DATE: 22.11.20

P-3

# Pure Maths-3

# Trigonometry Notes

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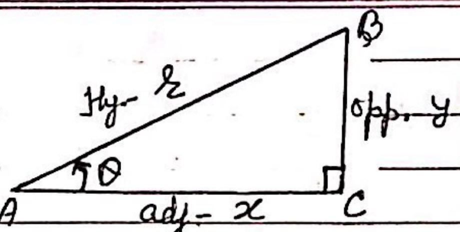
(P-3)

# Trigonometry - Notes.



## § Trigonometric Ratios:




Reciprocal trig - ratios.

(i) $\sin\theta = \frac{opp}{Hy} = \frac{y}{r} \longrightarrow \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{Hy}{opp} = \frac{r}{y}$

(ii) $\cos\theta = \frac{adj}{Hy} = \frac{x}{r} \longrightarrow \sec\theta = \frac{1}{\cos\theta} = \frac{Hy}{adj} = \frac{r}{x}$

(iii) $\tan\theta = \frac{opp}{adj} = \frac{y}{x} \longrightarrow \cot\theta = \frac{1}{\tan\theta} = \frac{adj}{opp} = \frac{x}{y}$

Note: $\frac{\sin\theta}{\cos\theta} = \tan\theta \rightarrow \frac{\cos\theta}{\sin\theta} = \cot\theta$

## § Trigonometric Identities:

1(i) $\sin^2\theta + \cos^2\theta = 1$
(ii) or $1 - \sin^2\theta = \cos^2\theta$
(iii) or $1 - \cos^2\theta = \sin^2\theta$

2.(i) $1 + \tan^2\theta = \sec^2\theta$
(ii) $\sec^2\theta - 1 = \tan^2\theta$
(iii) $\sec^2\theta - \tan^2\theta = 1.$

3.(i) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$
(ii) $\operatorname{cosec}^2\theta - 1 = \cot^2\theta$
(iii) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

(1) Proof: In the rt $\Delta ABC$,
Using Pythagoras theorem
$$y^2 + x^2 = r^2$$
Div. by $r^2 \rightarrow \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = 1$

or $\underline{\sin^2\theta + \cos^2\theta = 1}$ —— (1)

2. dividing (1) by $\cos^2\theta$
$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$
$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2$$
or $\tan^2\theta + 1 = \sec^2\theta$ —— (2)

3. dividing (1) by $\sin^2\theta$
$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$
or $\underline{1 + \cot^2\theta = \operatorname{cosec}^2\theta}$ —— (3)



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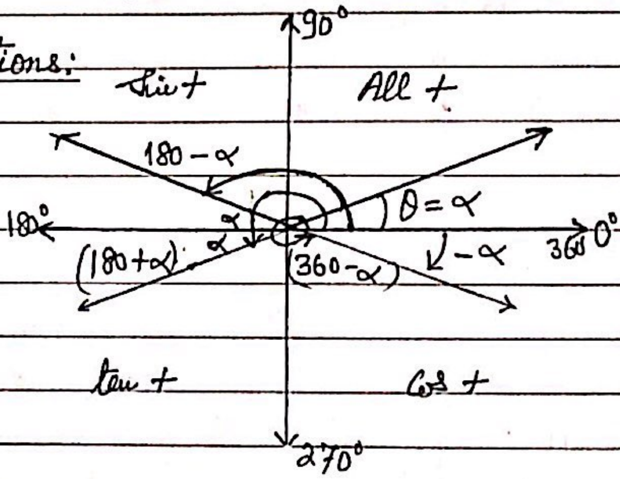
§ Values of trig. Ratios for some particular angles:

| $\theta$ | | $\frac{\pi}{6}$ rad | $\frac{\pi}{4}$ rad | $\frac{\pi}{3}$ rad | $\frac{\pi}{2}$ rad. | $\pi$ rad | $\frac{3}{2}\pi$ rad |
|---|---|---|---|---|---|---|---|
| | $0^\circ$ | $30^\circ$ | $45^\circ$ | $60^\circ$ | $90^\circ$ | $180^\circ$ | $270^\circ$ |
| $\sin\theta$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 | 0 | $-1$ |
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 | $-1$ | 0 |
| $\tan\theta$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | not def. | 0 | not def. |

Signs of Circular (Trig) Functions:

$\alpha$ is the basic angle.




$\begin{cases} \sin(180-\alpha) = +\sin\alpha \\ \cos(180-\alpha) = -\cos\alpha \\ \tan(180-\alpha) = -\tan\alpha. \end{cases}$

$\begin{cases} \sin(180+\alpha) = -\sin\alpha \\ \cos(180+\alpha) = -\cos\alpha \\ \tan(180+\alpha) = +\tan\alpha \end{cases}$

$\begin{cases} \sin(360-\alpha) = -\sin\alpha \\ \cos(360-\alpha) = +\cos\alpha \\ \tan(360-\alpha) = -\tan\alpha \end{cases}$ or $\begin{cases} \sin(-\alpha) = -\sin\alpha \\ \cos(-\alpha) = +\cos\alpha \\ \tan(-\alpha) = -\tan\alpha \end{cases}$

Example 1: Find the exact value of:

(i) $\sec 150^\circ = \frac{1}{\cos 150}$

$= \frac{1}{\cos(180-30)}$

$= \frac{1}{-\cos 30}$

$= \frac{-1}{\sqrt{3}/2} = \frac{-2}{\sqrt{3}}$ ✓

(ii) $\cot 225^\circ = \frac{1}{\tan 225^\circ}$

$= \frac{1}{\tan(180+45)}$

$= \frac{1}{\tan 45}$

$= \frac{1}{1} = \underline{1}$ ✓



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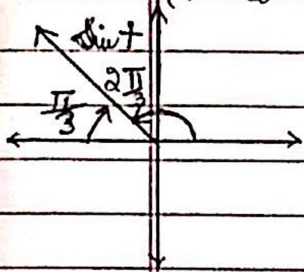
**Example 2.** Find the exact value of;

(i) $\operatorname{Cosec}\frac{2\pi}{3} = \frac{1}{\sin\left(\frac{2\pi}{3}\right)}$

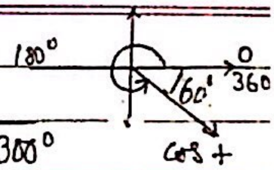



$= \frac{1}{\sin\left(\pi - \frac{\pi}{3}\right)}$

$= \frac{1}{\sin\frac{\pi}{3}}$

$= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$ ✓

(ii) $\sec 300^\circ = \frac{1}{\cos 300^\circ}$



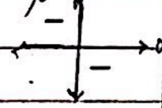

$= \frac{1}{\cos(360-60^\circ)}$

$= \frac{1}{\cos 60^\circ}$

$= \frac{1}{\frac{1}{2}} = 2$ ✓

**Example 3** Solve the equation for $0^\circ \leq \theta \leq 360^\circ$

(i) $\tan\theta = -1$



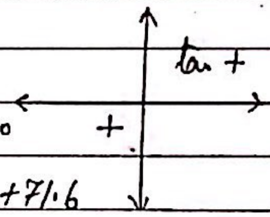

$= -\tan 45^\circ$

$\theta = 180-45^\circ$ or $360-45$

$\theta = 135^\circ\ ;\ 315^\circ$ ✓

Solve:

(ii) $\tan\theta = 3$



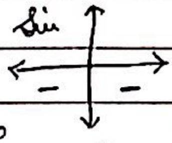

$= \tan 71.6^\circ$

$\therefore \theta = 71.6^\circ,\ 180+71.6$

$\therefore \theta = 71.6\ ;\ 251.6^\circ$ ✓

(iii) Solve;

$\operatorname{Cosec}\theta = -2$



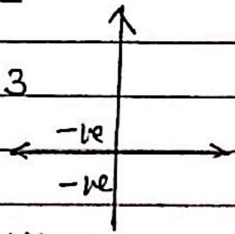

$\Rightarrow \sin\theta = -\frac{1}{2}$

$= -\sin 30^\circ$

$\theta = 180+30^\circ\ ;\ 360-30^\circ$

$\theta = 210^\circ$ or $330^\circ$ ✓

(iv) Solve $\sec\theta = -3$




$\Rightarrow \cos x = -\frac{1}{3}$

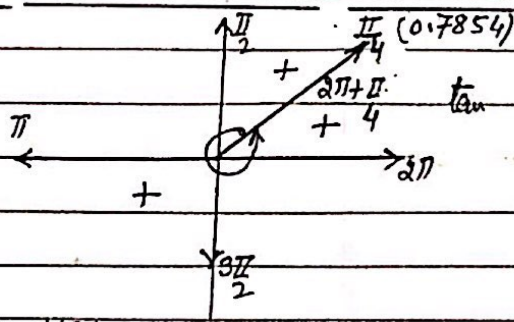
$= -\cos 70.5^\circ$

$\theta = 180-70.5^\circ$ or $180+70.5$

$\theta = 109.5^\circ$ or $250.5^\circ$ ✓

**Example 4** Solve:

$\cot\left(x+\frac{\pi}{4}\right) = 2 \qquad 0 \leq x \leq 2\pi$




$\Rightarrow \tan\left(x+\frac{\pi}{4}\right) = \frac{1}{2} \qquad \frac{\pi}{4} \leq x+\frac{\pi}{4} \leq 2\pi+\frac{\pi}{4}$

$\Rightarrow \tan x+\frac{\pi}{4} = \tan 0.4636$

$\Rightarrow x+\frac{\pi}{4} = 0.4636$ or $\pi+0.4636\ ;\ 2\pi+0.4636$

$x = (0.4636-0.7854)^{x},\ \pi+0.4636-0.7854,\ 2\pi+0.4636-0.7854$

$= -0.3218^{x}\ ;\quad 2.819\ ;\quad 5.96$

$\therefore x = 2.82\text{ rad}$ or $5.96\text{ rad}$ ✓



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**Example 5:** Solve the trig. equations for $-180^\circ \le x \le 180^\circ$.

(i) $\sec^2 x = 9$

$\Rightarrow \cos^2 x = \frac{1}{9}$

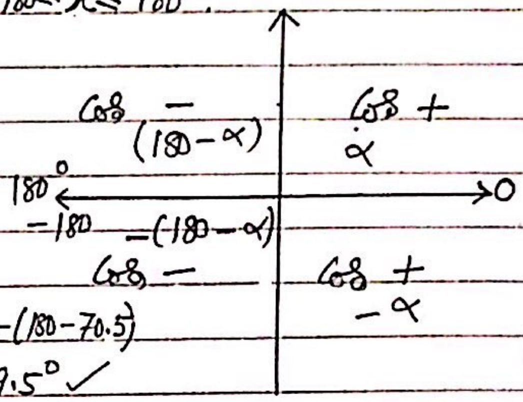
$\Rightarrow \cos x = \pm\sqrt{\frac{1}{9}}$

$\cos x = \frac{1}{3}$ or $\cos x = -\frac{1}{3}$

$= \cos 70.5$ or $\cos x = -\cos 70.5$

$x = 70.5, (-70.5)$ or $(180-70.5), -(180-70.5)$

$x = 70.5, -70.5^\circ$ or $109.5^\circ; -109.5^\circ$ ✓




(ii) $9\cot^2 \frac{1}{2}x = 4$ $\qquad -180^\circ \le x \le 180^\circ$

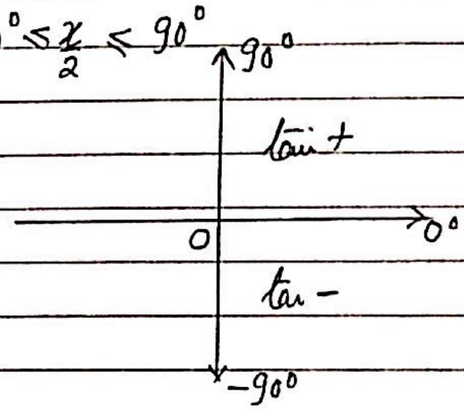
$\Rightarrow \tan^2 \frac{x}{2} = \frac{9}{4}$ $\qquad \therefore -90^\circ \le \frac{x}{2} \le 90^\circ$

$\Rightarrow \tan\frac{x}{2} = \pm\sqrt{\frac{9}{4}} = \pm\frac{3}{2}$

$\tan\frac{x}{2} = \pm\tan 56.3^\circ$

$\frac{x}{2} = 56.3$ or $-56.3$

$\Rightarrow x = 112.6^\circ$ or $-112.6^\circ$




**Example 6:** By expressing the equation $\operatorname{cosec}\theta = 3\sin\theta + \cot\theta$ in terms of $\cos\theta$ only, solve the equation for $0^\circ < \theta < 180^\circ$. ---[5]

[S-16/31/Q3]

**Solution:** $\operatorname{cosec}\theta = 3\sin\theta + \cot\theta$ $\qquad : 0 < \theta < 180^\circ$

$\Rightarrow \frac{1}{\sin\theta} = 3\sin\theta + \frac{\cos\theta}{\sin\theta}$

$\Rightarrow 1 = 3\sin^2\theta + \cos\theta$

$\Rightarrow 1 = 3(1-\cos^2\theta) + \cos\theta$

$\Rightarrow 3\cos^2\theta - \cos\theta - 2 = 0$

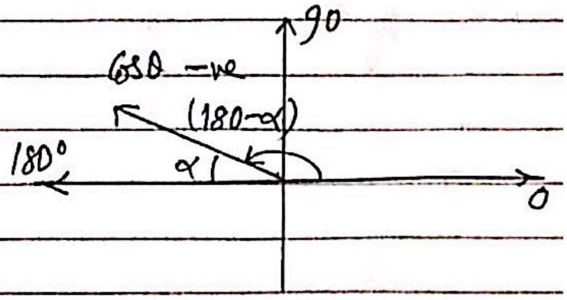
$\Rightarrow (\cos\theta - 1)(3\cos\theta + 2) = 0$

$\Rightarrow \cos\theta = 1$ or $\cos\theta = -\frac{2}{3}$

$\Rightarrow \theta = 0^\times$ (as $0 < \theta < 180^\circ$) or $\cos\theta = -\cos 48.2^\circ$

$\therefore \theta = 180 - 48.2^\circ = 131.8^\circ$

$\therefore \underline{\theta = 131.8^\circ}$ ✓



**Example 7:** Express the equation $\sec\theta = 3\cos\theta + \tan\theta$ as a quadratic equation in $\sin\theta$. Hence solve this equation for $-90^\circ < \theta < 90^\circ$ ---[5]

[W-16/31/Q3]

**Solution:** $\sec\theta = 3\cos\theta + \tan\theta$ for $-90^\circ < \theta < 90^\circ$

$\Rightarrow \dfrac{1}{\cos\theta} = 3\cos\theta + \dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow 1 = 3\cos^2\theta + \sin\theta$

$\Rightarrow 1 = 3(1 - \sin^2\theta) + \sin\theta$

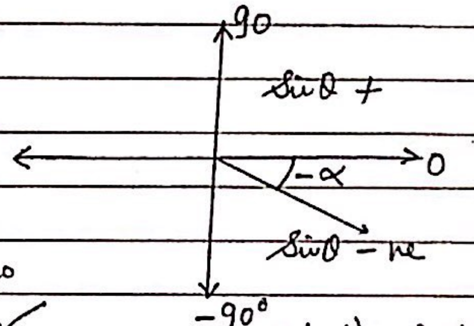
$\Rightarrow 3\sin^2\theta - \sin\theta - 2 = 0$

$\Rightarrow (\sin\theta - 1)(3\sin\theta + 2) = 0$

$\Rightarrow \sin\theta = 1$ or $\sin\theta = -2/3$

$\theta = 90^\circ$ ✗ $\qquad \sin\theta = -\sin 41.8^\circ$

(as $-90 < \theta < 90$) $\qquad \therefore \theta = -41.8^\circ$ ✓




$\because \sin(-\alpha) = -\sin\alpha$

**Example 8:** Prove the identity:

$$\frac{1+\sin\theta}{1-\sin\theta} = 2\sec^2\theta + 2\sec\theta\tan\theta - 1$$ ---[3]

[S-17/32/Q7(ii)]

**Solution:** L.H.S. $\dfrac{1+\sin\theta}{1-\sin\theta} = \dfrac{1+\sin\theta}{1-\sin\theta} \times \dfrac{1+\sin\theta}{1+\sin\theta}$

$= \dfrac{(1+\sin\theta)^2}{1^2 - \sin^2\theta}$

$= \dfrac{1 + \sin^2\theta + 2\sin\theta}{\cos^2\theta}$

$= \dfrac{1}{\cos^2\theta} + \dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{2\sin\theta}{\cos\theta} \times \dfrac{1}{\cos\theta}$

$= \sec^2\theta + \tan^2\theta + 2\tan\theta \cdot \sec\theta$

$= \sec^2\theta + (\sec^2\theta - 1) + 2\sec\theta\tan\theta \quad \because (1 + \tan^2\theta = \sec^2\theta)$

$= 2\sec^2\theta + 2\sec\theta\tan\theta - 1$ ✓

$=$ R.H.S



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Example 9. Prove that: $\sqrt{\sec^2\theta+\operatorname{cosec}^2\theta} = \tan\theta+\cot\theta$

Solution: L.H.S $\sqrt{\sec^2\theta+\operatorname{cosec}^2\theta} = \sqrt{1+\tan^2\theta+1+\cot^2\theta}$

$$=\sqrt{\tan^2\theta+\cot^2\theta+2}$$

$$=\sqrt{\tan^2\theta+\cot^2\theta+2\tan\theta\cdot\cot\theta}$$

$$=\sqrt{(\tan\theta+\cot\theta)^2}$$

$$= \tan\theta+\cot\theta = R.H.S$$

Example 10: Prove that: $\dfrac{\tan A+\sec A-1}{\tan A-\sec A+1} = \dfrac{1+\sin A}{\cos A}$

Solution: L.H.S. $\dfrac{\tan A+\sec A-1}{\tan A-\sec A+1} = \dfrac{(\tan A+\sec A)-(\sec^2 A-\tan^2 A)}{(\tan A-\sec A+1)}$

$$=\frac{(\tan A+\sec A)-(\sec A+\tan A)(\sec A-\tan A)}{(\tan A-\sec A+1)}$$

$$=\frac{(\sec A+\tan A)[1-(\sec A-\tan A)]}{(\tan A-\sec A+1)}$$

$$=\left(\frac{1}{\cos A}+\frac{\sin A}{\cos\theta}\right)\times\frac{\cancel{(1-\sec A+\tan A)}}{\cancel{(\tan A-\sec A+1)}}$$

$$=\frac{1+\sin A}{\cos A} = R.H.S$$

Example 11: Prove that: $(1+\cot A+\tan A)(\sin A-\cos A) = \dfrac{\sec A}{\operatorname{cosec}^2 A} - \dfrac{\operatorname{cosec} A}{\sec^2 A}$.

Solution: L.H.S. $(1+\cot A+\tan A)(\sin A-\cos A)$

$$=\left(1+\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)(\sin A-\cos A)$$

$$=\frac{(\cos A\sin A+\cos^2 A+\sin^2 A)(\sin A-\cos A)}{\sin A\cdot\cos A}$$

$$=\frac{\sin^3 A-\cos^3 A}{\sin A\cos A} \qquad [\because (a-b)(a^2+b^2+ab) = a^3-b^3]$$

$$=\frac{\sin^3 A}{\sin A\cos A}-\frac{\cos^3 A}{\sin A\cos A} = \frac{\sec A}{\operatorname{cosec}^2 A}-\frac{\operatorname{cosec} A}{\sec^2 A} = R.H.S$$



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## § Compound Angle Formulae:

(i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$
(ii) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(iii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$
(iv) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$.

$\tan(45^\circ + A) = \dfrac{1+\tan A}{1-\tan A}$

$\tan(45^\circ - A) = \dfrac{1-\tan A}{1+\tan A}$

## § Multiples of angles Formulae:

(i) $\sin 2A = 2 \sin A \cdot \cos A$

(ii) $\cos 2A = \cos^2 A - \sin^2 A$
$= 1 - 2\sin^2 A \Rightarrow 1 - \cos 2A = 2\sin^2 A$
$= 2\cos^2 A - 1 \Rightarrow 1 + \cos 2A = 2\cos^2 A$

(iii) $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

## Submultiple Formulae.

$\sin A = 2 \sin A/_2 \cos A/_2$

$\cos A = \cos^2 A/_2 - \sin^2 A/_2$
$= 1 - 2\sin^2 A/_2$
$= 2\cos^2 A/_2 - 1$

$1 + \cos A = 2\cos^2 A/_2$
$1 - \cos A = 2\sin^2 A/_2$

$\tan A = \dfrac{2\tan A/_2}{1 - \tan^2 A/_2}$

## §

(i) $\sin 3A = 3\sin A - 4\sin^3 A$
(ii) $\cos 3A = 4\cos^3 A - 3$

(iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

## §

(i) $\sin 4A = \sin 2(2A) = 2\sin 2A \cos 2A$
(ii) $\cos 4A = \cos 2(2A) = \cos^2 2A - \sin^2 2A$
$= 1 - 2\sin^2 2A$
$= 2\cos^2 2A - 1$



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Example 12: Find the exact value of the following:

(i) $\cos 75^\circ$ $\quad [\cos(A+B) = \cos A \cos B - \sin A \sin B]$

$= \cos(45+30^\circ) = \cos 45 \cos 30 - \sin 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{(\sqrt{3}-1)}{2\sqrt{2}} = \frac{(\sqrt{3}-1)}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{(\sqrt{6}-\sqrt{2})}{4}$ ✓

(ii) $\sin 105^\circ$

$= \sin(60+45^\circ)$ $\quad [\because \sin(A+B) = \sin A \cos B + \cos A \sin B]$

$= \sin 60 \cdot \cos 45^\circ + \cos 60^\circ \sin 45^\circ$

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{(\sqrt{3}+1)}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{(\sqrt{6}+\sqrt{2})}{4}$ ✓

(iii) $\sin 50^\circ \cdot \cos 10^\circ + \cos 50^\circ \cdot \sin 10^\circ$

$= \sin(50^\circ + 10^\circ)$

$= \sin 60^\circ = \frac{\sqrt{3}}{2}$ ✓

(iv) $\frac{\tan 25^\circ + \tan 20^\circ}{1 - \tan 25^\circ \cdot \tan 20}$ $\quad [\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}]$

$= \tan(25+20^\circ)$

$= \tan 45^\circ = 1$ ✓

(V) $\sin \frac{\pi}{12}$ $\quad (\because \frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4})$

$= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ $\quad [\because \sin(A-B) = \sin A \cos B - \cos A \sin B]$

$= \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4}$

$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{(\sqrt{6}-\sqrt{2})}{4}$ ✓

**Example 13:** The angles A and B are such that:

$\sin(A+45^\circ) = 2\sqrt{2}\cos A$ and $4\sec^2 B + 5 = 12\tan B$.

Without using a calculator, find the exact value of $\tan(A-B)$. --[8]

[W-15/33/Q6]

**Solution:** Given, $\sin(A+45^\circ) = 2\sqrt{2}\cos A$

$\Rightarrow \sin A\cos 45^\circ + \cos A\sin 45^\circ = 2\sqrt{2}\cos A$

$\Rightarrow \frac{1}{\sqrt{2}}\sin A + \frac{1}{\sqrt{2}}\cos A = 2\sqrt{2}\cos A$

$\Rightarrow \sin A + \cos A = 4\cos A$

$\Rightarrow \sin A = 3\cos A$

$\Rightarrow \tan A = 3$ —— ①

Given, $4\sec^2 B + 5 = 12\tan B$

$\Rightarrow 4(1+\tan^2 B) + 5 = 12\tan B$

$\Rightarrow 4 + 4\tan^2 B + 5 = 12\tan B$

$\Rightarrow 4\tan^2 B - 12\tan B + 9 = 0$

$\Rightarrow (2\tan B - 3)^2 = 0$

$\Rightarrow 2\tan B - 3 = 0$

or $\tan B = 3/2$ —— ②

Now to find, $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\tan B}$

$= \dfrac{3 - 3/2}{1 + 3\times 3/2} = \dfrac{3}{11}$ ✓ [∵ from ① and ②]

**Example 14** Solve the equation:

$\cos(x+30^\circ) = 2\cos x$ for $-180^\circ < x < 180^\circ$ ---[5]

[S-14/32/Q3]

**Solution:** Given, $\cos(x+30^\circ) = 2\cos x$

$\Rightarrow \cos x\cos 30^\circ - \sin x\sin 30^\circ = 2\cos x$ [∵ $\cos(A+B) = \cos A\cos B - \sin A\sin B$]

$\Rightarrow \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = 2\cos x$

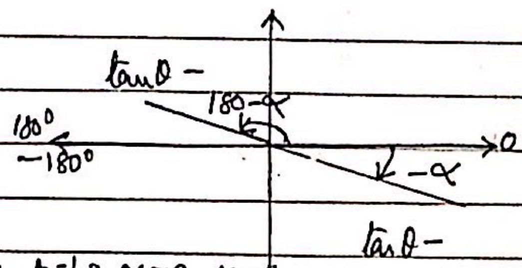
$\Rightarrow \sqrt{3}\cos x - \sin x = 4\cos x$

$\Rightarrow (4-\sqrt{3})\cos x = -\sin x$

or $\tan x = (\sqrt{3}-4)$

$= -2.2679$

$= -\tan 66.2^\circ$ [$\alpha = \tan^{-1} 2.2679 = 66.2^\circ$]




$\therefore x = (180 - 66.2)^\circ$ or $-66.2^\circ$ [∵ $\tan x$ is −ve in IInd & IVth quad.

$x = \underline{113.8^\circ}$ or $\underline{-66.2^\circ}$ ✓

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**Example 15 (i)** Show that the equation, $\tan(x-60^\circ)+\cot x=\sqrt{3}$
Can be written in the form $2\tan^2 x+\sqrt{3}\tan x-1=0$ --[3]
(ii) Hence solve the equation, $\tan(x-60^\circ)+\cot x=\sqrt{3}$, for $0^\circ<x<180^\circ$ --[3].

[S-14/33/Q3]

**Solution:** Given, $\tan(x-60^\circ)+\cot x=\sqrt{3}$

$$\Rightarrow \frac{\tan x-\tan 60^\circ}{1+\tan x\cdot\tan 60^\circ}+\frac{1}{\tan x}=\sqrt{3} \qquad \left(\because \tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}\right)$$

$$\Rightarrow \frac{\tan x-\sqrt{3}}{1+\sqrt{3}\tan x}+\frac{1}{\tan x}=\sqrt{3}$$

$$\Rightarrow \frac{\tan x(\tan x-\sqrt{3})+1+\sqrt{3}\tan x}{\tan x(1+\sqrt{3}\tan x)}=\sqrt{3}$$

$$\Rightarrow \tan^2 x-\cancel{\sqrt{3}\tan x}+1+\cancel{\sqrt{3}\tan x}=\sqrt{3}\tan x+3\tan^2 x$$

$$\Rightarrow 2\tan^2 x+\sqrt{3}\tan x-1=0 \qquad \left\{\begin{array}{l} a=2\\ b=\sqrt{3}\\ c=-1\\ b^2-4ac=3+8=11\end{array}\right.$$

$$\tan x=\frac{-\sqrt{3}\pm\sqrt{11}}{4}$$

$$=\frac{1.5845}{4} \text{ or } -\frac{5.0486}{4}$$

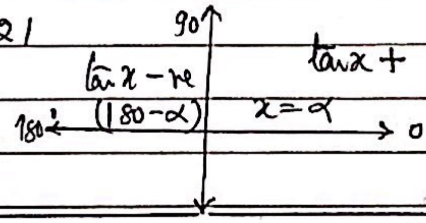
$0<x<180^\circ$

$$\tan x=0.3961 \quad\text{or}\quad -1.2621$$

$$=\tan 21.6^\circ \quad\text{or}\quad -\tan 51.6^\circ$$

$$x=21.6^\circ \text{ or } (180-51.6^\circ)$$

$$x=21.6^\circ \text{ or } 128.4^\circ$$




solve the equation:

**Example 16:** $\sin(\theta+45^\circ)=2\cos(\theta-30^\circ)$ for $0^\circ<\theta<180^\circ$ --[5]

[S-12/33/Q3]

**Solution:** Given $\sin(\theta+45^\circ)=2\cos(\theta-30^\circ)$

$$\Rightarrow \sin\theta\cos45^\circ+\cos\theta\sin45^\circ=2(\cos\theta\cdot\cos30^\circ-\sin\theta\sin30^\circ)$$

$$\Rightarrow \frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta=2\left[\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta\right]$$

$0^\circ<\theta<180^\circ$

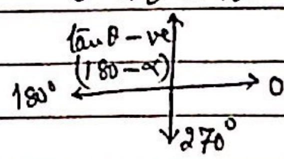
$$\Rightarrow \frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta=\sqrt{3}\cos\theta+\sin\theta$$

$$\Rightarrow \sin\theta\left(\frac{1}{\sqrt{2}}-1\right)=\cos\left(\sqrt{3}-\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \tan\theta=\frac{(\sqrt{6}-1)}{(1-\sqrt{2})}=-\frac{(\sqrt{6}-1)}{\sqrt{2}-1}=-3.5011$$

$$=-\tan 74.1^\circ$$

$$\therefore \theta=(180-74.1^\circ)=105.9^\circ$$



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**Example 17:** The angles $\theta$ and $\phi$ lie between $0°$ and $180°$, and are such that, $\tan(\theta-\phi)=3$ and $\tan\theta+\tan\phi=1$. Find if possible values of $\theta$ and $\phi$. ---[6]

[2017/SP-03/Q3]

**Solution:** $\tan(\theta-\phi)=3$ —— ①

$\left\{\begin{array}{l} 0°<\theta<180° \\ \text{and } 0°<\phi<180° \end{array}\right.$

$\Rightarrow \dfrac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=3$

$\Rightarrow \tan\theta-\tan\phi=3+3\tan\theta\cdot\tan\phi$

Also given $\tan\theta+\tan\phi=1$

$\Rightarrow \tan\phi=(1-\tan\theta)$ —— ②

$\Rightarrow \tan\theta-(1-\tan\theta)=3+3\tan\theta(1-\tan\theta)$ from ②

$\Rightarrow 2\tan\theta-1=3+3\tan\theta-3\tan^2\theta$

$\Rightarrow 3\tan^2\theta-\tan\theta-4=0$

$\Rightarrow (3\tan\theta-4)(\tan\theta+1)=0$

$\Rightarrow \tan\theta=-1$ or $\tan\theta=4/3=\tan 53.1$

$=-\tan45°$

$\theta=(180-45)=135°$;

Now $\tan\phi=1-\tan\theta$

$=1-(-1)=2$

$\phi=\tan^{-1}2=63.4°$

$\therefore \theta=135°$ and $\phi=63.4°$ or

$\therefore \theta=53.1$

$\tan\phi=1-\tan\theta$

$=1-4/3=-\frac{1}{3}$

$\tan\phi=-\tan 18.4°$

$\therefore \phi=180-18.4$

$=161.6°$

$\therefore \underline{\theta=135°\text{ and }\phi=63.4°}$ or $\underline{\theta=53.1°\text{ and }\phi=161.6°}$

**Example 18:** Solve the equation: $\cot\theta+\cot(\theta+45)=2$ for $0°<\theta<180°$ --[5]

[S-18/32/Q2]

**Solution:** Given $\cot\theta+\cot(\theta+45)=2$

$\Rightarrow \dfrac{1}{\tan\theta}+\dfrac{1}{\tan(\theta+45°)}=2$

$\Rightarrow \dfrac{1}{\tan\theta}+\dfrac{1}{\left(\frac{1+\tan\theta}{1-\tan\theta}\right)}=2$

$\Rightarrow \dfrac{1}{\tan\theta}+\dfrac{1-\tan\theta}{1+\tan\theta}=2$

$=\dfrac{(1+\tan\theta)+\tan(1-\tan\theta)}{\tan\theta(1+\tan\theta)}=2$

$\Rightarrow 1+\tan\theta+\tan\theta-\tan^2\theta=2\tan\theta+2\tan^2\theta$

$\Rightarrow 3\tan^2\theta=1$

$\Rightarrow \tan^2\theta=\frac{1}{3}$

$\tan\theta=\pm\frac{1}{\sqrt3}\Rightarrow \tan\theta=\frac{1}{\sqrt3}$ or $\tan\theta=-\frac{1}{\sqrt3}$

$=\tan30°$ or $=-\tan30°$

$\therefore \theta=30°$ or $180-30°$

$\theta=\underline{30°}$ or $\underline{150°}$



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**Example 19:** Prove the identity; $\tan 2\theta - \tan\theta = \tan\theta \cdot \sec 2\theta$ --- [4]

**Solution:** L.H.S $\tan 2\theta - \tan\theta = \dfrac{\sin 2\theta}{\cos 2\theta} - \dfrac{\sin\theta}{\cos\theta}$

$$= \frac{\sin 2\theta \cdot \cos\theta - \cos 2\theta \cdot \sin\theta}{\cos 2\theta \cdot \cos\theta}$$

$$= \frac{\sin(2\theta - \theta)}{\cos 2\theta \cdot \cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta} \times \frac{1}{\cos 2\theta}$$

$$= \tan\theta \cdot \sec 2\theta = \text{R.H.S}$$

**Example 20:** Express the equation $\cot 2\theta = 1 + \tan\theta$ as a quadratic equation in $\tan\theta$. Hence solve the equation for $0° < \theta < 180°$ -- [6]

[W-16/33/Q3]

**Solution:** $\cot 2\theta = 1 + \tan\theta$

$$\Rightarrow \frac{1}{\tan 2\theta} = 1 + \tan\theta$$

$$\Rightarrow \frac{1}{\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)} = 1 + \tan\theta \qquad \left(\because \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$\Rightarrow \frac{1-\tan^2\theta}{2\tan\theta} = 1 + \tan\theta$$

$$\Rightarrow 1 - \tan^2\theta = 2\tan\theta(1 + \tan\theta)$$

$$\Rightarrow 3\tan^2\theta + 2\tan\theta - 1 = 0$$

$$(\tan\theta + 1)(3\tan\theta - 1) = 0$$

$$\Rightarrow \tan\theta = \frac{1}{3} \quad \text{or} \quad \tan\theta = -1 \qquad 0° < \theta < 180°$$

$$= \tan 18.4° \quad \text{or} \quad \tan\theta = -\tan 45°$$

$$\theta = 18.4° \quad \text{or} \quad (180 - 45)$$

$$\theta = \underline{18.4°} \quad \text{or} \quad \underline{135°}$$



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**Example 21:** Solve the equation:

$$7\cos x - 6\sin 2x = 0 \quad \text{for} \quad 0 \le x \le \pi \quad \text{---[5]}$$

$\boxed{S\text{-}15/31/}$

**Solution:** To solve: $7\cos x - 6\sin 2x = 0$

$\Rightarrow 7\cos x - 6\times 2\sin x\cos x = 0 \quad (\because \sin 2x = 2\sin x\cos x)$

$\Rightarrow \cos x(7 - 12\sin x) = 0$

$\Rightarrow \cos x = 0$ or $\sin x = \frac{7}{12} \quad 0 \le x \le \pi$

$\Rightarrow x = \frac{\pi}{2}$ ✓ or $\sin x = \sin 0.623$

$x = \frac{\pi}{2}$ or $0.623$ ; $\pi - 0.623$

$\Rightarrow x = \frac{\pi}{2}$, $0.623$ or $2.52$ rad ✓

---

**Example 22:** Given that $3\cos 2\beta + 7\cos\beta = 0$, find the exact value of $\cos\beta$.

$\boxed{S\text{-}14/31/Q1/}$ ---[3]

**Solution:** Given; $3\cos 2\beta + 7\cos\beta = 0$

$\Rightarrow 3(2\cos^2\beta - 1) + 7\cos\beta = 0 \quad (\because \cos 2A = 2\cos^2 A - 1)$

$\Rightarrow 6\cos^2\beta + 7\cos\beta - 3 = 0$

$(2\cos\beta + 3)(3\cos\beta - 1) = 0$

$\Rightarrow \cos\beta = \frac{1}{3}$ ✓ or $\cos\beta = -\frac{3}{2}$ ✗ $(\because -1 \le \cos\beta \le 1)$

$\therefore \cos\beta = \frac{1}{3}$ ✓

---

**Example 23 (i)** Prove the identity $\cos 4\theta - 4\cos 2\theta = 8\sin^4\theta - 3$ ---[4]

(ii) Hence solve the equation: $\cos 4\theta = 4\cos 2\theta + 3$ for $0° \le \theta \le 360°$ ---[4]

$\boxed{S\text{-}16/32/Q5/}$

**Solution (i)** L.H.S. $\cos 4\theta - 4\cos 2\theta$

$= \cos 2(2\theta) - 4\cos 2\theta$

$= 1 - 2\sin^2 2\theta - 4(1 - 2\sin^2\theta)$

$= 1 - 2(2\sin\theta\cos\theta)^2 - 4(1 - 2\sin^2\theta)$

$= 1 - 8\sin^2\theta\cos^2\theta - 4 + 8\sin^2\theta$

$= 1 - 8\sin^2\theta(1 - \sin^2\theta) - 4 + 8\sin^2\theta$

$= 1 - 8\cancel{\sin^2\theta} + 8\sin^4\theta - 4 + 8\cancel{\sin^2\theta}$

$= 8\sin^4\theta - 3$ —— (1) = R.H.S ✓

(ii) To solve $\cos 4\theta = 4\cos 2\theta + 3$

or $\cos 4\theta - 4\cos 2\theta = 3$

from Part (i) $\Rightarrow 8\sin^4\theta - 3 = 3$

$\Rightarrow \sin^4\theta = 6/8 = \frac{3}{4}$

$\sin^2\theta = \frac{\sqrt{3}}{2}$ or $\sin^2\theta = -\frac{\sqrt{3}}{2}$ ✗

$\sin\theta = \pm\sqrt{0.866} = \pm 0.93$

$\sin\theta = \sin 68.5$ or $\sin\theta = -68.5$

$\theta = 68.5, (180 - 68.5)$, or $(180 + 68.5), (360 - 68.5)$

$\theta = 68.5;\ 111.5°$ or $248.5°;\ 291.5°$



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Example 24(i) Show that: $\cos(\theta-60^\circ)+\cos(\theta+60^\circ)=\cos\theta$ --- [3]

(ii) Given that; $\dfrac{\cos(2x-60^\circ)+\cos(2x+60^\circ)}{\cos(x-60^\circ)+\cos(x+60^\circ)}=3$ find the exact value of $\cos x$.

W-14 | 33 | Q4 --- [4]

Solution (i) L.H.S. $\cos(\theta-60^\circ)+\cos(\theta+60^\circ)$

$=(\cos\theta\cos60^\circ+\sin\theta\sin60^\circ)+(\cos\theta\cos60^\circ-\sin\theta\sin60^\circ)$

$=2\cos\theta\cdot\cos60^\circ$

$=2\times\frac{1}{2}\times\cos\theta=\cos\theta=$ R.H.S ✓

(ii) Now Given. $\dfrac{\cos(2x-60^\circ)+\cos(2x+60^\circ)}{\cos(x-60^\circ)+\cos(x+60^\circ)}=3$

$\Rightarrow \dfrac{\cos2x}{\cos x}=3$ (∵ Using Part (i))

$\Rightarrow \cos2x=3\cos x$ $(\cos2A=2\cos^2A-1)$

$\Rightarrow 2\cos^2x-1=3\cos x$

$\Rightarrow 2\cos^2x-3\cos x-1=0$ | $a=2,\ b=-3,\ c=-1$

$\Rightarrow \cos x=\dfrac{3\pm\sqrt{17}}{4}$ | $b^2-4ac=9+8=17$

$\cos x=\dfrac{3-\sqrt{17}}{4}$ or $\dfrac{3+\sqrt{17}}{4}$ ✗ $=1.7^{✗}$ $(\because -1\le\cos x\le1)$

$\cos x=\left(\dfrac{3-\sqrt{17}}{4}\right)$ ✓

---

Example 25: Solve the equation; $\operatorname{cosec}2\theta=\sec\theta+\cot\theta$ for $0<\theta<360^\circ$ --- [6]

S-12 | 32 | Q4

Solution: Given $\operatorname{cosec}2\theta=\sec\theta+\cot\theta$

$\Rightarrow \dfrac{1}{\sin2\theta}=\dfrac{1}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}$

$\Rightarrow \dfrac{1}{2\sin\theta\cos\theta}=\dfrac{\sin\theta+\cos^2\theta}{\sin\theta\cos\theta}$

$\Rightarrow 2(\sin\theta+\cos^2\theta)=1$

$\Rightarrow 2\sin\theta+2(1-\sin^2\theta)=1$

$\Rightarrow 2\sin^2\theta-2\sin\theta-1=0$

$\sin\theta=\dfrac{2\pm\sqrt{12}}{4}$ | $a=2,\ b=-2,\ c=-1$; $b^2-4ac=12$

$\sin\theta=\dfrac{2\pm2\sqrt{3}}{4}$

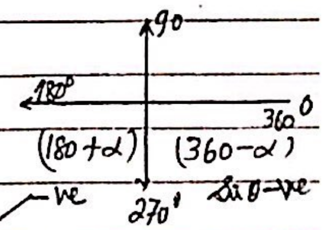
$\sin\theta=\dfrac{1-\sqrt{3}}{2}$ or $\left(\dfrac{1+\sqrt{3}}{2}\right)=1.36^{✗}$ $(\because -1\le\sin\theta\le1)$

$\sin\theta=-0.366$

$=-\sin21.5^\circ$

$\theta=180+21.5^\circ$ or $(360-21.5^\circ)$

$\theta=201.5^\circ$ or $339.5^\circ$ ✓ −ve



**Example 26(i).** By first expanding $\sin(2\theta+\theta)$, show that:

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \quad \text{---[4]}$$

(ii) Show that, after making the substitution $x = \frac{2\sin\theta}{\sqrt{3}}$, the equation $x^3 - x + \frac{1}{6}\sqrt{3} = 0$ can be written in the form $\sin 3\theta = \frac{3}{4}$ --[1]

(iii) Hence solve the equation $x^3 - x + \frac{1}{6}\sqrt{3} = 0$ giving your answer correct to 3 s.f. [N-14/31/Q8] --[4]

**Solution (i)**

$$\sin 3\theta = \sin(2\theta+\theta)$$
$$= \sin 2\theta\cdot\cos\theta + \cos 2\theta\cdot\sin\theta$$
$$= 2\sin\theta\cos\theta\cdot\cos\theta + \sin\theta(1-2\sin^2\theta)$$
$$= 2\sin\theta\cdot\cos^2\theta + \sin\theta(1-2\sin^2\theta)$$
$$= 2\sin\theta(1-\sin^2\theta) + \sin\theta(1-2\sin^2\theta)$$
$$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta = \text{R.H.S} \checkmark$$

(ii) $x^3 - x + \frac{1}{6}\sqrt{3} = 0$, put $x = \frac{2\sin\theta}{\sqrt{3}}$

$$\Rightarrow \left(\frac{2\sin\theta}{\sqrt{3}}\right)^3 - \frac{2\sin\theta}{\sqrt{3}} + \frac{1}{6}\sqrt{3} = 0$$
$$\Rightarrow \frac{8\sin^3\theta}{3\sqrt{3}} - \frac{2\sin\theta}{\sqrt{3}} + \frac{1}{6}\sqrt{3} = 0$$
$$\Rightarrow 8\sin^3\theta - 6\sin\theta + \frac{9}{6} = 0$$
$$\Rightarrow 6\sin\theta - 8\sin^3\theta = 3/2$$
$$\Rightarrow 2(3\sin\theta - 4\sin^3\theta) = 3/2$$
$$\Rightarrow \sin 3\theta = \frac{3}{4} \quad [\text{from part (i)}]$$

(iii) To solve: $x^3 - x + \frac{1}{6}\sqrt{3} = 0$ ——(1)

Using part (i), equation (1) reduces to $\sin 3\theta = \frac{3}{4}$ when $x = \frac{2\sin\theta}{\sqrt{3}}$

$$= \sin 48.6$$

$3\theta = 48.6^\circ,\ 180-48.6^\circ,\ -360+48.6^\circ$

$3\theta = 48.6^\circ,\ 131.4^\circ,\ -311.4^\circ$

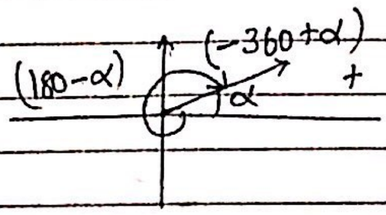
$\theta = 16.2^\circ,\ 43.8^\circ,\ -103.8^\circ$

Now $x = \frac{2\sin\theta}{\sqrt{3}}$

$x = \frac{2}{\sqrt{3}}\sin 16.2,\ \frac{2}{\sqrt{3}}\sin 4\cdot$ and $\frac{2}{\sqrt{3}}\sin(-103.8^\circ)$

$x = \underline{0.322}\checkmark,\ \underline{0.799}\checkmark,\ \underline{-1.12}\checkmark$

Cubic equation (1) has 3 real roots. $\checkmark$



## § Linear Trigonometric Functions:

(i) $a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$ ; $\begin{cases} a = R\cos\alpha \\ b = R\sin\alpha \end{cases}$ $\begin{cases} R = \sqrt{a^2+b^2} \\ \tan\alpha = \frac{b}{a} \Rightarrow \alpha = \tan^{-1}\frac{b}{a} \end{cases}$

(ii) $a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$, : $R > 0$ and $0^\circ < \alpha < 90^\circ$

**Example 27(a)** By first expanding $\cos(x+45^\circ)$, express:
$\cos(x+45^\circ) - \sqrt{2}\sin x$ in the form $R\cos(x+\alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. Give the value of $R$ correct to 4 s.f, and the value of $\alpha$ correct to 2 d.p. ---[5]

(b) Hence solve the equation, $\cos(x+45^\circ) - \sqrt{2}\sin x = 2$, $0 < x < 360^\circ$ [4]

[SP-20/03/Q7]

**Solution (a)** $\cos(x+45^\circ) = \cos x \cdot \cos 45^\circ - \sin x \sin 45^\circ$

$= \frac{\cos x}{\sqrt{2}} - \frac{\sin x}{\sqrt{2}}$ ——— ①

$\therefore \cos(x+45) - \sqrt{2}\sin x$

$= \frac{\cos x}{\sqrt{2}} - \frac{\sin x}{\sqrt{2}} - \sqrt{2}\sin x$ from ①

$= \frac{1}{\sqrt{2}}\cos x - \frac{3}{\sqrt{2}}\sin x$

$= R\cos\alpha\cos x - R\sin\alpha\sin x$

$= R\cos(x+\alpha)$ ✓

where $R = 2.236$ and $\alpha = 71.57^\circ$

Put $R\cos\alpha = \frac{1}{\sqrt{2}}$
$R\sin\alpha = \frac{3}{\sqrt{2}}$
Sq and add $R = \sqrt{\frac{1}{2}+\frac{9}{2}} = 2.236$ ✓
and $\tan\alpha = \frac{3}{\sqrt{2}} / \frac{1}{\sqrt{2}} = 3$
$\alpha = 71.57^\circ$ ✓

(b) To Solve. $\cos(x+45^\circ) - \sqrt{2}\sin x = 2$

$\Rightarrow R\cos(x+\alpha) = 2$ from Part (a)

or $2.236\cos(x+71.57) = 2$

$\Rightarrow \cos(x+71.57^\circ) = \frac{2}{2.236} = 0.8944$

~~$= \cos 26.56$~~

$\therefore x + 71.57 = 26.57, (360-26.56)$

$= 26.57, 333.44^\circ$

or $x = (26.57 - 71.57)$ or $(333.44 - 71.57)$

or $x = -45.01^\circ$ ✗ or $262^\circ$

$x = 262^\circ$ Only (as $0^\circ < x < 360^\circ$)



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Example 28(i) Express $3\sin\theta + 2\cos\theta$ in the form $R\sin(\theta+\alpha)$, where $R>0$, $0^\circ<\alpha<90^\circ$, stating the exact value of $R$ and giving the value of $\alpha$ correct to 2 decimal places. ---[3]

(ii) Hence solve the equation; $3\sin\theta + 2\cos\theta = 1$, for $0^\circ<\theta<180^\circ$ ---[3]

[S-15/32/Q4]

Solution: $3\sin\theta + 2\cos\theta$

(i) $= R\cos\alpha\sin\theta + R\sin\alpha\cos\theta$

$= R\sin(\theta+\alpha)$

$= \sqrt{13}\sin(\theta + 33.69^\circ)$ —— ①

Put $R\cos\alpha = 3$

$R\sin\alpha = 2$

$R = \sqrt{3^2+2^2} = \sqrt{13}$

$\tan\alpha = 2/3 \Rightarrow \alpha = \tan^{-1}\frac{2}{3}$

$= 33.69$

Now to solve:

(ii) $3\sin\theta + 2\cos\theta = 1$

$\Rightarrow \sqrt{13}\sin(\theta+33.69^\circ) = 1$ (from Part (i))

$\sin(\theta+33.69) = \frac{1}{\sqrt{13}} = \sin 16.1$

$\therefore \theta + 33.69 = 16.1$ or $(180 - 16.1^\circ)$

$\theta = (16.1 - 33.69)$ ✗ or $169.9 - 33.69$ ✓

$\theta = 130.2^\circ$ ✓

---

Example 29: Express $\sqrt{3}\cos x + \sin x$ in the form $R\cos(x-\alpha)$, where $R>0$ and $0<\alpha<\frac{\pi}{2}$, giving exact value of $R$ and $\alpha$.

[S-13/33/Q4] ---[3]

Solution: $\sqrt{3}\cos x + \sin x$

$= R\cos\alpha\cos x + R\sin\alpha\sin x$

$= R(\cos x\cos\alpha + \sin x\sin\alpha)$

$= R\cos(x-\alpha)$

$= 2\cos\left(x - \frac{\pi}{6}\right)$ ✓

Put $R\cos\alpha = \sqrt{3}$

and $R\sin\alpha = 1$

$\Rightarrow R = \sqrt{(\sqrt{3})^2+1^2} = 2$

$\tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$



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**Example 29:** It is given that $\tan 3x = k \tan x$, where $k$ is a constant and $\tan x \neq 0$

(i) By first expanding $\tan(2x+x)$, show that:

$$(3k-1)\tan^2 x = k-3 \qquad \text{-- [4]}$$

(ii) Hence solve the equation: $\tan 3x = k\tan x$, when $k=4$ for $0 < x < 180°$ [3]

(iii) Show that the equation $\tan 3x = k\tan x$, has no roots in the interval $0 < x < 180°$ when $k=2$. [S-12/33/Q6] --- [1]

**Solution:** Given, $\tan 3x = k \tan x$

(i)

$$\tan(2x+x) = k\tan x$$

$$\Rightarrow \frac{\tan 2x + \tan x}{1 - \tan 2x \cdot \tan x} = k \tan x \qquad \left(\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right)$$

$$\Rightarrow \frac{\frac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \frac{2\tan x}{1-\tan^2 x}\cdot \tan x} = k\tan x \qquad \left(\because \tan 2A = \frac{2\tan A}{1-\tan^2 A}\right)$$

$$\Rightarrow \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} = k\tan x$$

$$\Rightarrow \frac{\cancel{\tan x}(3 - \tan^2 x)}{1 - 3\tan^2 x} = k\cancel{\tan x} \qquad [\because \tan x \neq 0]$$

$$\Rightarrow 3 - \tan^2 x = k - 3k\tan^2 x$$

$$\Rightarrow \underline{(3k-1)\tan^2 x = k-3} \checkmark \quad 1$$

~~Now to solve~~

(ii) $\tan 3x = k\tan x, \quad 0 < x < 180°$

$\Rightarrow (3k-1)\tan^2 x = k-3$ (from Part (i))

$\Rightarrow 11\tan^2 x = 1$ (for $k=4$)

$\Rightarrow \tan^2 x = \frac{1}{11} \Rightarrow \tan x = \pm\frac{1}{\sqrt{11}}$

$\tan x = \frac{1}{\sqrt{11}}$ or $\tan x = -\frac{1}{\sqrt{11}}$

$\tan x = \tan 16.8°$ or $\tan x = -\tan 16.8°$

$x = 16.8°$ or $x = 180 - 16.8°$

$\therefore x = \underline{16.8°}$ or $\underline{163.2°} \checkmark$

(iii) Now for $k=2$

$\tan 3x = k\tan x$

$\Rightarrow (3k-1)\tan^2 x = k-3$ (from Part (i))

$\Rightarrow 5\tan^2 x = -1$ (for $k=2$)

$\Rightarrow \tan^2 x = -\frac{1}{5}$ × $[\because a^2 \geq 0$ for a real$]$

Not possible

$\therefore$ Eqn has no roots.



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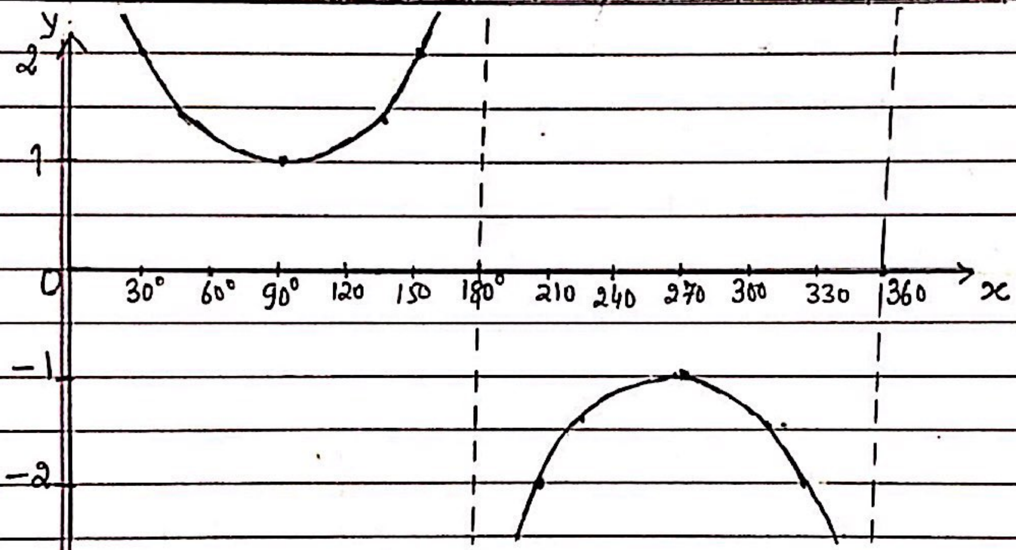
§ Graph. $y = \operatorname{Cosec} x$. $\quad 0 < x < 360^\circ$

| | $x=\alpha$ | | | | $x=180-\alpha$ <br> $\sin x = \sin\alpha$ | | | $x=180+\alpha$ <br> $\sin x = -\sin\alpha$ | | | | $x=360-\alpha$ <br> $\sin x = -\sin\alpha$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | $0^+$ | $30^\circ$ | 45 | 90 | $135^\circ$ | $150^\circ$ | $180^-$ | $180^+$ | $210^\circ$ | 225 | $270^\circ$ | $315^\circ$ | $330^\circ$ | $360^-$ |
| $\sin x$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt2}$ | 1 | $\frac{1}{\sqrt2}$ | $\frac{1}{2}$ | 0 | 0 | $-\frac{1}{2}$ | $-\frac{1}{\sqrt2}$ | $-1$ | $-\frac{1}{\sqrt2}$ | $-\frac{1}{2}$ | 0 |
| $\operatorname{Cosec} x$ | $+\infty$ | 2 | $\sqrt2 = 1.4$ | 1 | $\sqrt2$ | 2 | $+\infty$ | $-\infty$ | $-2$ | $-\sqrt2$ | $-1$ | $-\sqrt2$ | $-2$ | $-\infty$ |




$y = \operatorname{Cosec} x$    Domain $= R - n\pi : n \in$ Integer

Range $= \operatorname{Cosec} x \le -1$ or $\operatorname{Cosec} x \ge 1$

Period $= 360^\circ$

Amplitude is not defined.



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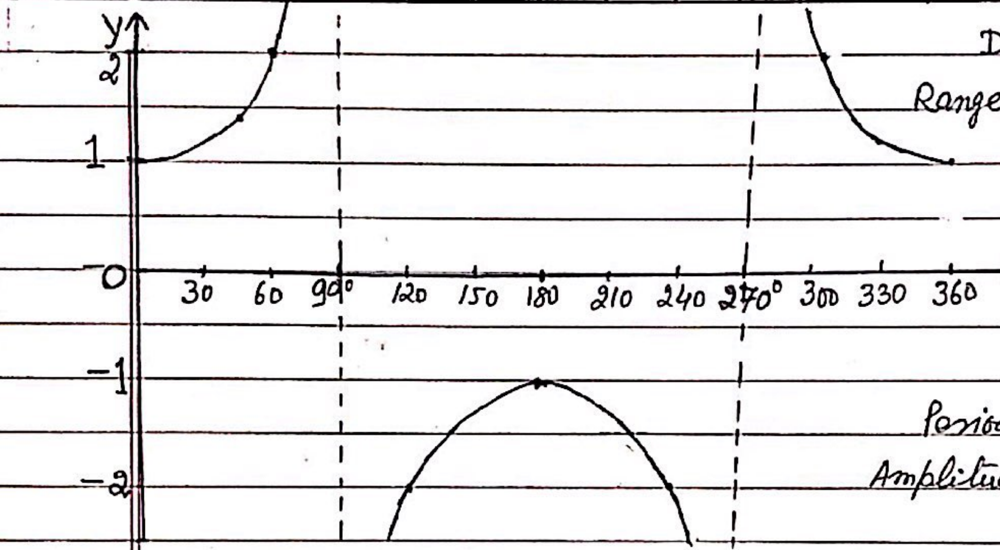
§ Graph: $y = \sec x$. $\quad 0 \leq x \leq 360°$

| | $x=\alpha$ | | | | $x=180-\alpha$, $\cos x = -\cos\alpha$ | | | | $x=180+\alpha$, $\cos x = -\cos\alpha$ | | | $x=360-\alpha$, $\cos x = \cos\alpha$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 45° | 60 | $90^-$ | $90^+$ | 120° | 135° | 180° | 225° | 240 | $270^-$ | $270^+$ | 300 | 315° | 360° |
| $\cos x$ | 1 | $\frac{1}{\sqrt2}$ | $\frac12$ | 0 | 0 | $-\frac12$ | $-\frac{1}{\sqrt2}$ | $-1$ | $-\frac{1}{\sqrt2}$ | $-\frac12$ | 0 | 0 | $\frac12$ | $\frac{1}{\sqrt2}$ | 1 |
| $\sec x$ | 1 | $\sqrt2$ | 2 | $+\infty$ | $-\infty$ | $-2$ | $-\sqrt2$ | $-1$ | $-\sqrt2$ | $-2$ | $-\infty$ | $+\infty$ | 2 | $\sqrt2$ | 1 |




Domain $= R - 90\times(2n+1)$

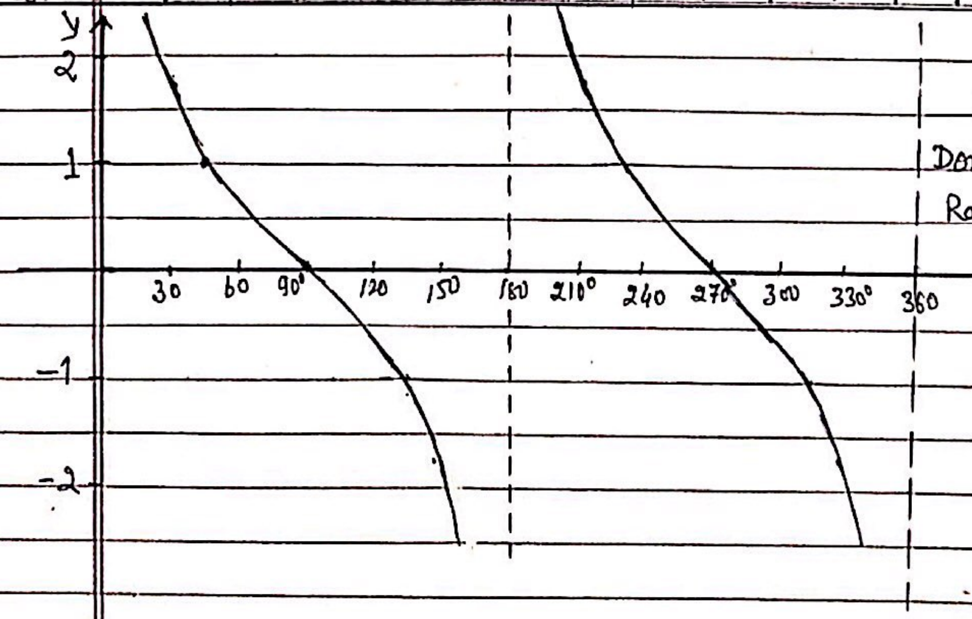
Range: $\sec x \leq -1$ or $\sec x \geq 1$

Period $= 360°$.

Amplitude is not defined.

§ Graph $y = \cot x$

| | $x=\alpha$ | | | | $x=180-\alpha$, $\cot x = -\cot\alpha$ | | | $x=180+\alpha$, $\cot x = \cot\alpha$ | | | | $x=360-\alpha$, $\cot x = -\cot\alpha$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | $0^+$ | 30° | 45° | 90° | 135° | 150° | $180^{-0}$ | $180^+$ | 210° | 225° | 270 | 315° | 330° | $360^-$ | $360^+$ |
| $\cot x$ | $\infty$ | $\sqrt3$ | 1 | 0 | $-1$ | $-\sqrt3$ | $-\infty$ | $\infty$ | $\sqrt3$ | 1 | 0 | $-1$ | $-\sqrt3$ | $-\infty$ | $\infty$ |




Period $= 180°$

Domain $= R - n\pi$

Range: $-\infty < \cot x < \infty$



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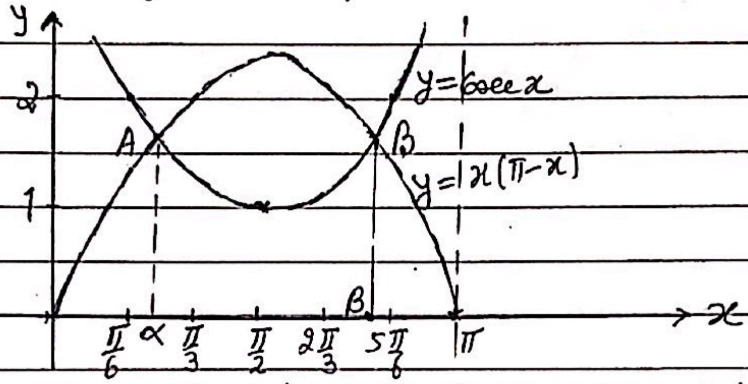
**Example 30.** By sketch the graph of $y = \operatorname{cosec} x$ and $y = x(\pi - x)$ for $0 < x < \pi$, Show that the equation $\operatorname{cosec} x = x(\pi - x)$ has exactly two real roots in the interval $0 < x < \pi$. ---[3]

[S-14/31/Q8(i)]

**Solution:** Sketch $y = \operatorname{cosec} x$ — ① for $0 < x < \pi$

$y = x(\pi - x)$ — ②




Graphs of Equations ① and ② intersects at A and B points.

$\therefore$ equation $\operatorname{cosec} x = x(\pi - x)$

has exactly two roots $x = \alpha$ and $x = \beta$.

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**Example 31** (i) Prove that: $\tan(45 + x) + \tan(45 - x) = 2\sec 2x$ ---[4]

(ii) Hence sketch the graph of $y = \tan(45° + x) + \tan(45° - x)$ for $0 \le x \le 90°$. ---[3]

[W-17/31/Q4]

**Solution** (i) L.H.S. $\tan(45 + x) + \tan(45 - x)$

$$= \frac{\tan 45° + \tan x}{1 - \tan 45° \tan x} + \frac{\tan 45° - \tan x}{1 + \tan 45° \cdot \tan x}$$

$$= \frac{1 + \tan x}{1 - \tan x} + \frac{1 - \tan x}{1 + \tan x}$$

$$= \frac{(1 + \tan x)^2 + (1 - \tan x)^2}{(1 - \tan x)(1 + \tan x)}$$

$$= \frac{1 + \tan^2 x + 2\tan x + 1 - 2\tan x + \tan^2 x}{1 - \tan^2 x}$$

$$= 2\left[\frac{1 + \tan^2 x}{1 - \tan^2 x}\right] = 2\frac{\left[1 + \frac{\sin^2 x}{\cos^2 x}\right]}{\left[1 - \frac{\sin^2 x}{\cos^2 x}\right]}$$

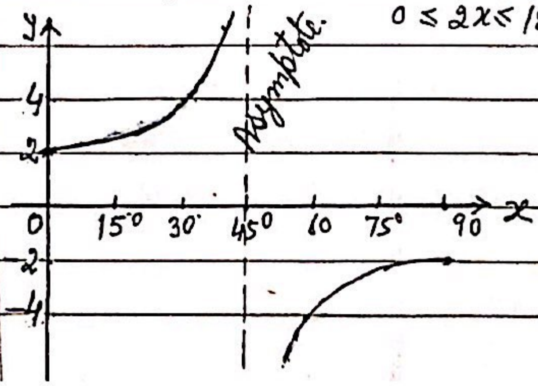
$$= \frac{2(\cos^2 x + \sin^2 x)}{\cos^2 x - \sin^2 x}$$

$$= \frac{2}{\cos 2x}$$

$$= 2\sec 2x = R.H.S$$

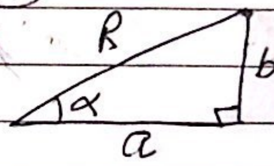
(ii) Graph of $y = \tan(45 + x) + \tan(45 - x)$ is Graph of $y = 2\sec 2x$ $\quad 0 \le x \le 90$

$0 \le 2x \le 180$



## Linear Trig. Function:

(i) $a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$ --- (1)




Proof: Put $a = R\cos\alpha$ —— (i)

$b = R\sin\alpha$ —— (ii)

Square and add (i) & (ii)

$$R^2(\cos^2\alpha + \sin^2\alpha) = a^2 + b^2$$

$$R^2 = a^2 + b^2$$

$$R = \sqrt{a^2+b^2}$$

Divide (ii) ÷ (i) $\dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{b}{a} \Rightarrow \tan\alpha = \dfrac{b}{a}$

$$\alpha = \tan^{-1}\frac{b}{a}$$

for (1) $R\cos\alpha \cdot \sin\theta \pm R\sin\alpha\cos\theta =$

$$= R(\sin\theta\cos\alpha \pm \cos\theta\sin\alpha)$$

$$= R\sin(\theta \pm \alpha)$$

(ii) $a\cos\theta \pm b\sin\theta$ Put $a = R\cos\alpha$, $b = R\sin\alpha$

$R\cos\alpha \cdot \cos\theta \pm R\sin\alpha\sin\theta$

$= R[\cos\theta\cos\alpha \pm \sin\theta\sin\alpha]$

$= R\cos(\theta \mp \alpha)$