show that: $\int_{\pi/4}^{\pi/3} 12\sin^2 x\cos^2 x\,dx = \frac{\pi}{8} + \frac{3\sqrt{3}}{16}$

S-13/21/Q3

Solution (i) L.H.S; $12\sin^2 x\cdot\cos^2 x$

$= 3\times(2\sin x\cos x)^2$

$= 3\cdot\sin^2 2x$

$= 3\times\left(\frac{1-\cos 2\times 2x}{2}\right)$

$= \frac{3}{2}(1-\cos 4x)$

$=$ R.H.S

$(1-\cos 2\theta = 2\sin^2\theta)$

(ii) $\int_{\pi/4}^{\pi/3} 12\sin^2 x\cos^2 x\,dx$

$= \int_{\pi/4}^{\pi/3}\frac{3}{2}(1-\cos 4x)\,dx$ (from Part (i))

$= \frac{3}{2}\left[x - \frac{\sin 4x}{4}\right]_{\pi/4}^{\pi/3}$

$= \frac{3}{2}\left[\left(\frac{\pi}{3} - \frac{\sin 4\pi/3}{4}\right) - \left(\frac{\pi}{4} - \frac{\sin\pi}{4}\right)\right]$

$= \frac{3}{2}\left[\frac{\pi}{3} - \frac{\pi}{4} - \frac{1}{4}\times\left(-\frac{\sqrt{3}}{2}\right) + 0\right]$

$= \frac{3}{2}\left(\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right) = \frac{\pi}{8} + \frac{3\sqrt{3}}{16}$ ✓

18 Find. $\int \frac{1+\cos^4 2x}{\cos^2 2x}\,dx$

$\boxed{S-16/21/Q7}$

Solution: $\int \frac{1+\cos^4 2x}{\cos^2 2x}\,dx = \int (\sec^2 2x + \cos^2 2x)\,dx$

$= \int \left(\sec^2 2x + \frac{1+\cos 4x}{2}\right)dx \quad \left(\cos^2\theta = \frac{1+\cos 2\theta}{2}\right)$

$= \frac{1}{2}\tan 2x + \frac{1}{2}x + \frac{1}{2}\times\frac{1}{4}\sin 4x + C$

$= \frac{1}{2}\tan 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x + c$ ✓

19.(i) Prove that if $y = \frac{1}{\cos\theta}$, then $\frac{dy}{d\theta} = \sec\theta\cdot\tan\theta$ ---[2]

(ii) Prove the identity: $\frac{1+\sin\theta}{1-\sin\theta} = 2\sec^2\theta + 2\sec\theta\tan\theta - 1$ --[3]

(iii) Hence find the exact value of: $\int_0^{\pi/4} \frac{1+\sin\theta}{1-\sin\theta}\,d\theta$ ---[4]

$\boxed{S-17/32/Q7}$

Solution(i) $y = \frac{1}{\cos\theta} \Rightarrow \frac{dy}{d\theta} = \frac{\cos\theta\cdot\frac{d}{d\theta}1 - 1\times\frac{d}{d\theta}\cos\theta}{\cos^2\theta}$

$= \frac{0-(-\sin\theta)}{\cos^2\theta} = \frac{1}{\cos\theta}\times\frac{\sin\theta}{\cos\theta} = \sec\theta\cdot\tan\theta$

$\Rightarrow \frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta$

(ii) L.H.S $\frac{1+\sin\theta}{1-\sin\theta}\times\frac{1+\sin\theta}{1+\sin\theta}$

$= \frac{(1+\sin\theta)^2}{1-\sin^2\theta}$

$= \frac{1+2\sin\theta+\sin^2\theta}{\cos^2\theta}$

$= \frac{1}{\cos^2\theta} + 2\frac{\sin\theta}{\cos\theta}\times\frac{1}{\cos\theta} + \frac{\sin^2\theta}{\cos^2\theta}$

$= \sec^2\theta + 2\tan\theta\sec\theta + \tan^2\theta$

$= \sec^2\theta + 2\sec\theta\tan\theta + \sec^2\theta - 1$

$= 2\sec^2\theta + 2\sec\theta\tan\theta - 1.$

(iii) $\int_0^{\pi/4} \frac{1+\sin\theta}{1-\sin\theta}\,d\theta$

$= \int_0^{\pi/4} (2\sec^2\theta + 2\sec\theta\tan\theta - 1)\,d\theta$ (Part (ii))

$= \left[2\tan\theta + 2\sec\theta - \theta\right]_0^{\pi/4}$

$= \left[\left(2\tan\frac{\pi}{4} + 2\sec\frac{\pi}{4} - \frac{\pi}{4}\right) - (2\tan 0 + 2\sec 0 - 0)\right]$

$= \left(2\times 1 + 2\cdot\sqrt{2} - \pi/4\right) - (0 + 2\times 1 - 0)$

$= 2 + 2\sqrt{2} - \frac{\pi}{4} - 2$

$= 2\sqrt{2} - \frac{1}{4}\pi$ ✓

20. (i) Prove the identity: $\tan 2\theta - \tan\theta = \tan\theta \cdot \sec 2\theta$ ---[4]

(ii) Hence show that $\int_0^{\frac{1}{6}\pi} \tan\theta \cdot \sec 2\theta \, d\theta = \frac{1}{2}\ln\frac{3}{2}$ ---[4]

W-16/31/Q5

Solution (i) L.H.S. $\tan 2\theta - \tan\theta$

$$= \frac{\sin 2\theta}{\cos 2\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sin 2\theta \cos\theta - \cos 2\theta \cdot \sin\theta}{\cos\theta \cdot \cos 2\theta}$$

$$= \frac{\sin(2\theta - \theta)}{\cos\theta \cdot \cos 2\theta}$$

$$= \frac{\sin\theta}{\cos\theta} \times \frac{1}{\cos 2\theta} = \underline{\tan\theta \cdot \sec 2\theta}$$

(ii) $\int_0^{\frac{\pi}{6}} \tan\theta \cdot \sec 2\theta \, d\theta$

$$= \int_0^{\frac{\pi}{6}} (\tan 2\theta - \tan\theta)\, d\theta \quad \text{(from Part (i))}$$

$$= \left[\frac{\ln \sec 2\theta}{2} - \ln \sec\theta\right]_0^{\pi/6}$$

$$= \frac{1}{2}\left[\ln \sec 2\theta - 2\ln \sec\theta\right]_0^{\pi/6}$$

$$= \frac{1}{2}\left[\left(\ln \sec\frac{\pi}{3} - 2\ln \sec\frac{\pi}{6}\right) - \left(\ln \sec 0 - 2\ln \sec 0\right)\right]$$

$$= \frac{1}{2}\left[\left(\ln 2 - 2\ln\frac{2}{\sqrt{3}}\right) - (0)\right]$$

$$= \frac{1}{2}\left[\ln 2 - \ln\frac{4}{3}\right] = \frac{1}{2}\ln 2 \times \frac{3}{4} = \underline{\frac{1}{2}\ln\frac{3}{2}}$$

21(a) Prove that: $\frac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$ ---[2]

(b) Hence find the exact value of: $\int_{\pi/6}^{\pi/3} \frac{1-\cos 2\theta}{1+\cos 2\theta}\, d\theta$ ---[4]

S-31/31/Q4

Solution (a) L.H.S: $\frac{1-\cos 2\theta}{1+\cos 2\theta}$

$$= \frac{\cancel{2}\sin^2\theta}{\cancel{2}\cos^2\theta} = \underline{\tan^2\theta} \text{ (R.H.S)}$$

(b) $\int_{\pi/6}^{\pi/3} \frac{1-\cos 2\theta}{1+\cos 2\theta}\, d\theta$

$$= \int_{\frac{\pi}{6}}^{\pi/3} \tan^2\theta \, d\theta \quad \text{(Part (a))}$$

$$= \int_{\pi/6}^{\pi/3} (\sec^2\theta - 1)\, d\theta = \left[\tan\theta - \theta\right]_{\pi/6}^{\pi/3}$$

$$= \left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(\tan\frac{\pi}{6} - \frac{\pi}{6}\right)$$

$$= \left(\sqrt{3} - \frac{\pi}{3}\right) - \left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right)$$

$$= \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) - \left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{2}{\sqrt{3}} - \frac{\pi}{6}$$

$$= \underline{\frac{2\sqrt{3}}{\sqrt{3}} - \frac{\pi}{6}}$$