classmate
Date 05.12.23.
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P-3

Pure Maths-3.

Iteration

Numerical Solution of Equations

Exercise-1. Solution (Revision)

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Iteration _ Numerical Solution of Equations.
Revision

Example 1: The parametric equations of a curve are,
$x = e^{2t-3}$, $y = 4\ln t$, where $t>0$. When $t=a$, the gradient of curve is 2

(a) Show that a satisfies the equation $a = \frac{1}{2}(3-\ln a)$ --[4]

(b) Verify by calculation that this equation has a root between 1 and 2. --[2]

(c) Use the iterative formula $a_{n+1} = \frac{1}{2}(3-\ln a_n)$ to calculate a correct to 2 d.p., showing the result of each iteration to 4 d.p. --[3]

SP-20 | 03 | Q4

Solution:

(a) $x = e^{2t-3}$; $y = 4\ln t$

$\frac{dx}{dt} = 2e^{2t-3}$; $\frac{dy}{dt} = \frac{4}{t}$

$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{4}{t} / 2e^{2t-3}$

$\Rightarrow \frac{dy}{dx} = \frac{2}{t\, e^{(2t-3)}}$

gradient at $t=a$

$\left(\frac{dy}{dx}\right)_{t=a} = \frac{2}{a \cdot e^{(2a-3)}} = 2$ given

$\Rightarrow a\, e^{(2a-3)} = 1$

$a = \frac{1}{e^{2a-3}}$

$\Rightarrow a = e^{(3-2a)}$

$\ln a = 3-2a$

$\Rightarrow \underline{a = \frac{1}{2}(3-\ln a)}$ ✓

(c) Let us take $a_1 = 1.5$ (between 1 & 2)

$a_1 = 1.5$	$a_{n+1} = \frac{1}{2}(3-\ln a_n)$
	$a_2 = 1.2972$
	$a_3 = 1.3698$
	$a_4 = 1.3426$
	$a_5 = 1.3526$
	$a_6 = 1.3489$
	$a_7 = 1.3503$
	$a_8 = 1.3498$ / 1.3500
	$a_9 = 1.3499$
	$a_{10} = 1.3499$

repeats

$\therefore a = 1.3499 = 1.35$ to 2dp

$\underline{a = 1.35}$ ✓

(b) $a = \frac{1}{2}(3-\ln a)$ —— ①

Consider $f(a) = a - \frac{1}{2}(3-\ln a)$ — ①

at $a=1 \Rightarrow$ $f(1) = 1 - \frac{1}{2}(3-0)$

$f(1) = -\frac{1}{2}$ ✓

and

at $a=2$; $f(2) = 2 - \frac{1}{2}(3-\ln 2)$

$= 2 - \frac{1}{2} \times 2.30$

$= 2 - 1.15 = 0.85$ ✓

$\therefore$ f(a) changes sign between $a=1$ and $a=2$

$\therefore$ the eqn ① has a root between 1 and 2.

Example 2(a) By sketching a suitable pair of graph, show that the equation $\sec x = 2 - \frac{1}{2}x$ has exactly one root in the interval $0 \le x \le \frac{1}{2}\pi$ --[2]

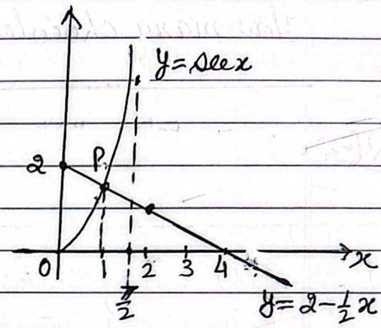
(b) Verify by calculation that this root lies between 0.8 and 1. --[2]

(c) Use the iterative formula $x_{n+1} = \cos^{-1}\left(\frac{2}{4-x_n}\right)$ to determine the root correct to 2 d.p. Give your result of each iteration to 4 d.p. [M-20/32/Q3] --[3]

Solution: $y = \sec x$ —① $0 \le x \le \frac{\pi}{2}$

(a) $y = 2 - \frac{1}{2}x$ —②

The graphs of ① and ② intersect at P, exactly one point, in the interval, $0 \le x \le \frac{\pi}{2}$ ✓



(b) Now $\sec x = 2 - \frac{1}{2}x$

$$\frac{1}{\cos x} = \frac{4-x}{2}$$

$$\Rightarrow \cos x = \frac{2}{(4-x)}$$

$$\Rightarrow \cos x - \frac{2}{(4-x)} = 0 \quad \text{—③}$$

Consider $f(x) = \cos x - \frac{2}{(4-x)}$

$f(0.8) = \cos 0.8 - \frac{2}{(4-.8)}$

$= 0.075$ ✓

and $f(1) = \cos 1 - \frac{2}{(4-1)} = -1.3$ ✓

Change of sign for $f(x)$. Hence the given eqn ③ has a root between 0.8 and 1.

(c) $x_{n+1} = \cos^{-1}\left(\frac{2}{4-x_n}\right)$

Let $x_1 = 0.8$

$x_1 = 0.8$	$x_{n+1} = \cos^{-1}\left(\frac{2}{4-x_n}\right)$
	$x_2 = 0.8956$
	$x_3 = 0.8707$
	$x_4 = 0.8774$
	$x_5 = 0.8756$
	$x_6 = 0.8761$
	$x_7 = 0.8759$
	$x_8 = 0.8760$
	$x_9 = 0.8760$ ✓

$\therefore x = 0.8760$

or $x = 0.88$ ✓ to 2 d.p.

3. Let $f(x) = \dfrac{e^{2x}+1}{e^{2x}-1}$ for $x > 0$.

(a) The equation $x = f(x)$ has one root, denoted by a.
Verify by calculation that a lies between 1 and 1.5. ---[2]

(b) Use an iterative formula base on the equation in part (a) to determine a correct to 2 decimal places. Give your result of each iteration to 4 decimal places. ---[3]

(c) Find $f'(x)$. Hence find the exact value of x for which $f'(x) = -8$ --[6]

[M-21/32/Q9]

Solution (a) $f(x) = x \Rightarrow \dfrac{e^{2x}+1}{e^{2x}-1} = x \Rightarrow e^{2x}+1 = xe^{2x} - x$

$\Rightarrow e^{2x} + x + 1 - xe^{2x} = 0$ ---(1)

Now let $p(x) = e^{2x} + x - 1 - xe^{2x}$

$p(1) = e^2 + 1 + 1 - e^2 = 2$ ----(2)

and $p(1.5) = e^3 + 1.5 + 1 - 1.5e^3 = e^3(1-1.5) + 2.5 = 2.5 - 0.5e^3$

$= 2.5 - 10.0427 = -7.543$ ---(3)

for (2) and (3) $\Rightarrow$ $P(x)$ changes sign between $x = 1$ and $x = 1.5$

$\Rightarrow$ one of its root 'a' lies between 1 and 1.5

(b) $x = \dfrac{e^{2x}+1}{e^{2x}-1} \Rightarrow x_{n+1} = \dfrac{e^{2x_n}+1}{e^{2x_n}-1}$,

Calculator	
1	(execute)
$\left(\dfrac{e^{2ans}+1}{e^{2.ans}-1}\right)$	1 (execute) 1.3130 1.5601

Let $x_0 = 1 \Rightarrow x_1 = \dfrac{e^2+1}{e^2-1} = 1.3130$

$x_2 = 1.5601$, $x_3 = 1.2199$, $x_4 = 1.1910$

$x_5 = 1.2035$, $x_6 = 1.9799$, $x_6 = 1.2004$, $x_7 = 1.1993$

$x_8 = 1.1998$ $x_9 = 1.1996$, $x_{10} = 1.1997$, $x_{11} = 1.1996$

$x_{12} = 1.1996 \Longrightarrow x = 1.20$ (2dp)

(c) $f(x) = \dfrac{e^{2x}+1}{e^{2x}-1}$

$\Rightarrow f'(x) = \dfrac{(e^{2x}-1)\cdot 2e^{2x} - (e^{2x}+1)\cdot 2e^{2x}}{(e^{2x}-1)^2}$

$= \dfrac{-4e^{2x}}{(e^{2x}-1)^2} = -8$ (Given)

$\Rightarrow e^{2x} = 2(e^{2x}-1)^2$, let $e^{2x} = a$

$\Rightarrow a = 2(a-1)^2$

$\Rightarrow a = 2(a^2 - 2a + 1)$

$\Rightarrow 2a^2 - 5a + 2 = 0$

$(2a-1)(a-2) = 0$

$\Rightarrow a = \frac{1}{2}$; $a = 2$

$e^{2x} = \frac{1}{2}$; $e^{2x} = 2$

$\Rightarrow 2x = \ln 0.5 < 0$ as $x > 0$; $2x = \ln 2$

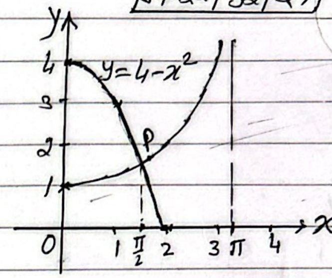
$x = \frac{1}{2}\ln 2$

4.(a) By sketching a suitable pair of graphs, show that the equation $4-x^2 = \sec\frac{1}{2}x$ has exactly one root in the interval $0 \le x < \pi$ --- [2]

(b) Verify by calculation that this root lies between 1 and 2. --- [2]

(c) Use the iterative formula $x_{n+1} = \sqrt{4 - \sec\frac{1}{2}x_n}$ to determine the root correct to 2 decimal places. --- [3]
Give the result of each iteration to decimal places.

M-22/32/Q7



Solution (a) $y = 4 - x^2$ — ① $\quad 0 \le x < \pi$

and $y = \sec\frac{1}{2}x$ — ②

from the graph of ① & ② we find that the eqn $4-x^2 = \sec\frac{1}{2}x$ ③ has only one root (as the two graphs intersect at one point P).

(b) Now consider the eqn $\sec x/_2 + x^2 - 4 = 0$

and let $y = x^2 + \sec x/_2 - 4$

for $x = 1$, $y = 1^2 + \sec \frac{1}{2} - 4 = 1 + 1.14 - 4 = -1.86$

for $x = 2$, $y = 2^2 + \sec 1 - 4 = 4 + 1.8518 - 4 = +1.8518$

we find that y changes sign ($-$ to $+$) as x changes 1 to 2.

$\therefore$ the root of the eqn ③ lies between 1 and 2. ✓

(c) Consider $x_{n+1} = \sqrt{4 - \sec\frac{1}{2}x_n}$

Let $x_0 = 1 \Rightarrow$

$x_1 = 1.6913$
$x_2 = 1.5786$
$x_3 = 1.6062$
$x_4 = 1.6000$
$x_5 = 1.6014$
$x_6 = 1.6011$
$x_7 = 1.6012$
$x_8 = 1.6011$
$x_9 = 1.6011$
$x_{10} = 1.6011$ ✓

Calculator:

1

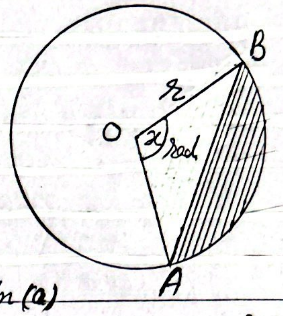
1

$\sqrt{4 - \frac{1}{\cos .5(\text{ans})}}$

1.6913

$\therefore x = 1.60$ ✓

5. The diagram shows a circle with centre O and radius r. The angle of minor sector AOB of the circle is x radians. The area of major sector of the circle is 3 times the area of the shaded region.



(a) Show that $x = \frac{3}{4}\sin x + \frac{\pi}{2}$ ---[4]

(b) Show by calculation that the root of the equation in (a) lies between 2 and 2.5. ---[2]

(c) Use an iterative formula based of the equation in (a) to calculate this root correct to 2 d.p. Give the result of each iteration to 4 d.p. ---[3]

M-23/32/Q7

Solution (a) Area of the major sector $= \frac{1}{2}r^2(2\pi - x)$ --- ①

Area of the shaded segment of the circle $= \left(\frac{1}{2}\pi r^2 x - \frac{1}{2}r^2\sin x\right)$ --- ②

from ① & ② and given relation:

$$\frac{1}{2}r^2(2\pi - x) = 3\left[\frac{1}{2}r^2 x - \frac{1}{2}r^2\sin x\right] \Rightarrow 2\pi - x = 3x - 3\sin x$$

$$\Rightarrow 4x = 3x + 2\pi \Rightarrow \underline{x = \frac{3}{4}\sin x + \frac{\pi}{2}} \checkmark \text{ --- ③}$$

(b) Consider $f(x) = x - \frac{3}{4}\sin x - \frac{\pi}{2}$

for $x = 2 \rightarrow f(2) = 2 - \frac{3}{4}\sin 2 - \frac{\pi}{2} = -0.2527\ldots < 0$

for $x = 2.5 \rightarrow f(2.5) = 2.5 - \frac{3}{4}\sin 2.5 - \frac{\pi}{2} = 0.4803 > 0$

as $f(x)$ changes sign $\Rightarrow$ x lies between 2 and 2.5.

(c) Consider an iterative formula $x_{n+1} = \frac{3}{4}\sin x_n + \frac{\pi}{2}$ ---④

On calculator:

2	enter
	2
$\frac{3}{4}\sin \text{Ans} + \frac{\pi}{2}$	
	2.2528
	2.1530
	⋮

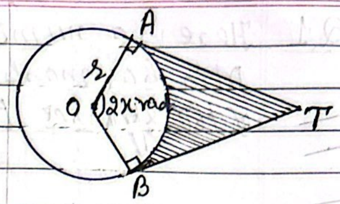
Let $x_0 = 2$

from ④

$x_1 = 2.2528$	$x_8 = 2.1839$
$x_2 = 2.1530$	$x_9 = 2.1841$
$x_3 = 2.1967$	$x_{10} = 2.1840$
$x_4 = 2.1786$	$x_{11} = 2.1841$
$x_5 = 2.1866$	$x_{12} = 2.1840$
$x_6 = 2.1830$	$x_{13} = 2.1841$
$x_7 = 2.1846$	$x_{14} = 2.1841$
	$x_{15} = 2.1841$

Hence $x = \underline{2.18} \checkmark$

Example 6: The diagram shows a circle with centre O and radius r. The tangents to the circle at the points A and B meet at T, and angle AOB is 2x radians. The shaded region is bounded by the tangents AT and BT, and by minor arc AB. The area of the shaded region is equal to the area of the circle.



(a) Show that x satisfies the equation $\tan x = \pi + x$ ---[3]

(b) This equation has one root in the interval $0 < x < \frac{1}{2}\pi$. Verify by calculation that this root lies between 1 and 1.4. ---[2]

(c) Use iterative formula $x_{n+1} = \tan^{-1}(\pi + x_n)$, to determine the root correct to 2 d.p. Give the result of each iteration to 4 d.p.

[S-20/31/Q6] ---[3]

Solution: In rt ΔOAT
$AT = r \tan x$

(a) Area of $\Delta OAT = \frac{1}{2} \times r \times r \tan x$

$\therefore$ Area of quadrilateral $OATB = 2 \times ar \Delta OAT$
$= 2 \times \frac{1}{2} r^2 \tan x = r^2 \tan x$ ✓

Area of Sector $AOB = \frac{1}{2} r^2 \times 2x = r^2 x$ ✓

$\therefore$ Shaded area $= (r^2 \tan x - r^2 x)$

Give Shaded area = area of circle

$r^2(\tan x - x) = \pi r^2$

$\Rightarrow \tan x - x = \pi \Rightarrow \underline{\tan x = \pi + x}$ ✓

(b) $\tan x = \pi + x$ ——— ①

Consider $f(x) = \tan x - \pi - x$

$f(1) = \tan 1 - \pi - 1 = -2.58$ ✓
$f(1.4) = \tan 1.4 - \pi - 1.4 = 1.65$ ✓

$f(x)$ changes sign for $x = 1$ and 1.4

$\therefore$ ① has a root between 1 and 1.4 ✓

(c) $x_{n+1} = \tan^{-1}(\pi + x_n)$

Let $x_1 = 1$

$x_1 = 1$	$x_{n+1} = \tan^{-1}(\pi + x_n)$
	$x_2 = 1.3338$
	$x_3 = 1.3509$
	$x_4 = 1.3517$
	$x_5 = 1.3518$
	$x_6 = 1.3518$

$\therefore x = \underline{1.35}$ ✓ to 2 d.p

Calculator

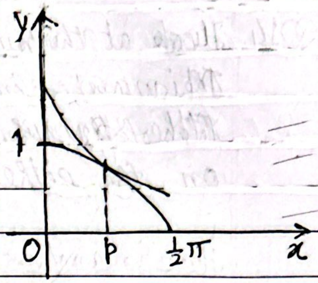
1 enter

1

$\tan^{-1}(\pi + Ans)$

1.3338

Example 7: The diagram shows the curves $y = \cos x$ and $y = \frac{k}{1+x}$, where k is a constant, for $0 \le x \le \frac{1}{2}\pi$
The curves touch at the point where $x = p$.



(a) Show that the p satisfies the equation,
$\tan p = \frac{1}{1+p}$ ---[5]

(b) Use the iterative formula $p_{n+1} = \tan^{-1}\left(\frac{1}{1+p_n}\right)$, to determine the value of p correct to 3 d.p. Give the result of each iteration to 5 d.p.
[S-20/32/Q9] --[3]

Solution: $y = \cos x$ —①

(a) $y = \frac{k}{1+x}$ —②

both the curves ① & ② meet at $x = p$

$\therefore \cos p = \frac{k}{1+p}$ —③

for ① diff $\frac{dy}{dx} = -\sin x \Rightarrow \left(\frac{dy}{dx}\right)_p = -\sin p$

for ② $\frac{dy}{dx} = -\frac{k}{(1+x)^2} \Rightarrow \left(\frac{dy}{dx}\right)_{x=p} = -\frac{k}{(1+p)^2}$

both the curves touch

$\therefore \cancel{-}\sin p = \cancel{-}\frac{k}{(1+p)^2}$ —③

fr ③ $[k = (1+p)\cos p$

$\sin p = \frac{(1+p)\cos p}{(1+p)^2}$

$\Rightarrow \tan p = \frac{1}{(1+p)}$

(b) Now $p_{n+1} = \tan^{-1}\frac{1}{(1+p_n)})$

Now $0 \le p \le \frac{1}{2}\pi$

Set $p_1 = 1$

$p_{n+1} = \tan^{-1}\left(\frac{1}{1+p_n}\right)$

$p_2 = 0.46364$
$p_3 = 0.59937$
$p_4 = 0.55877$
$p_5 = 0.57039$
$p_6 = 0.56702$
$p_7 = 0.56800$
$p_8 = 0.56771$
$p_9 = 0.56780$
$p_{10} = 0.56777$
$p_{11} = 0.56778$
$p_{12} = 0.56778$

$\therefore x = 0.56778$
$= 0.568$ to 3 d.p

$x = 0.568$

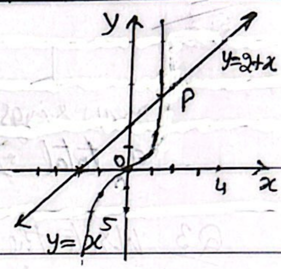
Example 8 (a) By sketching a suitable pair of graphs, show that the equation $x^5 = 2 + x$ has exactly one real root. --[2]

(b) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{4x_n^5 + 2}{5x_n^4 - 1}$$

converges, then it converges to the root of equation in part (a). --[2]

(c) Use the iterative formula with initial value $x_1 = 1.5$ to calculate the root correct to 3 d.p. Give the results of each iteration to 5 d.p. [S-20/33/Q6] --[3]



Solution: (a)

$y = 2 + x$ —— ①
$y = x^5$ —— ②

$\Big\}\ x^5 = 2 + x$ at P.

The graphs of eqn's ① and ② intersect at only one point P. Hence eqn $x^5 = 2 + x$ has exactly one root.

(b) Now consider the equation

$$x = \frac{4x^5 + 2}{5x^4 - 1}$$

$\Rightarrow x(5x^4 - 1) = 4x^5 + 2$

$\Rightarrow 5x^5 - x = 4x^5 + 2$

$\Rightarrow x^5 = x + 2$ is the equation in part (a).

(c) $x_{n+1} = \dfrac{4x_n^5 + 2}{5x_n^4 - 1}$

Now let $x_1 = 1.5$

$x_{n+1} = \left(\dfrac{4x_n^5 + 2}{5x_n^4 - 1}\right)$

$x_2 = 1.331619$

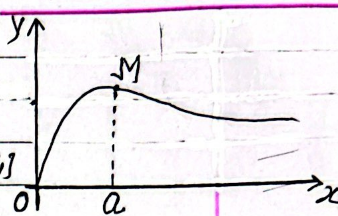
$x_3 = 1.273516$

$x_4 = 1.267236$

$\left.\begin{array}{l} x_5 = 1.267168 \\ x_6 = 1.267168 \end{array}\right\} \checkmark$

$x = \underline{1.267}$ ✓

9. The diagram shows the curve $y = \dfrac{\tan^{-1}x}{\sqrt{x}}$ and its maximum point M, where $x = a$



(a) Show that a satisfies the equation
$$a = \tan\left(\frac{2a}{1+a^2}\right) \quad \text{---[4]}$$
(b) Verify by calculation that a lies, between 1.3 and 1.5 ---[2]

(c) Use an iterative formula base on the equation in part (a) to determin a Correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

S-21/31/Q7 ---[3]

Solution:
(a) $y = \dfrac{\tan^{-1}x}{\sqrt{x}} \Rightarrow \dfrac{dy}{dx} = \dfrac{\sqrt{x}\cdot\frac{d}{dx}\tan^{-1}x - \tan^{-1}x\cdot\frac{d}{dx}\sqrt{x}}{(\sqrt{x})^2} = \dfrac{\sqrt{x}\cdot\frac{1}{1+x^2} - \tan^{-1}x\cdot\frac{1}{2\sqrt{x}}}{x}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2x - (1+x^2)\cdot\tan^{-1}x}{2\sqrt{x}(1+x^2)} \Rightarrow \left(\dfrac{dy}{dx}\right)_{x=a} = \dfrac{2a-(1+a^2)\tan^{-1}a}{2\sqrt{a}(1+a^2)} = 0$ as Max at $x = a$

$\Rightarrow 2a - (1+a^2)\tan^{-1}a = 0 \Rightarrow \tan^{-1}a = \dfrac{2a}{1+a^2}$

$\Rightarrow a = \tan\left(\dfrac{2a}{1+a^2}\right)$ ✓

(b) Consider $\tan\left(\dfrac{2a}{1+a^2}\right) - a = 0$; Let $p(a) = \tan\left(\dfrac{2a}{1+a^2}\right) - a$ —(1)

from (1)
$p(1.3) = \tan\left(\dfrac{2.6}{2.69}\right) - 1.3 = 0.1484$ —(2)

and $p(1.5) = \tan\left(\dfrac{3}{3.25}\right) - 1.5 = -0.178$ —(3)

$p(a)$ changes sign when a varies from 1.3 to 1.5 (from (2) & (3))

Hence a lies between 1.3 and 1.5.

(c) Now using the Iterative process;
$$a_{n+1} = \tan\left(\frac{2a_n}{1+a_n^2}\right)$$
Let $a_0 = 1.3 \Rightarrow a_1 = \tan\left(\dfrac{2.6}{1+(1.3)^2}\right) = \tan\left(\dfrac{2.6}{2.69}\right) = 1.4484$

(*) Hence $a = 1.39$ ✓

Calculor entries:

1.3	(Execute)
	1.3
$\tan\left(\dfrac{2\,ans}{1+ans^2}\right)$	(Execute)
	1.4484
	1.4484
	1.3552

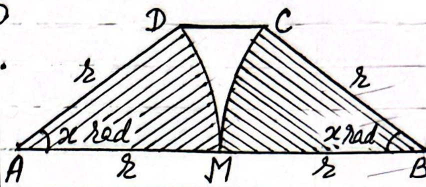
$a_2 = 1.4484$
$a_3 = 1.3552$
$a_4 = 1.4148$
$a_5 = 1.3769$
$a_6 = 1.4011$
$a_7 = 1.3857$

$a_8 = 1.3955$
$a_9 = 1.3892$
$a_{10} = 1.3933$
$a_{11} = 1.3907$
$a_{12} = 1.3923$
$a_{13} = 1.3913$

$a_{14} = 1.3920$
$a_{15} = 1.3915$
$a_{16} = 1.3918$
$a_{17} = 1.3916$
$a_{18} = 1.3917$ ✓
$a_{19} = 1.3917$ ✓
$a_{20} = 1.3917$ ✓

$\therefore a = 1.39$ (*)

10. The diagram shows a trapezium ABCD in which $AD = BC = r$ and $AB = 2r$. The acute angles BAD and ABC are both equal to x-radians. Circular arcs of radius r with centre A and B meet at M, the mid point of AB.

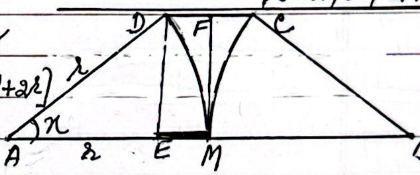


(a) Given that the sum of the areas of the shaded sectors is 90% of the area of the trapezium, show that x satisfies the equation:
$$x = 0.9(2-\cos x)\sin x \quad ---[3]$$
(b) Verify by calculation that x lies between 0.5 and 0.7 ---[2]

(c) Show that if a sequence of values in the interval $0 < x < \frac{1}{2}\pi$, given by the iterative formula: $x_{n+1} = \cos^{-1}\left(2 - \dfrac{x_n}{0.9\sin x_n}\right)$ converges, then it converges to the root of the equation in part (a). ---[2]

(d) Use this iterative formula to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. --[3]

[S-21/32/Q10]



Solution (a) $CD = 2DF = 2EM = 2(r - r\cos x)$ ✓

$\begin{cases} \text{Area of Trap } ABCD = \frac{1}{2} \times r\sin x\,[2(r - r\cos x) + 2r] \\ \text{Shaded area} = 2 \times \frac{1}{2} r^2 x \end{cases}$

$\Rightarrow r^2 x = \frac{90}{100} \times r^2 \sin x\,(2-\cos x)$

$\Rightarrow x = 0.9\sin x(2-\cos x)$ ✓

(b) Consider $p(x) = x - 0.9\sin x\,(2-\cos x)$

$P(.5) = 0.01569$, $P(.7) = -0.161$

$P(x)$ changes sign as x changes from 0.5 to 0.7 $\Rightarrow 0.5 < x < 0.7$ ✓

(c) $\cos x = 2 - \dfrac{x}{0.9\sin x}$

$\Rightarrow \dfrac{x}{0.9\sin x} = (2-\cos x)$

$\Rightarrow x = 0.9(2-\cos x)\sin x$

(as given in part (a)) ✓

(d) Iterative formulas
$$x_{n+1} = \cos^{-1}\left[2 - \frac{x_n}{0.9\sin x_n}\right]$$
Let $x_0 = 0.5$, $x_1 = \cos^{-1}\left[2 - \dfrac{0.5}{0.9\sin 0.5}\right]$
$= 0.5712$ ✓

$x_2 = 0.5987$
$x_3 = 0.6100$
$x_4 = 0.6148$
$x_5 = 0.6169$
$x_6 = 0.6178$
$x_7 = 0.6181$
$x_8 = 0.6183$
$x_9 = 0.6184$ ✓
$x_{10} = 0.6184$ ✓
$x_{11} = 0.6184$ ✓

$\therefore x = 0.6184$

$x = 0.62$ (2dp) ✓

11 (a) By sketching a suitable pair of graphs, show that the equations $\cot\frac{1}{2}x = 1+e^{-x}$ has exactly one root in the interval $0<x\leq\pi$. --- [2]

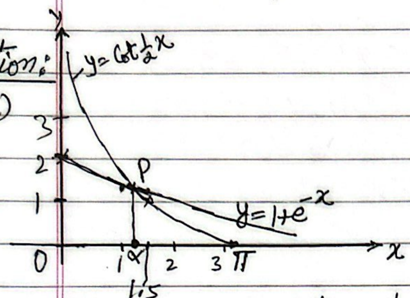
(b) Verify by calculation that this root lies between 1 and 1.5 --- [2]

(c) Use iterative formula $x_{n+1} = 2\tan^{-1}\left(\frac{1}{1+e^{-x_n}}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. --- [3]

[S-21/33/Q6]

Solution:

(a)



The two curves intersect at exactly one point $0<x\leq\pi$. ✓

(b) $\cot x/_2 = 1+e^{-x}$

consider $P(x) = \cot x/_2 - 1 - e^{-x}$

$$\begin{cases} P(1) = 1.83 - 1 - 0.3678 = 0.462 \\ P(1.5) = 1.073 - 1 - 0.223 = -0.15 \end{cases}$$

$P(x)$ changes sign when x changes values from 1 to 1.5.

$\therefore$ $1 < x < 1.5$ ✓

Calculator:

	execute
1	1
$2\tan^{-1}\left(\frac{1}{1+e^{-Ans}}\right)$	1.2625
	⋮
	$x_8 = 1.3403$ ✓

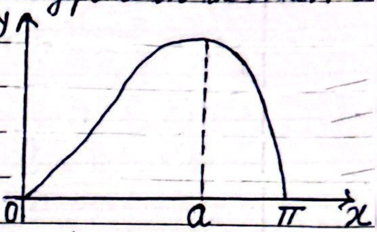
(c) Iterative formula

$$x_{n+1} = 2\tan^{-1}\left(\frac{1}{1+e^{-x}}\right)$$

Let $x_0 = 1$,
$x_1 = 1.2625$
$x_2 = 1.3241$
$x_3 = 1.3370$
$x_4 = 1.3396$
$x_5 = 1.3402$
$x_6 = 1.3403$ ✓
$x_7 = 1.3403$ ✓
$x_8 = 1.3403$ ✓

$\therefore$ $x = 1.34$ ✓ (2dp)

12 The curve $y = x\sqrt{\sin x}$ has one stationary point in the interval $0 < x < \pi$, where $x = a$.



(a) Show that $\tan a = -\frac{1}{2}a$ ----[4]

(b) Verify by calculation that a lies between 2 and 2.5. ---[2]

(c) Show that if a sequence of values in the interval $0 < x < \pi$ given by the iterative formula: $x_{n+1} = \pi - \tan^{-1}\left(\frac{1}{2}x_n\right)$ converges to a, the root of the equation in part (a). ---[2]

(d) Use the iterative formula given in part (c) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. --[3]

S-22/31/Q10

<u>Solution:</u> $y = x\sqrt{\sin x}$ ---- ①

(a) $\frac{dy}{dx} = 1\cdot\sqrt{\sin x} + \frac{x\cdot\cos x}{2\sqrt{\sin x}} = 0$ for stationary pt.

$\Rightarrow \sqrt{\sin x} = -\frac{x\cos x}{2\sqrt{\sin x}}$

$\Rightarrow \tan x = -x/2$

at $x = a$, $\tan a = -a/2$ ✓

(b) Consider $f(x) = \tan x + \frac{1}{2}x$

at $x = 2$, $f(2) = \tan 2 + \frac{1}{2}\times 2$
$= -2.18 + 1 = -1.18$

at $x = 2.5$, $f(2.5) = \tan 2.5 + \frac{1}{2}\times 2.5$
$= -0.747 + 1.25$
$= 0.503$

$f(x)$ changes sign as x changes from 2 to 2.5 $\Rightarrow 2 < a < 2.5$

(c) $\tan x = -\frac{1}{2}x$

$\Rightarrow x = \pi - \tan^{-1}\frac{1}{2}x$

$\Rightarrow x_{n+1} = \pi - \tan^{-1}\left(\frac{1}{2}x_n\right)$ ✓

(d) using iterative formula.

$x_{n+1} = \pi - \tan^{-1}\left(\frac{1}{2}x_n\right)$

Let $x_0 = 2$

$x_1 = 2.3561$
$x_2 = 2.2746$
2.2920
2.2882
2.2890
2.2888
2.2889
2.2889
2.2889 ✓

$\therefore a = 2.29$ (2 s.f)

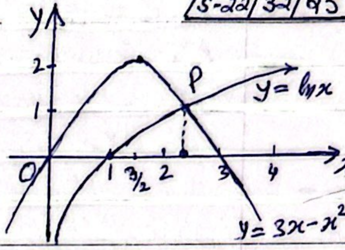
Calculator window

2	
	2
$\pi - \tan^{-1}(\frac{1}{2}\text{ans})$	
	2.3561

13(a) By sketching a suitable pair of graphs, show that the equation $\ln x = 3x - x^2$ has one real root. --[2]
(b) Verify by calculation that the root lies between 2 and 2.8. --[2]
(c) Use the iterative formula $x_{n+1} = \sqrt{3x_n - \ln x_n}$ to determine the root correct to 2 decimal place. Give the result of each iteration to 4 d.p. ---[3]

[S-22/32/Q5]

Solution (a) $\ln x = 3x - x^2$ $\begin{cases} \text{Consider } y = \ln x \\ \text{and } y = 3x - x^2 \end{cases}$



The two graphs intersect at one point P.
$\therefore$ given equation has one root.

(b) Consider $f(x) = \ln x - (3x - x^2)$

$f(2) = \ln 2 - (3\times 2 - 2^2) = 0.6931 - 2 = -1.306$

$f(2.8) = \ln 2.8 - (3\times 2.8 - 2.8^2) = 1.029 - 0.56) = 0.469$

$f(x)$ changes sign from −ve to + as x change from 2 to 2.8

$\Longrightarrow$ $2 < \text{root} < 2.8$

(c) consider the iterative formula:

$x_{n+1} = \sqrt{3x_n - \ln x_n}$

Now let $x_0 = 2$ $\quad x_1 = \sqrt{3\times 2 - \ln 2} =$

$\Rightarrow x_1 = 2.3036$

$x_2 = 2.4650$

$x_3 = 2.5481$

$x_4 = 2.5901$

$x_5 = 2.6112$

$x_6 = 2.6218$

$x_7 = 2.6270$

$x_8 = 2.6297$

$x_9 = 2.6310$

$x_{10} = 2.6316$

$x_{11} = 2.6319$

$x_{12} = 2.6321$

$x_{13} = 2.6322$

$x_{14} = 2.6322$

$x_{15} = 2.6322$

Calculator

2	2
$\sqrt{3\text{ans} - \ln\text{ans}}$	2.3036

$\therefore$ Required root $= 2.63$ ✓

14. The constant a is such that, $\int_1^a x^2 \ln x \, dx = 4$

(a) Show that $a = \left(\dfrac{35}{3\ln a - 1}\right)^{1/3}$ ...[5]

(b) Verify by calculation that a lies between 2.4 and 2.8 ...[2]

(c) Use an iterative formula based on the equation in part (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 d.p. ...[3]

S-22/33/Q10

Solution:

(a) Consider $\int x^2 \ln x \, dx = \ln x \cdot \int x^2 dx - \int\left(\frac{d}{dx}\ln x \cdot \int x^2 dx\right)dx$

$= \ln x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} dx = \ln x \cdot \frac{x^3}{3} - \frac{1}{3}\int x^2 dx$

$= \frac{x^3}{3}\ln x - \frac{1}{3} \cdot \frac{x^3}{3} = \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right)$

$\therefore \int_1^a x^2 \ln x \, dx = \left[\frac{x^3}{3}\left(\ln x - \frac{1}{3}\right)\right]_1^a =$

$= \frac{a^3}{3}\left(\ln a - \frac{1}{3}\right) - \frac{1}{3}\left(0 - \frac{1}{3}\right) = 4$ (given)

$\Rightarrow \frac{a^3}{9}(3\ln a - 1) + \frac{1}{9} = 4$

$\Rightarrow a^3(3\ln a - 1) = 9\left(4 - \frac{1}{9}\right) = 35$

$\therefore a^3 = \frac{35}{(3\ln a - 1)} \Rightarrow a = \left(\frac{35}{3\ln a - 1}\right)^{1/3}$ ✓

(b) Consider $f(y) = y^3(3\ln y - 1) - 35$ ---- ①

for $y = 2.4 \Rightarrow f(2.4) = (2.4)^3[3 \times \ln 2.4 - 1] - 35 = -12.51$

for $y = 2.8 \Rightarrow f(2.8) = (2.8)^3[3\ln 2.8 - 1] - 35 = 10.85$

Here we find $f(y)$ changes sign from $-$ve to $+$ve as y changes from 2.4 to 2.8, hence $2.4 < a < 2.8$ ✓

(c) Using the iterative formula: $a_{n+1} = \left(\frac{35}{3\ln a - 1}\right)^{1/3}$

Let $a_0 = 2.4 \Rightarrow a_1 = 2.7815$

$a_2 = 2.5670$	$a_9 = 2.6377$
$a_3 = 2.6751$	$a_{10} = 2.6365$
$a_4 = 2.6173$	$a_{11} = 2.6371$
$a_5 = 2.6472$	$a_{12} = 2.6368$
$a_6 = 2.6315$	$a_{13} = 2.6369$
$a_7 = 2.6397$	$a_{14} = 2.6369$
$a_8 = 2.6354$	$a_{15} = 2.6369$

✓

$\therefore a = 2.64$ (2 dp)

15. The constant a is such that $\int_0^a x e^{-2x} = \frac{1}{8}$

(a) Show that $a = \frac{1}{2}\ln(4a+2)$ ---[5]

(b) Verify by calculation that a lies between 0.5 and 1. ---[2]

(c) Use iterative formula based on the equation in (a) to determine a correct to 2d.p. Give result of each iteration to 4 decimal places. --[3]

S-23 | 31 | Q 9

Solution (a) Consider $\int x e^{-2x}\,dx$

$\left\{\int u\frac{dv}{dx} = u.v - \int\left(v.\frac{du}{dx}\right)dx\right.$

$\left\{\text{Let } u = x \Rightarrow \frac{du}{dx} = 1\right.$

$\left\{\text{and } \frac{dv}{dx} = e^{-2x} \Rightarrow v = \frac{e^{-2x}}{-2}\right.$

$$= x.\frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2} \times 1\,dx$$

$$= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx$$

$$= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\times\frac{e^{-2x}}{-2} = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} \quad ---(1)$$

Hence $\int_0^a xe^{-2x}dx = \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^a = \frac{1}{8}$ (Given and (1))

$\Rightarrow \left(-\frac{1}{2}ae^{-2a} - \frac{1}{4}e^{-2a}\right) - \left(0 - \frac{1}{4}\right) = \frac{1}{8} \Rightarrow -\frac{1}{2}e^{-2a}\left(a+\frac{1}{2}\right) = -\frac{1}{8}$

$\Rightarrow e^{-2a}\left(a+\frac{1}{2}\right) = \frac{1}{4} \Rightarrow 4a+2 = e^{2a} \Rightarrow 2a = \ln(4a+2)$

$\Rightarrow a = \frac{1}{2}\ln(4a+2)$ --(2)

(b) from (2)

Consider $f(a) = \frac{1}{2}\ln(4a+2) - a$

$\Rightarrow f(0.5) = \frac{1}{2}\ln 4 - 0.5 = 0.193 > 0$ ---(3)

And $f(1) = \frac{1}{2}\ln 6 - 1 = -0.104 < 0$ ---(4)

when a lies between 0.5 and 1 $f(a)$ changes sign. (from (3) & (4))

$\therefore$ a lies between 0.5 and 1.

(c) Now using iterative formula $a_{n+1} = \frac{1}{2}\ln(4a_n+2)$ ---(5)

Let $a_0 = 0.5$ from (5)

$a_1 = 0.6931$
$a_2 = 0.7814$
$a_3 = 0.8171$
$a_4 = 0.8309$
$a_5 = 0.8360$

$a_6 = 0.8380$
$a_7 = 0.8387$
$a_8 = 0.8390$
$a_9 = 0.8391$
$\left.\begin{aligned} a_{10} &= 0.8391 \\ a_{11} &= 0.8391 \end{aligned}\right\}$

Calculator window.
0.5
0.5
$\frac{1}{2}\ln(4\,\text{ans}+2)$
0.6931

$\Rightarrow a = 0.84$ ✓

16\. The equation $\cot \frac{1}{2}x = 3x$ has one root in the interval $0 < x < \pi$, denoted by α.

(a) Show by calculation that α lies between 0.5 and 1. ---[2]

(b) Show that, if a sequence of positive values given by the iterative formula, $x_{n+1} = \frac{1}{3}\left(x_n + 4\tan^{-1}\left(\frac{1}{3x_n}\right)\right)$ converges, then it converges to α. ---[2]

(c) Use this iterative formula to calculate α correct to 2 d.p. Give the result of each iteration to 4 decimal places. ---[3]

S-23/32/Q6

Solution: $\cot x/2 = 3x$ ---①

(a) Consider $f(x) = 3x - \cot x/2$ ---②

Now $f(0.5) = 3\times0.5 - \cot\frac{0.5}{2} = -2.93$

$f(1) = 3\times1 - \cot 0.5 = +1.169$

When x lies between 0.5 and 1 $f(x)$ changes sign.

Hence α lies between 0.5 and 1.

(b) Consider:

$$x = \frac{1}{3}\left(x + 4\tan^{-1}\left(\frac{1}{3x}\right)\right) \Rightarrow 3x = x + 4\tan^{-1}\left(\frac{1}{3x}\right)$$

$$\Rightarrow 2x = 4\tan^{-1}\left(\frac{1}{3x}\right) \Rightarrow \tan x/2 = \frac{1}{3x}$$

$$\Rightarrow \cot x/2 = 3x \text{ given result}$$

Hence the given iterative formula will converge to α.

(c) Given iterative formula $x_{n+1} = \frac{1}{3}\left[x_n + 4\tan^{-1}\left(\frac{1}{3x_n}\right)\right]$

Take $x_0 = 0.5$

$\Rightarrow x_1 = 0.9506$

$x_2 = 0.7665$

$x_3 = 0.8024$

$x_4 = 0.7924$

$x_5 = 0.7950$

$x_6 = 0.7944$

$x_7 = 0.7945$

$x_8 = 0.7945$

$x_9 = 0.7945$

$x_{10} = 0.7945$

Calculator Window:

0.5

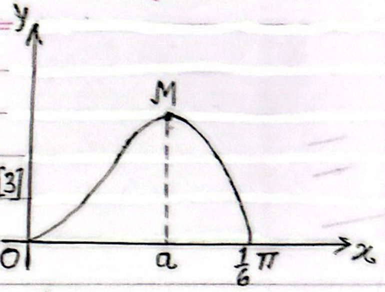
0.5

$\frac{1}{3}\left[\text{ans} + 4\tan^{-1}\left(\frac{1}{3\,\text{ans}}\right)\right]$

0.9506

$\therefore \alpha = 0.79$ ✓

17 The diagram shows the part of the curve $y = x^2 \cdot \cos 3x$ for $0 \le x \le \frac{1}{6}\pi$, and its maximum point at M, where $x = a$.



(a) Show that a satisfies the equation, ---[3]

$$a = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a}\right).$$

(b) Use an iterative formula on the equation in (a) to determine a correct to 2 d.p. Give the result of each iteration to 4 decimal places. ---[3]

S-23/33/Q5

Solution (a) $y = x^2 \cdot \cos 3x$

$$\frac{dy}{dx} = 2x\cos 3x - 3x^2 \sin 3x$$

$$\left(\frac{dy}{dx}\right)_{x=a} = 2a\cos 3a - 3a^2\sin 3a = 0 \text{ for Max.}$$

$$\Rightarrow 3a^2\sin 3a = 2a\cos 3a \Rightarrow \tan 3a = \frac{2}{3a} = 3a = \tan^{-1}\left(\frac{2}{3a}\right)$$

$$\Rightarrow a = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a}\right)$$

(b) from part (a) Using the iterative formula: $a_{n+1} = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a_n}\right)$

$0 < 0.5 < \frac{1}{6}\pi$

Let $a_0 = 0.5$

$a_1 = 0.3091$
$a_2 = 0.3789$
$a_3 = 0.3513$
$a_4 = 0.3619$
$a_5 = 0.3578$
$a_6 = 0.3594$
$a_7 = 0.3587$
$a_8 = 0.3590$

$\left.\begin{array}{l} a_9 = 0.3589 \\ a_{10} = 0.3589 \\ a_{11} = 0.3589 \end{array}\right\}$

Calculator window:

0.5

0.5

$\frac{1}{3}\tan^{-1}\left(\frac{2}{3\,\text{ans}}\right)$

0.3091

Hence $\underline{a = 0.36}$

18(a) By sketching a suitable pair of graphs, show that the equation $\operatorname{cosec} x = 1 + e^{-\frac{1}{2}x}$ has exactly two roots in the interval $0 < x < \pi$ --- [2]

(b) The sequence of values given by the iterative formula.

$$x_{n+1} = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}x_n} + 1}\right)$$

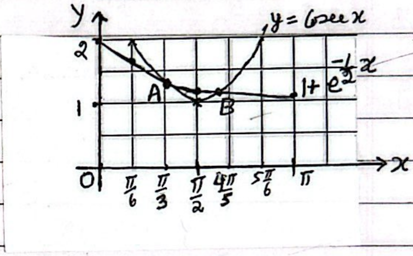
with initial value $x_1 = 2$, converges to one of these roots. Use the formula to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [W-20/31/Q5] --[3]

Solution:

(a) By sketing the graphs

$y = \operatorname{cosec} x, \quad 0 < x < \pi$

and $y = 1 + e^{-\frac{1}{2}x} \quad 0 < x < \pi$



The two curves intersect at two points A and B, has the equation $\operatorname{cosec} x = 1 + e^{-\frac{1}{2}x}$ has two roots.

(b)

$$x_{n+1} = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}x_n} + 1}\right)$$

$x_1 = 2$ $\quad x_{n+1} = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}x_n} + 1}\right)$

$x_2 = 2.3217$

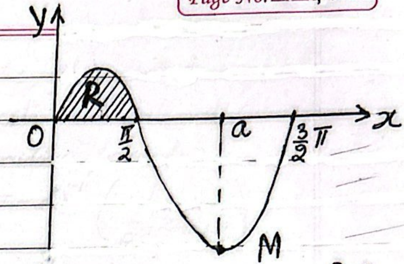
$x_3 = 2.2760$

$$\left.\begin{aligned} x_4 &= 2.2824 \\ x_5 &= 2.2815 \\ x_6 &= 2.2816 \end{aligned}\right\}$$

$\therefore \quad x = 2.28$ to 2 decimal places.

Scanned with CamScanner

19. The diagram shows the curve $y=\sqrt{x}\cos x$, for $0 \leqslant x \leqslant \frac{3}{2}\pi$, and its minimum point M, where $x=a$. The shaded region between the curve and the x-axis is denoted by R.



(a) Show that a satisfies the equation $\tan a = \frac{1}{2a}$ --- [3]

(b) The sequence of values given by the iterative formula $a_{n+1} = \pi + \tan^{-1}\left(\frac{1}{2a_n}\right)$, with initial value $x_1 = 3$, converges to a.
Use this formula to determine a correct to 2 decimal places.
Give the result of each iteration to 4 decimal places. ---[3]

(c) Find the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π. ---[6]

[W-20/32/Q 10]

Solution: $y = \sqrt{x}\cdot\cos x$ for $0 \leqslant x \leqslant \frac{3}{2}\pi$

(a) diff. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\cos x + \sqrt{x}(-\sin x)$

Now $\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{2\sqrt{a}}\cos a - \sqrt{a}\sin a = 0$ for Min at M.

$\Rightarrow \sqrt{a}\sin a = \frac{1}{2\sqrt{a}}\cos a$

$\Rightarrow \tan a = \frac{1}{2a}$ ✓

(b) $x_1 = 3$, $a_{n+1} = \pi + \tan^{-1}\left(\frac{1}{2a_n}\right)$

$x_2 = 3.3067$

$x_3 = 3.2917$

$x_4 = 3.2923$

$x_5 = 3.2923$

$\therefore$ $a = 3.29$

(c) Volume $= \pi\int y^2\,dx$

$= \pi\int(\sqrt{x}\cos x)^2\,dx$

$= \pi\int x\cdot\cos^2 x\,dx$

$= \pi\int x\left(\frac{1+\cos 2x}{2}\right)dx$

$= \frac{\pi}{2}\int(x + x\cos 2x)\,dx$

$= \frac{\pi}{2}\left[\frac{x^2}{2} + x\int\cos 2x\,dx - \int\left(\frac{d}{dx}x\int\cos 2x\,dx\right)dx\right.$

$= \frac{\pi}{2}\left[\frac{x^2}{2} + \frac{x\cdot\sin 2x}{2} - \int 1\times\frac{\sin 2x}{2}\,dx\right.$

$= \frac{\pi}{2}\left[\frac{x^2}{2} + \frac{x\sin 2x}{2} - \left(-\frac{\cos 2x}{4}\right)\right]$

$= \frac{\pi}{2}\left[\frac{x^2}{2} + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]$

$\therefore$ The required Volume

$= \frac{\pi}{2}\left[\frac{x^2}{2} + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}\left[\left(\frac{\pi^2}{8} + 0 - \frac{1}{4}\right) - \left(\frac{1}{4}\right)\right]$

$= \frac{\pi}{2}\left[\frac{\pi^2}{8} - \frac{2}{4}\right] = \frac{\pi}{16}\cdot(\pi^2 - 4)$ ✓

20 The constant a is such that: $\int_1^a \frac{\ln x}{\sqrt{x}}\, dx = 6$

Note: $[\exp(x) = e^x]$

(a) Show that $a = \exp\left(\frac{1}{\sqrt{a}} + 2\right)$ ---- [5]

(b) Verify by calculation that a lies between 9 and 11. ---[2]

(c) Use an iterative formula base on the equation in part (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. --[3]

[W-21/31/Q8]

Solution (a) $\int_1^a \ln x \cdot x^{-\frac{1}{2}} dx = \left[2\ln x \cdot x^{1/2}\right]_1^a - \int_1^a \left(\frac{d}{dx}\ln x \cdot \frac{x^{1/2}}{1/2}\right) dx$

$= (2\sqrt{a}\ln a - 0) - \int_1^a \frac{1}{x} \cdot 2\sqrt{x}\, dx$

$= 2\sqrt{a}\ln a - 2\int_1^a x^{-1/2} dx = 2\sqrt{a}\ln a - 2\left[\frac{x^{+1/2}}{+1/2}\right]_1^a$

$= 2\sqrt{a}\ln a - 2[2\sqrt{x}]_1^a = 2\sqrt{a}\ln a - 4(\sqrt{a} - 1)$

$\Rightarrow 2\sqrt{a}\ln a - 4\sqrt{a} + 4 = 6$ (Given) $\Rightarrow 2\sqrt{a}\ln a = 2 + 4\sqrt{a}$

$\Rightarrow \ln a = \frac{1}{\sqrt{a}} + 2 \Rightarrow a = \mathrm{Exp}\left(\frac{1}{\sqrt{a}} + 2\right)$

(b) Consider $P(a) = a - \mathrm{Exp}\left(\frac{1}{\sqrt{a}} + 2\right)$

Now $P(9) = 9 - \mathrm{Exp}\left(\frac{1}{3} + 2\right) = 9 - 10.308 = -1.308$

and $P(11) = 11 - \mathrm{Exp}\left(\frac{1}{\sqrt{11}} + 2\right) = 11 - 9.889 = 1.010$

$P(a)$ changes sign as a changes from 9 to 11 $\Rightarrow$ $9 < a < 11$.

(c) Consider the Iterative formula: $a_{n+1} = \mathrm{Exp}\left(\frac{1}{\sqrt{a_n}} + 2\right)$

Let $a_0 = 9 \Rightarrow a_1 = \mathrm{Exp}\left(\frac{1}{\sqrt{9}} + 2\right) = 10.3122$

Calculator:

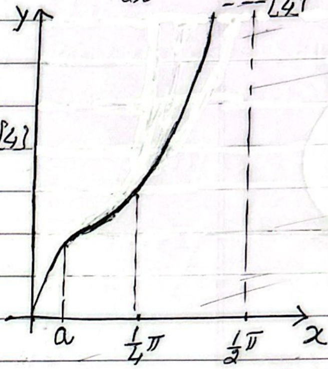
	execute
9	9
$\mathrm{Exp}\left(\frac{1}{\sqrt{\mathrm{ans}}} + 2\right)$	10.3122
	10.0885
	⋮

$a_2 = 10.0885$
$a_3 = 10.1232$
$a_4 = 10.1178$
$a_5 = 10.1186$
$a_6 = 10.1185$
$a_7 = 10.1185$
$a_8 = 10.1185$

$\Rightarrow a = 10.1185$ or $a = 10.12$ (2dp)

21. The equation of a curve is $y=\sqrt{\tan x}$, for $0 \leq x < \frac{1}{2}\pi$

(a) Express $\frac{dy}{dx}$ in terms of $\tan x$, and verify that $\frac{dy}{dx}=1$, where $x=\frac{1}{4}\pi$. --- [4]



The value of $\frac{dy}{dx}$ is also 1 at another point on the curve where $x=a$.

(b) Show that: $t^3+t^2+3t-1=0$, where $t=\tan a$ --- [4]

(c) Use iterative formula:

$$a_{n+1}=\tan^{-1}\left(\frac{1}{3}\left(1-\tan^2 a_n-\tan^3 a_n\right)\right)$$

to determine a correct to 2 decimal places, giving the result of each iteration to 4 decimal places. --- [3]

[W-21/32/Q 11]

Solution:

(a) $y=\sqrt{\tan x} \Rightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{\tan x}}\sec^2 x$

or $\frac{dy}{dx}=\frac{1+\tan^2 x}{2\sqrt{\tan x}}$ ✓

$\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}}=\frac{1+1}{2\sqrt{1}}=1$ ✓

(b) Given $\frac{dy}{dx}=\frac{1+\tan^2 x}{2\sqrt{\tan x}}=1$ (Given) —— (1)

$\Rightarrow (1+\tan^2 x)^2=4\tan x$

$\Rightarrow 1+\tan^4 x+2\tan^2 x=4\tan x$ —— (2)

$\Rightarrow t^4+2t^2-4t+1=0$ (Let $\tan x=t$)

Given $\tan a=t=1 \Rightarrow (t-1)$ is a factor of (2)

$$\begin{array}{l} t-1\,\big)\, t^4+2t^2-4t+1 \,\big(\, t^3+t^2+3t-1 \\ \quad -t^4 \mp t^3 \\ \quad t^3+2t^2 \\ \quad -t^3 \mp t^2 \\ \quad\quad 3t^2-4t \\ \quad\quad -3t^2 \mp 3t \\ \quad\quad\quad -t+1 \\ \quad\quad\quad -t+1 \\ \quad\quad\quad\quad \times \end{array}$$

from (2)

$\therefore (t-1)(t^3+t^2+3t-1)=0 \Rightarrow t^3+t^2+3t-1=0$ ✓

Use the Iterative formula:

$a_{n+1}=\tan^{-1}\left(\frac{1}{3}\left(1-\tan^2 a_n-\tan^3 a_n\right)\right)$

Let $a_0=0$

$a_1=0.3217$
$a_2=0.2800$
$a_3=0.2912$
$a_4=0.2884$
$a_5=0.2891$
$a_6=0.2890$ ✓
$a_7=0.2890$ ✓
$a_8=0.2890$ ✓

$\therefore a=0.2890$

or $a=0.29$ (2dp)

22 A large plantation of area 20 Km^2 is becoming infected with a plant disease. At time t years the area infected is $x\text{ Km}^2$ and the rate of increase of x is proportional to the ratio of the area infected to the area not yet infected. When $t=0$, $x=1$ and $\frac{dx}{dt}=1$.

(a) Show that x and t satisfy the differential equation: $\frac{dx}{dt}=\frac{19x}{20-x}$ --[2]

(b) Solve the diff. equation and show that when $t=1$, the value of x satisfy the equation; $x=e^{0.9+0.05x}$ --[5]

(c) Use an iterative formula based on the equation in part (b), with an initial value of 2, to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. --[3]

(d) Calculate the value of t at which the entire plantation becomes inflected. --[1]

W-21/33/Q 10

Solution (a) $\frac{dx}{dt}\propto\frac{x}{(20-x)} \Rightarrow \frac{dx}{dt}=\frac{k.x}{(20-x)}$ --- ①

$x=1, \frac{dx}{dt}=1 \Rightarrow 1=\frac{k\times1}{19} \Rightarrow k=19$

$\therefore$ from ① $\frac{dx}{dt}=\frac{19x}{(20-x)}$ ---②

(b) $\frac{dx}{dt}=\frac{19x}{20-x} \Rightarrow \int\frac{20-x}{19x}\,dx=\int dt$

$\Rightarrow \frac{1}{19}\int\left(\frac{20}{x}-1\right)dx=t+c$

$\Rightarrow \frac{1}{19}(20\ln x-x)=t+c$ --- ③

when $x=1$, $t=0$

$\frac{1}{19}(-1)=c \Rightarrow c=-\frac{1}{19}$

fr ③ $\frac{1}{19}(20\ln x-x)=t-\frac{1}{19}$

Now for $t=1$

$\Rightarrow \frac{1}{19}(20\ln x-x)=\frac{18}{19}$

$\Rightarrow 20\ln x-x=18$

$20\ln x=x+18$

$\ln x=0.9+0.05x$

$\Rightarrow x=e^{(0.9+0.05x)}$

(c) Using the Iterative formula;

$x_{n+1}=e^{(0.9+0.05x_n)}$

$x_0=2$, $x_1=2.7182$

$x_2=2.8176$

$x_3=2.8317$

$x_4=2.8337$

$x_5=2.8339$

$x_6=2.8340$

$x_7=2.8340$

$x_8=2.8340$

$\therefore$ $x=2.83$ (2dp)

(d) Entire plantation is infected for $x=20$

from ③ $t=\frac{1}{19}(20\ln x-x)+\frac{1}{19}$

$=\frac{1}{19}[20\ln x-x+1]$ --- ④

Now for $x=20$ in ④

$t=\frac{1}{19}[20.\ln 20-20+1]$

$=\frac{1}{19}[20.\ln 20-19]$

$t=2.15$

23 The equation of a curve is $y = \frac{x}{\cos^2 x}$, for $0^\circ \leq x < \frac{1}{2}\pi$. At the point where $x = a$, the tangent to the curve has gradient equal to 12.

(a) Show that $a = \left(\sqrt[3]{\frac{\cos a + 2a\sin a}{12}}\right.$ ...[3]

(b) Verify by calculation that a lies between 0.9 and 1. ---[2]

(c) Use an iterative formula based on the equation part (a) to determine a correct to 2d.p. Give result of each iteration to 4d.p. --[3]

W-22/31/Q7

Solution (a)

$$y = \frac{x}{\cos^2 x} \Rightarrow \frac{dy}{dx} = \frac{\cos^2 x \times 1 - x \cdot 2\cos x(-\sin x)}{\cos^4 x}$$

$$= \frac{\cos^2 x + 2\sin x \cos x}{\cos^4 x} = \frac{\cos x(\cos x + 2x\sin x)}{\cos^4 x}$$

∴ gradient at $x = a$

$$\left(\frac{dy}{dx}\right)_{x=a} = \frac{\cos a + 2a\sin a}{\cos^3 a} = 12 \text{ given}$$

$$\Rightarrow 12\cos^3 a = \cos a + 2a\sin a \Rightarrow \cos^3 a = \frac{\cos a + 2a\sin a}{12}$$

$$\Rightarrow \cos a = \left(\frac{\cos a + 2a\sin a}{12}\right)^{1/3} \Rightarrow a = \cos^{-1}\left(\sqrt[3]{\frac{\cos a + 2a\sin a}{12}}\right) ✓$$

(b) Consider $f(x) = \frac{\cos x + 2x\sin x}{\cos^3 x} - 12$

$x = 0.9 \to f(0.9) = \frac{\cos 0.9 + 2\times 0.9\sin 0.9}{\cos^3 0.9} - 12 = -3.5389$

$x = 1 \to f(1) = \frac{\cos 1 + 2\times 1\sin 1}{\cos^3 1} - 12 = 1.6719$

$f(x)$ change sign from −ve to + as x change from 0.9 to 1

Hence $0.9 < a < 1$ ✓

(c) Consider the iterative formula $a_{n+1} = \cos^{-1}\left(\sqrt[3]{\frac{\cos a_n + 2a_n \sin a_n}{12}}\right)$

Now for $x_0 = 0.9 \Rightarrow x_1 = 0.9845$

$x_2 = 0.9672$

$x_3 = 0.9707$

$x_4 = 0.9700$

$x_5 = .9702$

$x_6 = .9701$

$x_7 = .9701$

$x_8 = .9701$

∴ $a = 0.97$ ✓

On calculator

0.9 Enter

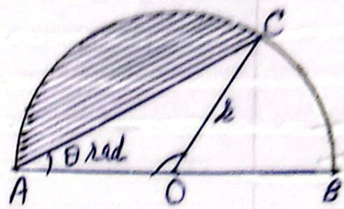
0.9

$\cos^{-1}\left(\sqrt[3]{\frac{\cos Ans + 2Ans \sin Ans}{12}}\right)$

0.9845

0.9672

24. The diagram shows a semicircle with diameter AB, centre O and radius r. The shaded region is the minor segment of the chord AC and its area is one-third of the area of the semicircle. The angle CAB is θ radians.



(a) Show that $\theta = \frac{1}{3}(\pi - 1.5\sin 2\theta)$ ---[4]

(b) Verify by calculation that $0.5 < \theta < 0.7$ ---[2]

(c) Use an iterative formula based on the equation in part (a) to determine θ correct to 3 d.p. Give the result of each iteration to 5 d.p.

W-22/32/Q9 --[3]

Solution:

(a) angle $AOC = (\pi - 2\theta)$; Area of the shaded segment

$= \text{area of Sector } AOC - \text{ar } \Delta AOC$

$= \frac{1}{2}r^2(\pi - 2\theta) - \frac{1}{2}r^2\sin(\pi - 2\theta)$

Given; Area of the shaded segment $= \frac{1}{3}$ area of semicircle

$\Rightarrow \frac{1}{2}r^2(\pi - 2\theta) - \frac{1}{2}r^2\sin(\pi - 2\theta) = \frac{1}{3}\times\frac{1}{2}\pi r^2$ $[\sin(\pi - 2\theta) = \sin 2\theta]$

$\Rightarrow (\pi - 2\theta) - \sin 2\theta = \frac{1}{3}\pi \Rightarrow 2\theta = \pi - \frac{1}{3}\pi - \sin 2\theta$

$2\theta = \frac{2}{3}\pi - \sin 2\theta$

$\Rightarrow \theta = \frac{1}{3}\pi - \frac{1}{2}\sin 2\theta \Rightarrow \theta = \frac{1}{3}(\pi - \frac{3}{2}\sin 2\theta)$

$\Rightarrow \theta = \frac{1}{3}(\pi - 1.5\sin 2\theta)$ ✓

(b)

Consider $f(\theta) = \frac{1}{3}(\pi - 1.5\sin 2\theta) - \theta$

$f(0.5) = \frac{1}{3}(\pi - 1.5\sin 1) - 0.5 = 0.1264$

and $f(0.7) = \frac{1}{3}(\pi - 1.5\sin 1.4) - 0.7 = -0.1455$

change of sign $\Rightarrow$ $0.5 < \theta < 0.7$ ✓

(c) On calculator

0.5 Enter 0.5

$\frac{1}{3}(\pi - 1.5\sin 2\text{Ans})$

0.62646

0.57224

(c) Using iterative formula: $\theta_{n+1} = \frac{1}{3}(\pi - 1.5\sin 2\theta_n)$

Put $\theta_0 = 0.5 \Rightarrow \theta_1 = 0.62646$

$\theta_2 = 0.57224$

$\theta_3 = 0.59194$

$\theta_4 = 0.58415$

$\theta_5 = 0.58715$

$\theta_6 = 0.58598$

$\theta_7 = 0.58643$

$\theta_8 = 0.58626$

$\theta_9 = 0.58633$

$\theta_{10} = 0.58630$

$\theta_{11} = 0.58631$

$\theta_{12} = 0.58631$

$\theta_{13} = 0.58631$ ✓

$\therefore \theta = 0.586$

25. The curve with equation $y = \frac{x^3}{e^x - 1}$ has a stationary point at $x = p$, where $p > 0$.

(a) Show that $p = 3(1 - e^{-p})$ ---[3]

(b) Verify by calculation that p lies between 2·5 and 3. ---[2]

(c) Use an iterative formula based on the equation in part (a) to determine p correct to 2 d.p. Give the result of each iteration to 4 d.p. ---[3]

[W-22/33/Q8]

Solution:

(a) $y = \frac{x^3}{e^x - 1} \Rightarrow \frac{dy}{dx} = \frac{(e^x-1)\cdot 3x^2 - x^3 \cdot e^x}{(e^x-1)^2}$

Stationary point at $x = p \Rightarrow \left(\frac{dy}{dx}\right)_{x=p} = \frac{(e^p-1)\cdot 3p^2 - p^3 e^p}{(e^p-1)^2} = 0$

$\Rightarrow p^2[3(e^p-1) - pe^p] = 0 \Rightarrow 3(e^p-1) - pe^p = 0 \Rightarrow pe^p = 3(e^p-1)$

$\Rightarrow p = \frac{3(e^p-1)}{e^p} \Rightarrow \underline{p = 3(1-e^{-p})}$ ✓

(b) Now consider $f(p) = 3(1-e^{-p}) - p$

$$\left.\begin{aligned} f(2.5) &= 3(1-e^{-2.5}) - 2.5 = 0.2537 \\ \text{and} \quad f(3) &= 3(1-e^{-3}) - 3 = -0.0497 \end{aligned}\right\}$$

Change of sign shows that → $\underline{2.5 < p < 3}$ ✓

(c) Using an iterative formula:

$$p_{n+1} = 3(1-e^{-p_n})$$

let $p_0 = 2.5 \Rightarrow$ $p_1 = 2.7531$

$p_2 = 2.8089$

$p_3 = 2.8191$

$p_4 = 2.8210$

$p_5 = 2.8213$

$$\left.\begin{aligned} p_6 &= 2.8214 \\ p_7 &= 2.8214 \\ p_8 &= 2.8214 \end{aligned}\right\}$$

Calculator

2.5 enter	2.5
$3(1-e^{-Ans})$	2.7534
	2.8089

$\therefore p = \underline{2.82}$ (2 d.p.)