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Pure Math - 3.

Further Calulus
Diff. and Integration
Notes

Content. Pages.

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§ (i) Differentiate: $\tan^{-1}x$ w.r.t x.

Let $y = \tan^{-1}x \Rightarrow x = \tan y$ ------ ①

Diff. w.r.t $x \Rightarrow 1 = \sec^2 y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \frac{1}{(1+x^2)}$

$\therefore \frac{d}{dx}\tan^{-1}x = \frac{1}{(1+x^2)}$ ✓

(ii) Find $\frac{dy}{dx}$: $y = \tan^{-1}\left(\frac{x}{a}\right)$

Diff. w.r.t x, $\frac{dy}{dx} = \frac{1}{1+\left(\frac{x}{a}\right)^2} \times \frac{1}{a} = \frac{1}{a} \times \frac{a^2}{a^2+x^2} = a \cdot \frac{1}{(a^2+x^2)}$

$\Rightarrow \frac{d}{dx}\tan^{-1}\frac{x}{a} = \frac{a}{a^2+x^2}$ ✓

Example 1. Show that $\frac{d}{dx}(x - \tan^{-1}x) = \frac{x^2}{1+x^2}$ --- [2]

SP-20/03/Q5(a)

Solution: $\frac{d}{dx}(x - \tan^{-1}x) = 1 - \frac{1}{1+x^2} = \frac{1+x^2-1}{1+x^2} = \frac{x^2}{(1+x^2)}$ ✓

Example 2: The equation of a curve is $y = x \cdot \tan^{-1}\left(\frac{1}{2}x\right)$ (a) Find $\frac{dy}{dx}$ -- [3]

(b) The tangent to the curve at the point where $x = 2$, meets y-axis at the coordinates $(0, p)$, find p. -- [3]

S-20/33/Q4

Solution: $y = x \cdot \tan^{-1}\left(\frac{1}{2}x\right)$ --- ①

(a) $\frac{dy}{dx} = 1 \times \tan^{-1}\left(\frac{x}{2}\right) + x \times \frac{1}{1+\left(\frac{x}{2}\right)^2} \times \frac{1}{2}$

$= \tan^{-1}x/2 + \frac{2x}{x^2+4}$ ✓ --②

(b) from ② $\left(\frac{dy}{dx}\right)_{x=2} = \tan^{-1}2/2 + \frac{2\times 2}{4+4}$

$= \tan^{-1}1 + \frac{1}{2} = \left(\frac{\pi}{4} + \frac{1}{2}\right)$ ✓

from ① at $x = 2$, $y = 2\tan^{-1}1 \Rightarrow y = \frac{2\times\pi}{4} = \frac{\pi}{2}$

$\therefore$ Equation of tangent to the curve at $\left(2, \frac{\pi}{2}\right)$ is; $y - \frac{\pi}{2} = \left(\frac{\pi}{4} + \frac{1}{2}\right)(x-2)$ --- ③

Tangent intersects y-axis at $x = 0$, put in ③

$\Rightarrow y - \frac{\pi}{2} = \left(\frac{\pi}{4} + \frac{1}{2}\right)(0-2)$

$\Rightarrow y - \frac{\pi}{2} = -\frac{\pi}{2} - 1 \Rightarrow y = -1$

$\therefore$ tangent intersect y-axis at $(0,-1) \equiv (0, p)$

$\Rightarrow p = -1$ ✓

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§ Integration:

(i) $$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C \qquad \left(\because \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}\right)$$

(ii) $$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \qquad \left(\because \frac{d}{dx}\tan^{-1}\frac{x}{a} = \frac{a}{a^2+x^2}\right)$$

Example 3(a) Find $\int \frac{1}{1+4x^2}\,dx$ (b) $\int \frac{1}{9x^2+16}\,dx$

Solution (a) $$\int \frac{1}{1+4x^2}\,dx = \int \frac{1}{1+(2x)^2}\,dx = \underline{\frac{1}{2}\tan^{-1}2x}$$

(b) $$\int \frac{1}{9x^2+16}\,dx \qquad \left\{\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right\}$$

$$= \int \frac{1}{4^2+(3x)^2}\,dx = \frac{1}{4}\times\frac{1}{3}\tan^{-1}\left(\frac{3x}{4}\right) = \underline{\frac{1}{12}\tan^{-1}\left(\frac{3x}{4}\right)}$$

---

§ Note: (i) $$\int \frac{1}{1+(bx)^2}\,dx = \frac{1}{b}\tan^{-1}(bx)$$

(ii) $$\int \frac{1}{a^2+(bx)^2}\,dx = \frac{1}{a}\times\frac{1}{b}\cdot\tan^{-1}\left(\frac{bx}{a}\right) = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right)$$

3(b) Alternate method

$$\int \frac{1}{16+9x^2}\,dx = \frac{1}{16}\int \frac{1}{1+\frac{9}{16}x^2}\,dx = \frac{1}{16}\int \frac{1}{1+\left(\frac{3}{4}x\right)^2}\,dx$$

$$= \frac{1}{16}\times\frac{1}{3/4}\cdot\tan^{-1}\left(\frac{3}{4}x\right)$$

$$= \underline{\frac{1}{12}\tan^{-1}\left(\frac{3x}{4}\right)}$$

Example 4: Find the value of $\int_0^3 \frac{1}{x^2+9}\,dx$

Solution: $$\int_0^3 \frac{1}{3^2+x^2}\,dx = \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3 = \frac{1}{3}\left[\tan^{-1}1 - \tan^{-1}0\right] = \frac{1}{3}\left(\frac{\pi}{4}-0\right)$$

$$= \underline{\frac{\pi}{12}}$$

§ $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$ $\left(\text{as } \frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}\right)$

Example 5: Show that, $\int \tan x\,dx = \ln \sec x + c$

Solution: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx$ $\left\{\begin{array}{l}\text{Let } f(x) = \cos x \\ f'(x) = -\sin x\end{array}\right.$

$= -\ln \cos x + c$ $\left\{\because \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|\right.$

$= \ln(\cos x)^{-1} + c$

$= \ln \sec x + c$

Example 6: Find: $\int \frac{x}{x^2+3}\,dx$

Solution: $\int \frac{x}{x^2+3}\,dx = \frac{1}{2}\int \frac{2x}{(x^2+3)}\,dx$ $\left\{\begin{array}{l}f(x) = (x^2+3) \\ f'(x) = 2x\end{array}\right.$

$= \frac{1}{2}\ln|x^2+3| + c$

Example 7: Find $\int \cot x\,dx$ $\int \frac{f'(x)}{f(x)} = \ln|f(x)|$

Solution: $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + c$ Let $f(x) = \sin x$, $f'(x) = \cos x$

Example 8: Show that $\int \operatorname{cosec} x\,dx = \ln(\operatorname{cosec} x - \cot x)$ (or $\ln \tan x/2$)

Solution: $\int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)\,dx}{(\operatorname{cosec} x - \cot x)}$ $\left\{\begin{array}{l}\text{Let } f(x) = (\operatorname{cosec} x - \cot x) \\ f'(x) = -\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x\end{array}\right.$

$= \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$ $\left\{\Rightarrow f'(x) = \operatorname{cosec} x(\operatorname{cosec} x - \cot x)\right.$

$= \ln|\operatorname{cosec} x - \cot x| + c$

$= \ln|\tan x/2| + c$

$\left\{\begin{array}{l}\because \operatorname{cosec} x - \cot x \\ = \frac{1}{\sin x} - \frac{\cos x}{\sin x} \\ = \frac{1-\cos x}{\sin x} = \frac{2\sin^2 x/2}{2\sin x/2 \cos x/2}\end{array}\right.$

$= \tan x/2$

§ Integration by Substitution:

Integration by substitution is the reverse process of the differentiation by chain rule.

Example: $\frac{d}{dx}\left(\frac{1}{3}\sin x^3\right) = \frac{1}{3} \times \cos x^3 \times 3x^2 = x^2 \cdot \cos x^3$

$$\Rightarrow \int x^2 \cdot \cos x^3 dx = \frac{1}{3}\sin x^3 + c.$$

Example 9. Find, $\int x^2 \cdot \cos x^3 dx$

Solution: $\int x^2 \cdot \cos x^3 dx$

$= \frac{1}{3}\int \cos x^3 \cdot 3x^2 dx$

$= \frac{1}{3}\int \cos u \cdot du = \frac{1}{3}\sin u = \frac{1}{3}\sin x^3 + c$

$\left\{ \begin{array}{l} \text{Put } u = x^3 \\ \text{diff. } \frac{du}{dx} = 3x^2 \\ \Rightarrow du = 3x^2 dx \end{array} \right.$

Example 10. Find, $\int \tan^2 x \cdot \sec^2 x dx$

Solution: $\int \tan^2 x \cdot \sec^2 x dx$

$= \int u^2 du = \frac{u^3}{3} = \frac{1}{3}\tan^3 x + c$

$\left\{ \begin{array}{l} \text{Put } u = \tan x \\ \frac{du}{dx} = \sec^2 x \\ \Rightarrow du = \sec^2 x dx \end{array} \right.$

Example 11. Evaluate. $\int \frac{x}{1+3x^4} dx$

Solution: $\int \frac{x}{1+3x^4} dx = \int \frac{x}{1+(\sqrt{3}x^2)^2} dx$

$= \frac{1}{2\sqrt{3}}\int \frac{2\sqrt{3}x\, dx}{1+(\sqrt{3}x^2)^2} = \frac{1}{2\sqrt{3}}\int \frac{1}{1+u^2} du$

$= \frac{1}{2\sqrt{3}}\tan^{-1} u = \frac{1}{2\sqrt{3}}\tan^{-1}(\sqrt{3}x^2) + c.$

$\left\{ \begin{array}{l} \text{Put } u = \sqrt{3}x^2 \\ \text{diff. } \frac{du}{dx} = 2\sqrt{3}x \\ du = 2\sqrt{3}x dx \end{array} \right.$

Example 12: Find. $\int \sin^2 x \cdot \cos^3 x\, dx$

Solution: $\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x \cdot \cos^2 x \cdot \cos x\, dx$

$= \int \sin^2 x (1-\sin^2 x) \cdot \cos x\, dx$

$\left\{ \begin{array}{l} \text{Put } u = \sin x \\ \frac{du}{dx} = \cos x \\ du = \cos x\, dx \end{array} \right.$

$= \int u^2(1-u^2)\, du = \int (u^2 - u^4)\, du = \frac{u^3}{3} - \frac{u^5}{5} + c$

$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$ ✓

Example 13: Find. $\int (3x^5+7x+1)^6 (15x^4+7)\, dx$

Solution: $\int (3x^5+7x+1)^6 (15x^4+7)\, dx$

$\left\{ \begin{array}{l} \text{Put } u = 3x^5+7x+1 \\ \frac{du}{dx} = 15x^4+7 \\ \Rightarrow du = (15x^4+7)\, dx \end{array} \right.$

$= \int u^6\, du$

$= \frac{1}{7}u^7 + c = \frac{1}{7}(3x^5+7x+1) + C$ ✓

§ Definite integration by substitution:

(Adjust the limits)

Example 14. Evalute. $\int_{\pi/3}^{\pi/2} \frac{\sin x}{(1+\cos x)}\, dx$ [S-18/32/ Q4(ii)] ... [4]

Solution: $\int_{\frac{\pi}{3}}^{\pi/2} \frac{\sin x}{(1+\cos x)}\, dx$

$\left\{ \begin{array}{l} \text{Let } u = 1+\cos x \quad \text{--- (1)} \\ \frac{du}{dx} = -\sin x \\ \Rightarrow du = -\sin x\, dx \\ \Rightarrow \sin x\, dx = -du \end{array} \right.$

$\left\{ \begin{array}{l} \text{from (1)} \\ x = \frac{\pi}{3} \Rightarrow u = 3/2 \\ x = \frac{\pi}{2} \Rightarrow u = 1 \end{array} \right.$

$= \int_{3/2}^{1} \frac{-1}{u}\, du$ ⊛

$= -[\ln u]_{3/2}^{1}$

$= -[\ln 1 - \ln 3/2]$

$= -[0 - \ln 3/2]$

$= \ln 3/2$ ✓

⊛ **Alternatively:**

$= \int_1^{3/2} \frac{1}{u}\, du = [\ln u]_1^{3/2}$

$\left\{ \begin{array}{l} \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \\ \text{Change of limit Property} \end{array} \right.$

$= [\ln 3/2 - \ln 1]$

$= \ln 3/2 - 0 = \ln 3/2$ ✓

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Example 15: Use the substitution $u = 4 - 3\cos x$, to find the exact value of $\int_0^{\frac{1}{2}\pi} \frac{9\sin 2x}{\sqrt{4-3\cos x}}\,dx$ ----[8]

W-15/33/Q5

Solution: $\int_0^{\frac{1}{2}\pi} \frac{9\sin 2x}{\sqrt{4-3\cos x}}\,dx$

Put $u = 4 - 3\cos x$ --- (1)

$\frac{du}{dx} = 3\sin x$

$\Rightarrow du = 3\sin x \cdot dx$ ✓

from (1) $3\cos x = (4-u)$ --- (2)

$x = 0 \rightarrow u = 1$

$x = \frac{\pi}{2} \rightarrow u = 4$

$= \int_0^{\frac{1}{2}\pi} \frac{9 \times 2\sin x \cos x}{\sqrt{4-3\cos x}}\,dx$

$= \int_0^{\frac{1}{2}\pi} \frac{2 \times 3\cos x \times 3\sin x}{\sqrt{4-3\cos x}}\,dx$

from (1) & (2)

$= \int_1^4 \frac{2(4-u)\,du}{\sqrt{u}} = 2\int_1^4 (4u^{-\frac{1}{2}} - u^{\frac{1}{2}})\,du = 2\left[4 \times \frac{u^{\frac{1}{2}}}{\frac{1}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$

$= 2\left[8\sqrt{u} - \frac{2}{3}u\sqrt{u}\right]_1^4 = 2\left[\left(8 \times 2 - \frac{2}{3} \times 4 \times 2\right) - \left(8 - \frac{2}{3}\right)\right]$

$= 2\left[\left(16 - \frac{16}{3}\right) - \left(\frac{22}{3}\right)\right] = \frac{20}{3}$ ✓ (or $6\frac{2}{3}$) ✓

Example 16: Let $I = \int_{\frac{1}{4}}^{\frac{3}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\,dx$

(i) Using substitution $x = \cos^2\theta$, Show that. $I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 2\cos^2\theta\,d\theta$ ---[4]

(ii) Hence find the value of I. --[4]

S-18/31/Q5

Solution (i) $\int_{\frac{1}{4}}^{\frac{3}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\,dx$

Put $x = \cos^2\theta$

$\Rightarrow \frac{dx}{d\theta} = -2\cos\theta \cdot \sin\theta$

$\Rightarrow dx = -2\cos\theta\sin\theta\,d\theta$ ✓

$x = \frac{1}{4} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$x = \frac{3}{4} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \pi/6$

$= \int_{\pi/3}^{\frac{\pi}{6}} \sqrt{\frac{\cos^2\theta}{1-\cos^2\theta}} \times -2\cos\theta\sin\theta\,d\theta$

$= -\int_{\pi/3}^{\pi/6} 2 \times \frac{\cos\theta}{\cancel{\sin\theta}} \times \cos\theta\,\cancel{\sin\theta}\,d\theta$

$= \int_{\frac{\pi}{6}}^{\pi/3} 2\cos^2\theta\,d\theta$ ✓

$\left\{ \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \right.$

(ii) $I = \int_{\pi/6}^{\pi/3} 2\cos^2\theta\,d\theta = \int_{\pi/6}^{\pi/3} (1 + \cos 2\theta)\,d\theta$ $\quad (\because \cos 2\theta = 2\cos^2\theta - 1)$

$= \left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/6}^{\pi/3} = \left[\left(\frac{\pi}{3} + \frac{1}{2} \times \frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{6} + \frac{1}{2} \times \frac{\sqrt{3}}{2}\right)\right] = \frac{1}{6}\pi$ ✓

§ Integration by parts:

$$\int u \cdot \frac{dv}{dx}\, dx = u \cdot v - \int v \cdot \frac{du}{dx}\, dx.$$

Using product rule for differentiation:

$$\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v\frac{du}{dx}$$

here $\int \frac{dv}{dx}\, dx = v$

$$\Rightarrow uv = \int \left(u\frac{dv}{dx} + v\frac{du}{dx}\right) dx$$

$$\Rightarrow \int u \cdot \frac{dv}{dx}\, dx = u \cdot v - \int v \cdot \frac{du}{dx}\, dx.$$

Order of u and $\frac{dv}{dx}$

$u \downarrow \frac{dv}{dx}$:
$\tan^{-1} x, \ln x$
$1, x, x^2, \ldots$
$\sin x, \cos x, e^x$

$I = \int e^x \cdot \sin x\, dx$

Example 17: Find, $\int x \cdot \sin x\, dx$

Solution: $\int x \cdot \sin x\, dx$

Let $\frac{dv}{dx} = \sin x$
$\Rightarrow v = \int \sin x\, dx$
$v = -\cos x$
and $u = x \rightarrow \frac{du}{dx} = 1$

$= \int u \cdot \frac{dv}{dx}\, dx$ (by parts)

$= u \cdot v - \int v \cdot \frac{du}{dx}\, dx$

$= x \cdot (-\cos x) - \int (-\cos x) \times 1\, dx$

$= -x\cos x + \int \cos x\, dx$

$= -x\cos x + \sin x + C$

Example 18: $\int x \cdot \cos 3x\, dx$

Let $u = x$; $\frac{dv}{dx} = \cos 3x$
$\frac{du}{dx} = 1$; $v = \int \cos 3x\, dx$
$v = \frac{1}{3}\sin 3x$.

$= \int u \cdot \frac{dv}{dx}\, dx$ (by parts)

$= u \cdot v - \int v \cdot \frac{du}{dx} \cdot dx$

$= x \times \frac{1}{3}\sin 3x - \int \frac{1}{3}\sin 3x \times 1\, dx$

$= \frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\, dx = \frac{1}{3}x\sin 3x - \frac{1}{3} \times \frac{1}{3}(-\cos 3x)$

$= \frac{1}{9}\left[3x\sin 3x + \cos 3x\right] + C$

18(a) Find the exact value of. $\int_0^{\frac{\pi}{2}} x^2 \cdot \sin 2x \, dx$ --- [5]

S-16 | 32 | Q3

Solution: consider $\int x^2 \cdot \sin 2x \, dx$

$$= \int u \cdot \frac{dv}{dx} \, dx$$

$$= u \cdot v - \int v \cdot \frac{du}{dx} \, dx$$

Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x$
and
$\frac{dv}{dx} = \sin 2x \Rightarrow v = -\frac{\cos 2x}{2}$

$$= x^2 \cdot \left(-\frac{\cos 2x}{2}\right) - \int \left(-\frac{\cos 2x}{\cancel{2}}\right) \cdot \cancel{2} x \, dx$$

$$= -\frac{1}{2} x^2 \cos 2x + \int x \cos 2x \, dx$$

Again apply. integ by part. ↓

$$= -\frac{1}{2} x^2 \cos 2x + \left[x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \times 1 \, dx \right.$$

$u = x$, $\frac{du}{dx} = 1$
$\frac{dv}{dx} = \cos 2x$, $v = \frac{\sin 2x}{2}$

$$= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x - \frac{1}{2} \times \left(-\frac{\cos 2x}{2}\right)$$

$$= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \quad \text{--- (1)}$$

from (1)

Hence $\int_0^{\frac{\pi}{2}} x^2 \sin 2x \, dx = \left[-\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \right]_0^{\pi/2}$

$$= \left(-\frac{1}{2} \times \frac{\pi^2}{4} \times (-1) + \frac{1}{2} \times \frac{\pi}{2} \times 0 + \frac{1}{4} \times (-1)\right) - \left(0 + 0 + \frac{1}{4}\right)$$

$$= \frac{\pi^2}{8} - \frac{1}{4} - \frac{1}{4} = \frac{\pi^2}{8} - \frac{1}{2}$$

$$= \frac{1}{8}(\pi^2 - 4)$$

Example 19. $\int \ln x \, dx$

$= \int \ln x \times 1 \, dx$

$= \int u \times \frac{dv}{dx} \, dx$

$= u \cdot v - \int v \cdot \frac{du}{dx} \cdot dx$

$= \ln x \cdot x - \int x \times \frac{1}{x} \, dx = x \cdot \ln x - \int 1 \, dx = x \ln x - x + C$

Let $u = \ln x$
$\frac{du}{dx} = \frac{1}{x}$
and $\frac{dv}{dx} = 1 \Rightarrow v = \int 1 \, dx = x$

Example 20 (i) Find: $\int \frac{\ln x}{x^3} \, dx$ ---[3]

(ii) Hence show that: $\int_1^2 \frac{\ln x}{x^3} \, dx = \frac{1}{16}(3 - \ln 4)$ ---[2]

$\boxed{W\text{-}18 | 32 | Q3}$

Solution (i) $\int \frac{\ln x}{x^3} \, dx = \int \ln x \cdot x^{-3} \, dx$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$
and $\frac{dv}{dx} = x^{-3}$
$\Rightarrow v = \int x^{-3} \, dx = \frac{x^{-2}}{-2} = \frac{-1}{2x^2}$

$= \int u \cdot \frac{dv}{dx} \, dx$

$= u \cdot v - \int v \cdot \frac{du}{dx} \, dx = \ln x \times \frac{-1}{2x^2} - \int \left(-\frac{1}{2x^2}\right) \times \frac{1}{x} \, dx$

$= -\frac{\ln x}{2x^2} + \frac{1}{2} \int x^{-3} \, dx$

$= -\frac{\ln x}{2x^2} + \frac{1}{2} \times \frac{x^{-2}}{-2}$

$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2}$ ------ (1)

(ii) $\int_1^2 \frac{\ln x}{x^3} \, dx = \left[-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right]_1^2$ from (1)

$= -\left[\left(\frac{\ln 2}{8} + \frac{1}{16}\right) - \left(0 + \frac{1}{4}\right)\right] = -\frac{1}{8} \ln 2 - \frac{1}{16} + \frac{1}{4}$

$= -\frac{1}{8} \ln 2 + \frac{3}{16} = \frac{1}{16}(3 - 2 \ln 2)$

$= \frac{1}{16}(3 - \ln 4)$

Example 21: The constant a is such that $\int_0^a x e^{-2x}\,dx = \frac{1}{8}$

Show that $a = \frac{1}{2}\ln(4a+2)$ ---[5]

S-23/31/Q9(a)

Solution: Consider $\int x \cdot e^{-2x}\,dx$ (by parts)

$$= \int u \cdot \frac{dv}{dx}\,dx = u \cdot v - \int v \cdot \frac{du}{dx}\,dx$$

$\left\{ \text{Let } u = x \rightarrow \frac{du}{dx} = 1 \right.$

$\left. \text{And } \frac{dv}{dx} = e^{-2x} \Rightarrow v = \int e^{-2x}\,dx = \frac{e^{-2x}}{-2} \right.$

$$= x \cdot \frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2} \times 1\,dx = -\frac{1}{2} \cdot x e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$$

$$= -\frac{1}{2} x e^{-2x} + \frac{1}{2} \cdot \frac{e^{-2x}}{-2}$$

$$= -\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} \text{ ---- } (1)$$

Now given

$$\int_0^a x e^{-2x}\,dx = \frac{1}{8} \Rightarrow \left[-\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x}\right]_0^a = \frac{1}{8} \quad (\text{from } (1))$$

$$= -\frac{1}{2} a e^{-2a} - \frac{1}{4} e^{-2a} + \frac{1}{4} = \frac{1}{8} \Rightarrow -\frac{1}{4} e^{-2a}(2a+1) = -\frac{1}{8}$$

$$\Rightarrow e^{2a} = (4a+2) \Rightarrow 2a = \ln(4a+2) \Rightarrow a = \frac{1}{2}\ln(4a+2)$$

Example 22 (a) Use the substitution $u = \cos x$ to show that: $\int_0^{\pi} \sin 2x\, e^{2\cos x}\,dx = \int_{-1}^{1} 2u e^{2u}\,du$ ---[4]

(b) Hence find the exact value of $\int_0^{\pi} \sin 2x\, e^{2\cos x}\,dx$ ---[4]

S-23/33/Q7

Solution (a) $\int_0^{\pi} \sin 2x \cdot e^{2\cos x}\,dx$

$\left\{ \text{Put } u = \cos x \right.$

$\left. \frac{du}{dx} = -\sin x \Rightarrow du = -\sin x\,dx \right.$

$$= \int_0^{\pi} 2\sin x \cdot \cos x \cdot e^{2\cos x}\,dx$$

Now for $\left\{ \begin{array}{l} x = 0 \rightarrow u = 1 \\ x = \pi \rightarrow u = -1 \end{array} \right.$

$$= \int_0^{\pi} 2\cos x\, e^{2\cos x} \cdot \sin x\,dx$$

$$= \int_1^{-1} 2u e^{2u} \cdot (-du)$$

$$= + \int_{-1}^{1} 2u e^{2u}\,du \text{ --- } (1)$$

(b) $\int_0^{\pi} \sin 2x\, e^{2\cos x}\,dx$

$$= \int_{-1}^{1} 2u e^{2u}\,du \text{ --- } (2)$$

from last (a)

Consider

$\int 2u\, e^{2u}\,du$ (by parts)

$\left\{ \text{Let } \frac{dv}{du} = e^{2u} \Rightarrow v = \frac{e^{2u}}{2} \right.$

$$= 2u \cdot \frac{e^{2u}}{2} - \int \frac{e^{2u}}{\cancel{2}} \times \cancel{2}\,du$$

$\left\{ \frac{d}{du} 2u = 2 \right.$

$$= u e^{2u} - \frac{e^{2u}}{2} \text{ --- } (3)$$

from (2) & (3)

$$\int_{-1}^{1} 2u e^{2u}\,du = \left[u e^{2u} - \frac{e^{2u}}{2}\right]_{-1}^{1} = \left(e^2 - \frac{e^2}{2}\right) - \left(-e^{-2} - \frac{e^{-2}}{2}\right)$$

$$= \frac{1}{2}e^2 + \frac{3}{2}e^{-2}$$

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Example 23: $\int \tan^{-1}x \, dx$

$\left\{ \begin{array}{l} \text{Let } u = \tan^{-1}x \\ \frac{du}{dx} = \frac{1}{(1+x^2)} \checkmark \end{array} \right.$

and $\frac{dv}{dx} = 1 \Rightarrow v = x \checkmark$

$= \int \tan^{-1}x \times 1 \, dx$

$= \int u \cdot \frac{dv}{dx} dx$ (By parts)

$= u \cdot v - \int v \cdot \frac{du}{dx} \cdot dx = \tan^{-1}x \cdot x - \int x \cdot \frac{1}{(1+x^2)} dx$

$= x \cdot \tan^{-1}x - \frac{1}{2}\int \frac{2x}{(1+x^2)} dx$ $\left\{ \begin{array}{l} f(x) = 1+x^2 \\ f'(x) = 2x \end{array} \right.$

$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c \checkmark$ $\left\{ \because \int \frac{f'(x)}{f(x)} dx = \ln f(x). \right.$

Example 24: $\int x \cdot \tan^{-1}x \, dx$

$\left\{ \begin{array}{l} \text{Let } u = \tan^{-1}x \Rightarrow \frac{du}{dx} = \frac{1}{(1+x^2)} \checkmark \\ \text{and Let } \frac{dv}{dx} = x \Rightarrow v = \frac{x^2}{2} \checkmark \end{array} \right.$

$= \int \tan^{-1}x \cdot x \, dx$

$= \int u \cdot \frac{dv}{dx} dx$

$= u \cdot v - \int v \cdot \frac{du}{dx} dx$

$= \tan^{-1}x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \times \frac{1}{(1+x^2)} dx$

$= \frac{x^2}{2} \cdot \tan^{-1}x - \frac{1}{2}\int \frac{(1+x^2)-1}{(1+x^2)} dx$

$= \frac{x^2}{2} \cdot \tan^{-1}x - \frac{1}{2}\left[\int \left(1 - \frac{1}{1+x^2}\right) dx\right.$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[x - \tan^{-1}x\right]$

$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + c$

Application of partial fractions to integration.

Example 25. Let $f(x) = \dfrac{2x^2+17x-17}{(1+2x)(2-x)^2}$

(a) Express $f(x)$ in partial fractions. ...[5]

(b) Hence show that $\int_0^1 f(x)\,dx = \frac{5}{2} - \ln 72$. ...[5]

S-23/32/Q9

Solution (a) $f(x) = \dfrac{2x^2+17x-17}{(1+2x)(2-x)^2} = \dfrac{a}{(1+2x)} + \dfrac{b}{(2-x)^2} + \dfrac{c}{(2-x)}$ ----①

multiply ① by Dr. $(1+2x)(2-x)^2 \Rightarrow$

$$2x^2+17x-17 = a(2-x)^2 + b(1+2x) + c(1+2x)(2-x) \text{ ---②}$$

for a; Put $(1+2x)=0 \Rightarrow x=-\frac{1}{2}$ in ② $\Rightarrow 2\left(\frac{-1}{2}\right)^2 + 17\times\left(-\frac{1}{2}\right) - 17 = a\left(2-\left(-\frac{1}{2}\right)\right)^2 + 0 + 0$

$\Rightarrow -25 = \frac{25}{4}a \Rightarrow a = -4$ ✓

for b, put $(2-x)=0 \Rightarrow x=2$ in ② $\Rightarrow 2\times 2^2 + 17\times 2 - 17 = 0 + b(1+2\times 2) + 0$

$\Rightarrow 25 = 5b \Rightarrow \underline{b=5}$ ✓

Put $a=-4$ & $b=5$ in ②

$$\Rightarrow 2x^2+17x-17 = -4(2-x)^2 + 5(1+2x) + c(1+2x)(2-x) \text{ ---③}$$

Put $x=0$ in ③ $\Rightarrow -17 = -4\times 2^2 + 5\times 1 + c\times 1\times 2 \Rightarrow 2c = -6 \Rightarrow \underline{c=-3}$ ✓

Hence the required partial fractions: $f(x) = \dfrac{-4}{(1+2x)} + \dfrac{5}{(2-x)^2} + \dfrac{-3}{(2-x)}$ ✓

(b) $\int f(x)\,dx = \int\left(\dfrac{-4}{(1+2x)} + 5(2-x)^{-2} - \dfrac{3}{(2-x)}\right)dx$

$$= -4\frac{\ln(1+2x)}{2} + \frac{5(2-x)^{-1}}{-1\times -1} - \frac{3\ln(2-x)}{-1}$$

$$= -2\ln(1+2x) + \frac{5}{(2-x)} + 3\ln(2-x).$$

Hence $\int_0^1 f(x)\,dx = \left[-2\ln(1+2x) + \dfrac{5}{(2-x)} + 3\ln(2-x)\right]_0^1$

$$= \left(-2\ln 3 + \frac{5}{2} + 3\ln 1\right) - \left(-2\ln 1 + 3\ln 2\right)$$

$$= -\ln 3^2 + \frac{5}{2} + 0 - \left(0 + \ln 2^3\right)$$

$$= \frac{5}{2} - \ln 9 - \ln 8 \quad = \frac{5}{2} - (\ln 9 + \ln 8)$$

$$= \frac{5}{2} - \ln 9\times 8 = \frac{5}{2} - \ln 72$$ ✓

26. Let $f(x)=\dfrac{7x+18}{(3x+2)(x^2+4)}$

(a) Express $f(x)$ in partial fractions. ---[5]

(b) Hence find the exact value of $\int_0^2 f(x)\,dx$ --[6]

W-20/32/Q9

Solution(a) $f(x)=\dfrac{7x+18}{(3x+2)(x^2+4)}=\dfrac{a}{(3x+2)}+\dfrac{bx+c}{x^2+4}$ ---- ①

multiply ① by $(3x+2)(x^2+4)$:

$\Rightarrow \quad 7x+18=a(x^2+4)+(bx+c)(3x+2)$ --- ②

for 'a' put $3x+2=0 \Rightarrow x=-\frac{2}{3}$ in ② $\Rightarrow 7\times(-\frac{2}{3})+18=a[(-\frac{2}{3})^2+4]+0$

$\Rightarrow \quad \frac{40}{3}=\frac{40}{9}a \Rightarrow \quad \underline{a=3}$ ✓

Put $a=3$ in ② $\Rightarrow \quad 7x+18=3(x^2+4)+(bx+c)(3x+2)$ --- ③

Comparing the constant term on both sides of ③ $\rightarrow 18=12+2c \Rightarrow c=3$ ✓

comparing the coefficient of x^2 on both sides of ③ $\rightarrow 0=3+3b \Rightarrow \underline{b=-1}$ ✓

∴ the required partial fraction: $f(x)=\dfrac{3}{(3x+2)}+\dfrac{-x+3}{x^2+4}$ --④ ✓

(b) Consider $\int f(x)\,dx=\int\left(\dfrac{3}{(3x+2)}+\dfrac{-x+3}{x^2+4}\right)dx$

$$=\int\left(\frac{3}{3x+2}-\frac{1}{2}\cdot\frac{2x}{(x^2+4)}+\frac{3}{2^2+x^2}\right)dx$$

$$=\frac{\cancel{3}\ln(3x+2)}{\cancel{3}}-\frac{1}{2}\ln(x^2+4)+3\times\frac{1}{2}\tan^{-1}\frac{x}{2},$$

$\left\{\begin{array}{l} f(x)=x^2+4 \\ f'(x)=2x \\ \int\frac{f'(x)}{f(x)}=\ln f(x)\end{array}\right.$

$\left\{\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}\right.$

Hence $\int_0^2 f(x)\,dx=\left[\ln(3x+2)-\frac{1}{2}\ln(x^2+4)+\frac{3}{2}\tan^{-1}\frac{x}{2}\right]_0^2$

$$=\left(\ln 8-\frac{1}{2}\ln 8+\frac{3}{2}\tan^{-1}1\right)-\left(\ln 2-\frac{1}{2}\ln 4+\frac{3}{2}\tan^{-1}0\right)$$

$$=\frac{1}{2}\ln 8-\cancel{\ln 2}+\cancel{\ln 2}+\frac{3}{2}\times\frac{\pi}{4}-0$$

$$=\frac{1}{2}\ln 2^3+\frac{3}{8}\pi$$

$$=\underline{\frac{3}{2}\ln 2+\frac{3}{8}\pi}.$$ ✓

27. Let $f(x) = \dfrac{4x^2+9x-8}{(x+2)(2x-1)}$

(a) Express $f(x)$ in the form: $A + \dfrac{B}{(x+2)} + \dfrac{C}{(2x-1)}$ ---[4]

(b) Hence show that: $\int_1^4 f(x)\,dx = 6 + \frac{1}{2}\ln\left(\frac{16}{7}\right)$ ---[5]

$\boxed{W\text{-}17/31/Q8}$

Solution:

(a) $f(x) = \dfrac{4x^2+9x-8}{(x+2)(2x-1)} = \dfrac{4x^2+9x-8}{2x^2+3x-2}$;

$$\begin{array}{r} 2 \\ 2x^2+3x-2 \overline{)\,4x^2+9x-8} \\ \underline{-4x^2 \pm 6x \mp 4} \\ 3x-4 \end{array}$$

$\boxed{\Rightarrow A = 2}$

$f(x) = 2 + \dfrac{3x-4}{(x+2)(2x-1)}$ ---(1)

Now Consider $\dfrac{3x-4}{(x+2)(2x-1)} = \dfrac{B}{(x+2)} + \dfrac{C}{(2x-1)}$ ---- (2)

multiply (2) by $D^r \to (x+2)(2x-1)$

$\Rightarrow 3x-4 = B(2x-1) + C(x+2)$ ---(3)

To get the Value of B, $x+2=0 \Rightarrow$ put $x=-2$ in (3) $\Rightarrow 3\times -2-4 = B(2\times -2-1)+0$

$\Rightarrow -10 = -5B \Rightarrow \underline{B=2}$ ✓

Again to get the value of C, $2x-1=0 \Rightarrow$ put $x=\frac{1}{2}$ in (3) $\Rightarrow 3\times\frac{1}{2}-4 = C\left(\frac{1}{2}+2\right)$

$\Rightarrow -\frac{5}{2} = \frac{5}{2}C \Rightarrow$ $C = -1$

Hence from (1) and (2)

$f(x) = 2 + \dfrac{2}{(x+2)} - \dfrac{1}{(2x-1)}$ ----- (4)

(b) $\int_1^4 f(x)\,dx = \int_1^4 \left(2 + \dfrac{2}{(x+2)} - \dfrac{1}{(2x-1)}\right)dx$

$= \left[2x + 2\ln(x+2) - \frac{1}{2}\ln(2x-1)\right]_1^4$

$= \left(8 + 2\ln 6 - \frac{1}{2}\ln 7\right) - \left(2 + 2\ln 3 - \frac{1}{2}\ln 1\right)$

$= 8-2 + 2(\ln 6 - \ln 3) - \frac{1}{2}\ln 7 + 0$

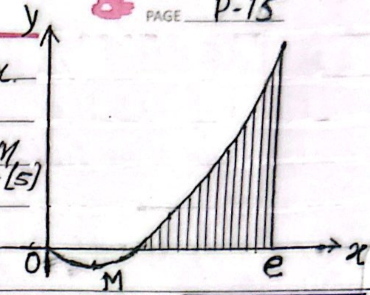
$= 6 + 2\ln 2 - \frac{1}{2}\ln 7$

$= 6 + \frac{1}{2}(4\ln 2 - \ln 7)$

$= 6 + \frac{1}{2}(\ln 16 - \ln 7)$

$= 6 + \frac{1}{2}\ln\left(\frac{16}{7}\right)$ ✓

28. The diagram shows the curve $y = x^2 . \ln x$ and its minimum point M.
(i) Find the exact values of the coordinates of M. ...[5]
(ii) Find the exact value of the area of the shaded region bounded by the curve, the x-axis and the line $x = e$. ...[5]



[W-11/31/Q9]

Solution: (i) Curve: $y = x^2 \ln x$ --- ①

$\frac{dy}{dx} = 2x . \ln x + x^2 \times \frac{1}{x}$
$= 2x \ln x + x$

for minimum point: $\frac{dy}{dx} = 0 \Rightarrow x(2\ln x + 1) = 0 \Rightarrow x = 0^*, \ \ln x = -\frac{1}{2}$

$\Rightarrow \begin{cases} x = e^{-1/2} = \frac{1}{\sqrt{e}}, \ y = (e^{-1/2})^2 . \ln e^{-1/2} \\ y = e^{-1} \times -\frac{1}{2} = -\frac{1}{2e} \end{cases}$

$\therefore M\left(\frac{1}{\sqrt{e}}, -\frac{1}{2}e\right)$

(ii) For lower limit of the shaded region on x-axis.
Put $y = 0 \Rightarrow x^2 \ln x = 0 \Rightarrow x = 0^*, \ x = 1$ ✓ $(\ln x = 0 \Rightarrow x = 1)$

$\therefore$ Area of the shaded region: $\int_1^e x^2 . \ln x \, dx$ --- ②

Consider $\int \ln x . x^2 \, dx$
$= \int u . \frac{dv}{dx} \, dx$
$= u.v - \int v . \frac{du}{dx} . dx$
$= \ln x . \frac{1}{3}x^3 - \int \frac{1}{3}x^3 . \frac{1}{x} \, dx$
$= \frac{1}{3}x^3 . \ln x - \frac{1}{3}\int x^2 \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3$ --- ③

$\begin{cases} u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \\ \frac{dv}{dx} = x^2 \Rightarrow v = \frac{1}{3}x^3 \end{cases}$

for ③ in ②
Req. Area $= \left[\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3\right]_1^e$
$= \left(\frac{1}{3}e^3 \times 1 - \frac{1}{9}e^3\right) - \left(0 - \frac{1}{9}\right)$
$= \frac{2}{9}e^3 + \frac{1}{9}$
$= \frac{1}{9}(2e^3 + 1)$ ✓