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# Pure Maths. 3.

## Vectors in 3D - Straight Lines.

## Notes-2

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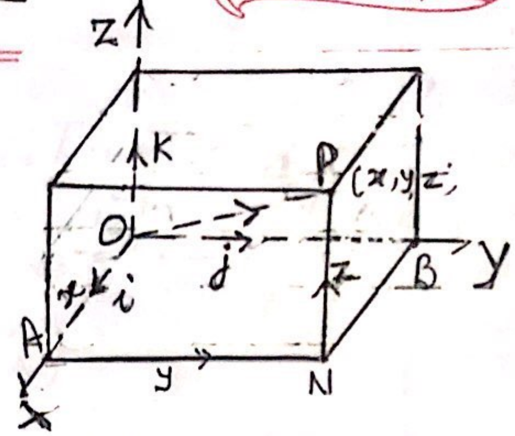
# Vectors in 3D




§ **Components of Vectors in 3D:**

Unit vectors along the axes OX, OY and OZ are denoted by $i, j, k$ respectively.

Let P be any point in space.

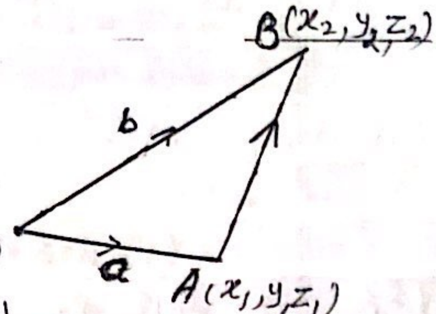
$$\vec{OP} = \vec{OA} + \vec{AN} + \vec{NP}$$

or $\vec{OP} = xi + yj + zk$

§ Let the position vector of any point P be denoted by $\vec{r} = xi + yj + zk = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

where $OA = x$, $AN = OB = y$; $NP = OC = z$

Distance $OP = |\vec{r}| = \sqrt{(x^2 + y^2 + z^2)}$.




§ **Position Vectors of given points:**

$A(x_1, y_1, z_1)$; $\vec{OA} = x_1 i + y_1 j + z_1 k = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = a$

$B(x_2, y_2, z_2)$; $\vec{OB} = x_2 i + y_2 j + z_2 k = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \vec{b}$

and $\vec{AB} = (\vec{b} - \vec{a}) = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$

In $\Delta OAB$
$\vec{OA} + \vec{AB} = \vec{OB}$
$\Rightarrow \vec{AB} = \vec{OB} - \vec{OA}$
$= b - a$

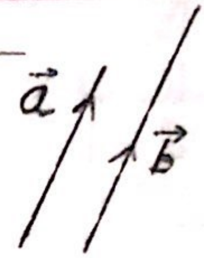
§ Magnitude of $\vec{a} = OA = |\vec{a}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$

§ Unit Vector along $\vec{a} = \hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{1}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}} \times (x_1 i + y_1 j + z_1 k)$

# Vectors in 3D

## § Parallel Vectors:

$$a = a_1 i + a_2 j + a_3 k = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$



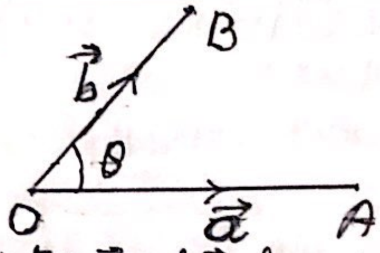

and $$b = b_1 i + b_2 j + b_3 k = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\underline{\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} = \lambda \vec{b}}, \quad \lambda \in R;\ \lambda \neq 0$$

or $$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \Leftrightarrow \vec{a} \parallel \vec{b}$$

## § Scalar Product of Vectors:

$$\underline{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta} \quad \text{----}(i)$$




or $$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \quad \text{----}(ii)$$

(Vectors $\vec{a}$ and $\vec{b}$ have a common initial point)

Now given $i, j, k$ are unit vectors along axes (are mutually perpendicular)

$i \cdot i = j \cdot j = k \cdot k = 1$ (as $i \cdot i = 1 \times 1 \times \cos 0^\circ = 1$)

or $i^2 = j^2 = k^2 = 1$ ----(iii) (Note $\vec{a} \cdot \vec{a} = |\vec{a}|^2 = \vec{a}^2$)

and $i \cdot j = j \cdot k = k \cdot i = 0$ ---(iv)

Note: $\underline{\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0}$, $\vec{a} \neq 0, \vec{b} \neq 0$ ---(v)

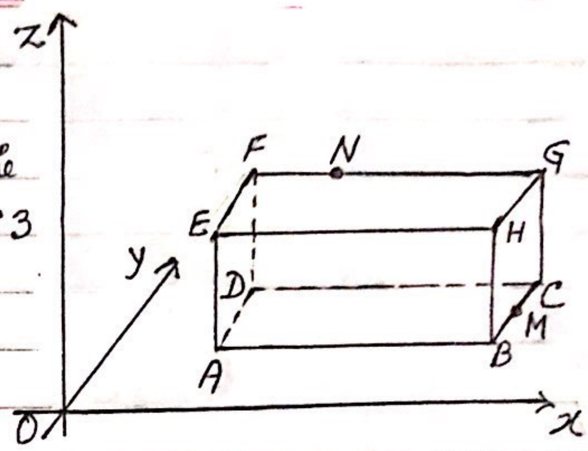
Now given $\vec{a} = a_1 i + a_2 j + a_3 k = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = b_1 i + b_2 j + b_3 k = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ ---(vi)

and $$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \underline{a_1 b_1 + a_2 b_2 + a_3 b_3} \checkmark$$

from (ii) $$\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)} \cdot \sqrt{(b_1^2 + b_2^2 + b_3^2)}} \quad \text{---}(vii)$$

1. The diagram shows a cuboid ABCD EFGH.
M is the mid point of BC and the point N on FG such that FN : NG = 1 : 3.
Given that $\vec{AG} = \begin{pmatrix} 12 \\ 4 \\ 2 \end{pmatrix}$, find the displacement vector:
(a) $\vec{AM}$ (b) $\vec{AN}$

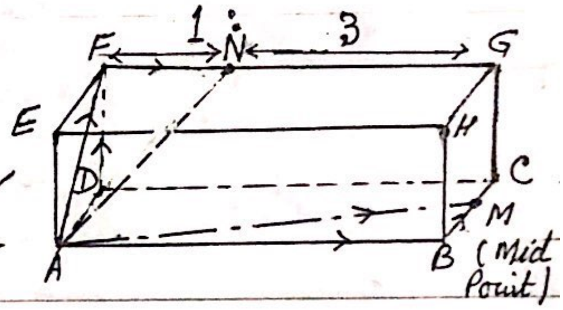



Solution: $\vec{AG} = \begin{pmatrix} 12 \\ 4 \\ 2 \end{pmatrix}$, $\vec{AB} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix}$, $\vec{BC} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$

(a) M is the mid point of BC, In $\Delta ABM$

$\vec{AM} = \vec{AB} + \vec{BM}$

$= \vec{AB} + \frac{1}{2}\vec{BC} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 2 \\ 0 \end{pmatrix}$ ✓




(b) Join AF and AN.

In $\Delta AFN$: $\vec{AN} = \vec{AF} + \vec{FN}$

$= (\vec{AD} + \vec{DF}) + \vec{FN}$ (In $\Delta ADF$, $\vec{AF} = \vec{AD} + \vec{DF}$)

$= \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ ✓

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2. Relative to an origin O, the position vectors of the points P and Q are:
$\vec{OP} = \begin{pmatrix} -6k \\ -2 \\ 8(1+k) \end{pmatrix}$ and $\vec{OQ} = \begin{pmatrix} 2k+13 \\ -8 \\ -32k \end{pmatrix}$ Given that OPQ is a straight line.

(a) Find the value of the constant k.
(b) Write each of $\vec{OP}$ and $\vec{OQ}$ in the form $(xi + yj + zk)$.
(c) Find the magnitude of the vector $\vec{PQ}$.

Solution: O → P → Q

(a) O, P, Q is a straight line:

$\Rightarrow \vec{OP} = \lambda \vec{OQ} \Rightarrow \begin{pmatrix} -6k \\ -2 \\ 8(1+k) \end{pmatrix} = \lambda \begin{pmatrix} 2k+13 \\ -8 \\ -32k \end{pmatrix}$

$\Rightarrow -2 = -8\lambda \Rightarrow \lambda = \frac{1}{4}$ ✓

$-6k = \frac{1}{4}(2k+13) \Rightarrow -24k = 2k+13$

$\Rightarrow k = -\frac{1}{2}$ ✓

(b) $\vec{OP} = \begin{pmatrix} -6\times -\frac{1}{2} \\ -2 \\ 8(1-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = (3i - 2j + 4k)$ ✓

and $\vec{OQ} = \begin{pmatrix} 2\times(-\frac{1}{2})+13 \\ -8 \\ -32\times -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 12 \\ -8 \\ 16 \end{pmatrix}$ ✓

$\vec{OQ} = (12i - 8j + 16k)$

(c) $\vec{PQ} = \vec{OQ} - \vec{OP}$

$= (12i - 8j + 16k) - (3i - 2j + 4k)$

$= (9i - 6j + 12k)$

$\therefore |\vec{PQ}| = \sqrt{9^2 + 6^2 + 12^2} = 3\sqrt{29}$

3. Relative to the origin O, the position vectors of the points P and Q are given by, $\vec{OP} = \begin{pmatrix} 5k \\ -3 \\ 7k+9 \end{pmatrix}$ and $\vec{OQ} = \begin{pmatrix} k \\ k+2 \\ -1 \end{pmatrix}$ where k is a constant.

(a) Find the value of k for which $\vec{OP}$ and $\vec{OQ}$ are perpendicular to one another.
(b) Given that $k=2$, find the angle $\theta$, between $\vec{OP}$ and $\vec{OQ}$.

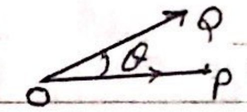
Solution: Given $\vec{OP} = \begin{pmatrix} 5k \\ -3 \\ 7k+9 \end{pmatrix}$, $\vec{OQ} = \begin{pmatrix} k \\ k+2 \\ -1 \end{pmatrix}$ $\left[ \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right]$

(a) $\vec{OP}$ is perp. to $\vec{OQ} \Rightarrow (5k \times k + (-3)(k+2) + (7k+9)(-1) = 0$ $\left\{ \vec{a} \perp \vec{b} = \vec{a} \cdot \vec{b} = 0 \right.$ $(a_1b_1 + a_2b_2 + a_3b_3 = 0$

$\Rightarrow 5k^2 - 3k - 6 - 7k - 9 = 0 \Rightarrow 5k^2 - 10k - 15 = 0$

$\Rightarrow k^2 - 2k - 3 = 0 \Rightarrow (k+1)(k-3) = 0 \Rightarrow \underline{k = -1; k = 3}$ ✓

(b) for $k=2 \Rightarrow \vec{OP} = \begin{pmatrix} 10 \\ -3 \\ 23 \end{pmatrix}$ and $\vec{OQ} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ $\left\{ \cos\theta = \dfrac{\vec{OP} \cdot \vec{OQ}}{|\vec{OP}||\vec{OQ}|} \right.$



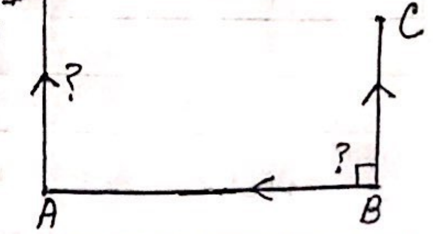

$\Rightarrow \vec{OP} \cdot \vec{OQ} = 10 \times 2 + (-3)(4) + 23(-1) = -15$ ✓

$|\vec{OP}| = \sqrt{10^2 + 3^2 + 23^2} = \sqrt{638}$ & $|\vec{OQ}| = \sqrt{2^2 + 4^2 + 1^2} = \sqrt{21}$

$\cos\theta = \dfrac{\vec{OP} \cdot \vec{OQ}}{|\vec{OP}||\vec{OQ}|} = \dfrac{-15}{\sqrt{638} \cdot \sqrt{21}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{-15}{\sqrt{638} \cdot \sqrt{21}}\right) = \underline{97.4^\circ}$ (1 dp) ✓

4. Relative to origin O, the points A, B, C and D have position vectors, $\vec{OA} = 4i + 2j - k$, $\vec{OB} = 2i - 2j + 5k$, $\vec{OC} = 2j + 7k$, $\vec{OD} = -6i + 22j + 9k$.

(a) Use a scalar product to show that angle ABC is a right angle.
(b) Show that $\vec{AD} = k\vec{BC}$, where k is a constant and explain what this explains.




Solution: $\vec{BA} = \vec{OA} - \vec{OB} = \begin{pmatrix} +2 \\ +4 \\ -6 \end{pmatrix}$; $\vec{BC} = \vec{OC} - \vec{OB} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$

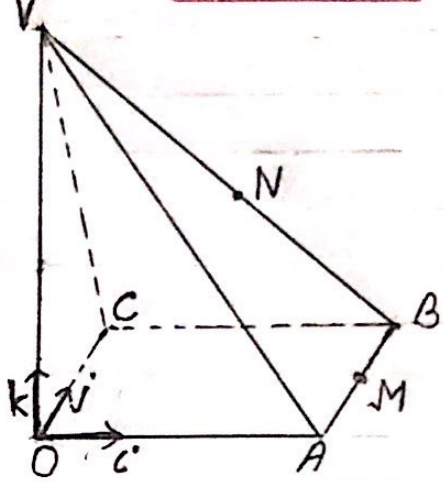
(a) $\vec{BA} \cdot \vec{BC} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} = 2 \times (-2) + 4 \times 4 + (-6) \times 2 = 0$

$\Rightarrow \vec{BA} \perp \vec{BC} \Rightarrow$ angle ABC is a right angle. ✓

(b) $\vec{AD} = \vec{OD} - \vec{OA} = \begin{pmatrix} -10 \\ 20 \\ 10 \end{pmatrix} = 5\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} = 5\vec{BC} \Rightarrow \vec{AD} = 5\vec{BC}$ ✓

Hence AD and BC are parallel.

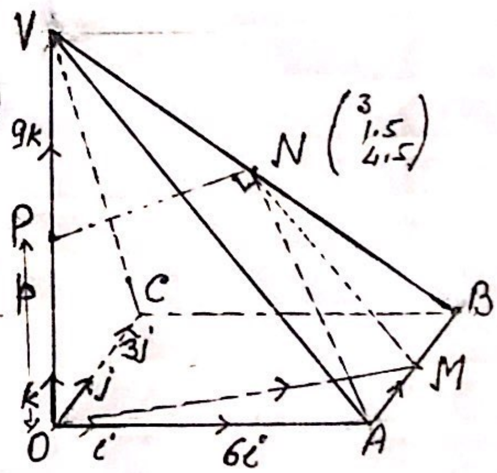
5. The diagram shows a pyramid VOABC. The base OABC, is a rectangle. The unit vectors $i$, $j$ and $k$ are parallel to $\vec{OA}$, $\vec{OC}$, and $\vec{OV}$ respectively. The position vectors of the points A, C and V are given by, $\vec{OA} = 6i$, $\vec{OC} = 3j$ and $\vec{OV} = 9k$.
The points M and N are the mid points of AB and VB respectively.




(a) Find the angle between the directions of $\vec{AN}$, $\vec{OC}$.
(b) Find the vector $\vec{MN}$
(c) The point P lies on OV and is such that angle PNM is a right angle. Find the position vector of the point P.

**Solution**
(a) $\vec{OV} = \begin{pmatrix}0\\0\\9\end{pmatrix}$; $\vec{OB} = \begin{pmatrix}6\\3\\0\end{pmatrix} \Rightarrow \vec{ON} = \frac{1}{2}(\vec{OV} + \vec{OB})$

$\vec{ON} = \frac{1}{2}\begin{pmatrix}6\\3\\9\end{pmatrix} = \begin{pmatrix}3\\1.5\\4.5\end{pmatrix}$ --- (1)




$\vec{AN} = \vec{ON} - \vec{OA} = \begin{pmatrix}3\\1.5\\4.5\end{pmatrix} - \begin{pmatrix}6\\0\\0\end{pmatrix} = \begin{pmatrix}-3\\1.5\\4.5\end{pmatrix}$ --- (2)

$\vec{OC} = \begin{pmatrix}0\\3\\0\end{pmatrix}$ --- (3)

$|\vec{AN}| = \sqrt{3^2 + 1.5^2 + 4.5^2} = \frac{3\sqrt{14}}{2}$

$|\vec{OC}| = 3$

Angle between $\vec{AN}$ and $\vec{OC}$, '$\theta$'; $\cos\theta = \dfrac{\vec{AN}\cdot\vec{OC}}{|\vec{AN}|\cdot|\vec{OC}|}$

$\Rightarrow \theta = \cos^{-1}\left[\dfrac{\begin{pmatrix}-3\\1.5\\4.5\end{pmatrix}\cdot\begin{pmatrix}0\\3\\0\end{pmatrix}}{\frac{3\sqrt{14}}{2}\times 3}\right] = \cos^{-1}\left(\dfrac{4.5}{4.5\sqrt{14}}\right) = 74.5°$ ✓

(b) $\vec{ON} = \begin{pmatrix}3\\1.5\\4.5\end{pmatrix}$; $\vec{OM} = \vec{OA} + \vec{AM} = \begin{pmatrix}6\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1.5\\0\end{pmatrix} = \begin{pmatrix}6\\1.5\\0\end{pmatrix}$ {In $\Delta$ OAM}

$\Rightarrow \vec{MN} = \vec{ON} - \vec{OM} = \begin{pmatrix}3\\1.5\\4.5\end{pmatrix} - \begin{pmatrix}6\\1.5\\0\end{pmatrix} = \begin{pmatrix}-3\\0\\4.5\end{pmatrix}$ ✓ ---- (4)

(c) P is a point of OV, let $\vec{OP} = \begin{pmatrix}0\\0\\p\end{pmatrix} \Rightarrow \vec{PN} = \vec{ON} - \vec{OP} = \begin{pmatrix}3\\1.5\\4.5-p\end{pmatrix}$ --- (5)

Given angle PNM is a right angle $\Rightarrow \vec{PN} \perp \vec{MN} \Rightarrow \vec{PN}\cdot\vec{MN} = 0 \Rightarrow \begin{pmatrix}3\\1.5\\4.5-p\end{pmatrix}\cdot\begin{pmatrix}-3\\0\\4.5\end{pmatrix} = 0$ from (4) & (5)

$\Rightarrow (3)(-3) + 0 + 4.5(4.5-p) = 0$

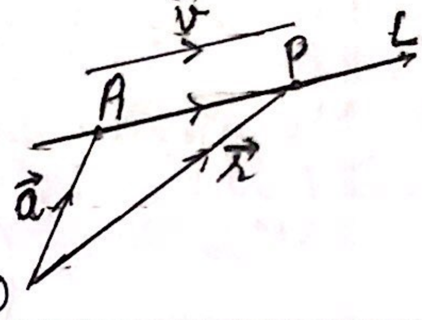
$\Rightarrow -9 + 20.25 - 4.5p = 0 \Rightarrow 4.5p = 11.25$

$\Rightarrow p = 2.5$

$\therefore \vec{OP} = \begin{pmatrix}0\\0\\2.5\end{pmatrix}$ (or $= 2.5k$)

## § Equation of a line 'L' passing through a given point A whose position vector is $\vec{a}$ and the direction of line is vector $\vec{v}$.

Let P is any point of the line 'L' and $\vec{OP} = \vec{r}$, Then equation of line 'L'




$$\vec{OP} = \vec{OA} + \vec{AP}$$

$$\vec{r} = \vec{a} + \lambda \vec{v} \quad \text{---(i)}$$

here given point $\vec{a} = x_1 i + y_1 j + z_1 k = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ as $A(x_1, y_1, z_1)$

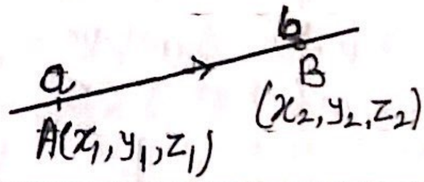
and variable point $\vec{r} = xi + yj + zk = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

and the direction of line 'L',

$$\vec{v} = v_1 i + v_2 j + v_3 k = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$\therefore$ Equation of line 'L' $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_2 \end{pmatrix}$ --(ii)

## § Equation of a line 'L' passing through two point $A(x_1, y_1, z_1)\vec{a}$ and $B(x_2, y_2, z_2)\vec{b}$




here $\vec{AB} = \vec{b} - \vec{a} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$

$\therefore$ Equation of line

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \quad \text{--(iii)}$$

or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$ fr (i) $\left\{ \begin{array}{l} \text{where} \\ \vec{v} = \vec{b} - \vec{a} \\ = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} \end{array} \right.$

(P3)

# Vectors in 3D



§ To Verify that two given lines $L_1$ and $L_2$ are Parallel (or Coincident / or Intersecting / or Skew lines):

$L_1: \vec{r} = \vec{a} + \mu \vec{v}$ ---(i) where $\vec{v} = v_1 i + v_2 j + v_3 k = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

and $L_2: \vec{r} = \vec{b} + \lambda \vec{u}$ ---(ii) where $\vec{u} = u_1 i + u_2 j + u_3 k = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

and $\vec{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ & $\vec{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

Case (a) $L_1 \parallel L_2 \Leftrightarrow \vec{u} = c\vec{v}$ : $c \in R$, $c \neq 0$

or $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = c \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ ✓

Case (b) $L_1$ and $L_2$ are coincident if

(i) $\vec{u} = c_1 \vec{v}$ ✓

and (ii) $\vec{b} - \vec{a} = c_2 \vec{u}$

Case (c) Intersecting if $\vec{u} \neq c_1 \vec{v}$ or (If not intersecting then skew lines)

and we can find out a common point (Point of Intersection).

for that

Any point on $L_1$, $\vec{r} = \begin{pmatrix} x_1 + \mu v_1 \\ y_1 + \mu v_2 \\ z_1 + \mu v_3 \end{pmatrix}$ --(iii) and Any point on $L_2$ $\vec{r} = \begin{pmatrix} x_2 + \lambda u_1 \\ y_2 + \lambda u_2 \\ z_2 + \lambda u_3 \end{pmatrix}$ ---(iv)

Equating them $\begin{pmatrix} x_1 + \mu v_1 \\ y_1 + \mu v_2 \\ z_1 + \mu v_3 \end{pmatrix} = \begin{pmatrix} x_2 + \lambda u_1 \\ y_2 + \lambda u_2 \\ z_2 + \lambda u_3 \end{pmatrix} \Rightarrow \begin{cases} x_1 + \mu v_1 = x_2 + \lambda u_1 \\ y_1 + \mu v_2 = y_2 + \lambda u_2 \\ z_1 + \mu v_3 = z_2 + \lambda u_3 \end{cases}$

$$\Rightarrow \begin{cases} \mu v_1 - \lambda u_1 = x_2 - x_1 & \text{---(v)} \\ \mu v_2 - \lambda u_2 = y_2 - y_1 & \text{---(vi)} \\ \mu v_3 - \lambda u_3 = z_2 - z_1 & \text{---(vii)} \end{cases}$$

Solve (v) & (vi) for $\lambda$ and $\mu$ ✓

Verify that these values of $\lambda$ and $\mu$ satisfy (vii) then Intersecting and Point of Intersection Put the value of $\lambda$ & $\mu$ in (iii) & (iv) ✓

Case (d) Line $L_1$ & $L_2$ are Skew if $L_1 \nparallel L_2$ and non-Intersecting

# Vectors in 3D



**Example 6** The points A and B have position vectors given by, $\vec{OA} = i - 2j + 2k$ and $\vec{OB} = 3i + j + k$. The line '$l$' has equation $r = 2i + j + mk + \mu(i - 2j - 4k)$, where $m$ is a constant.
Given that the line '$l$' intersects the line passing through A and B, find the value of $m$. [S-17/33/Q10(i)]

**Solution:** $\vec{OA} = i - 2j + 2k$ and $\vec{OB} = 3i + j + k$

$\therefore$ The direction of line AB, $\vec{v} = \vec{OB} - \vec{OA} = (2i + 3j - k)$

$\therefore$ equation of line AB, $\vec{r} = \vec{a} + \lambda v$

or $\vec{r} = (i - 2j + 2k) + \lambda(2i + 3j - k)$

Any point of line AB
or

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1+2\lambda \\ -2+3\lambda \\ 2-\lambda \end{pmatrix} \quad \text{—— (1)}$$

Equation of line '$l$'; $\vec{r} = 2i + j + mk + \mu(i - 2j - 4k)$

or
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2+\mu \\ 1-2\mu \\ m-4\mu \end{pmatrix} \quad \text{—— (2)}$$

Given that line '$l$' intersects line 'AB', i.e. equating the respective values of $x, y$ & $z$

$$1+2\lambda = 2+\mu \Rightarrow 2\lambda - \mu = +1 \quad \text{—— (3)}$$
$$-2+3\lambda = 1-2\mu \Rightarrow 3\lambda + 2\mu = 3 \quad \text{—— (4)}$$
$$2-\lambda = m-4\mu \Rightarrow -\lambda + 4\mu = m-2 \quad \text{—— (5)}$$

Solving (3) & (4) we get $\lambda = \frac{5}{7}$, $\mu = \frac{3}{7}$

Put the value of $\lambda$ & $\mu$ in (5) $-\frac{5}{7} + 4 \times \frac{3}{7} = m - 2$

$$\Rightarrow \underline{m = 3} \checkmark$$

Example 7. Two lines $l$ and $m$ have equations,
$r = 2i - j + k + s(2i + 3j - k)$ and $r = i + 3j + 4k + t(i + 2j + k)$
respectively. Show that the line are skew. [S-18/32/Q10(i)]

Solution: Eqⁿ of line '$l$' $r = 2i - j + k + s(2i + 3j - k)$ —— ①

$$\text{or} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2+2s \\ -1+3s \\ 1-s \end{pmatrix} \quad \text{—— ②}$$

Equation of line '$m$', $r = i + 3j + 4k + t(i + 2j + k)$ —— ③

$$\text{or} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+t \\ 3+2t \\ 4+t \end{pmatrix} \quad \text{—— ④}$$

Case(a) direction line $l = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and direction line $m = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

we find $l$ is not parallel to $m$ as $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \neq k\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

(where $k \in R$, $k \neq 0$

Case(b) To check that $l$ and $m$ don't intersect;
If at all $l$ and $m$ intersect, the two should have a common point, Then

$$\begin{pmatrix} 2+2s \\ -1+3s \\ 1-s \end{pmatrix} = \begin{pmatrix} 1+t \\ 3+2t \\ 4+t \end{pmatrix} \Rightarrow \begin{cases} 2s - t = -1 & \text{—— ⑤} \\ 3s - 2t = 4 & \text{—— ⑥} \\ -s - t = 3 & \text{—— ⑦} \end{cases}$$

Solving ⑤ & ⑥ we get $s = -6$, $t = -11$
Put these values of $s$ and $t$ in eqnat ⑦
$6 + 11 = 3$ false.

$\therefore$ The lines $l$ and $m$ dont intersect

$\therefore$ Two lines $l$ and $m$ are Skew lines
(Non-parallel and non-intersecting).

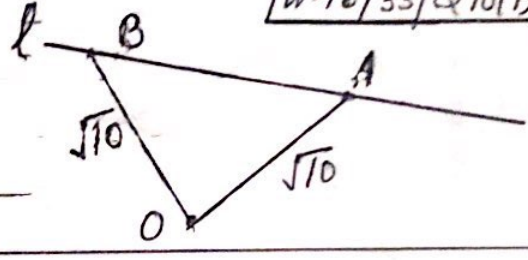
# Vectors in 3D

**Example 8** The line $l$ has vector equation:

$$r = i + 2j + k + \lambda(2i - j + k)$$

Find the position vector of two points on the line whose distance from origin is $\sqrt{10}$.

W-16/33/Q10(i)




**Solution:**

Any point of line $l$. (A or B)

Let A is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2\lambda \\ 2-\lambda \\ 1+\lambda \end{pmatrix}$ —— (1)

Given $OA$ (or $OB$) $= \sqrt{10}$

$\Rightarrow OA^2 = 10$

from (1) $OA^2 = (1+2\lambda)^2 + (2-\lambda)^2 + (1+\lambda)^2 = 10$

or $6\lambda^2 + 2\lambda - 4 = 0$

$\Rightarrow \lambda = -1, \ 2/3$

from (1) Two points $\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 7/3 \\ 4/3 \\ 5/3 \end{pmatrix}$

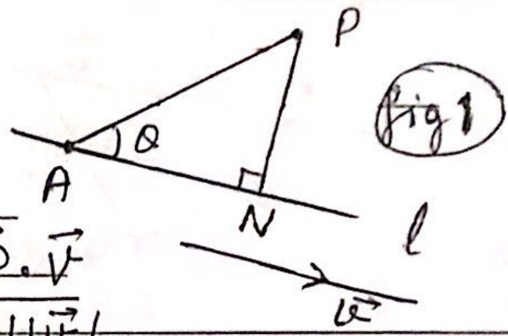
$\therefore$ Position vectors of two points are,

$-i + 3j$ and $\frac{7}{3}i + \frac{4}{3}j + \frac{5}{3}k$

# Vectors in 3D



## § To find the length of perpendicular from a point P to a line 'l'.




Let A is a given point of line l,
Let direction of line l is $\vec{v}$
and $PN \perp l$,

$$\frac{AN}{AP} = \cos\theta = \frac{\vec{AP}\cdot\vec{v}}{|\vec{AP}||\vec{v}|}$$

or $$AN = \frac{\vec{AP}\cdot\vec{v}}{|\vec{AP}||\vec{v}|} \times AP$$

Projection of $\vec{AP}$ on line $l = AN = \frac{\vec{AP}\cdot\vec{v}}{|\vec{v}|}$ —— ①

$\therefore PN = \sqrt{AP^2 - AN^2}$ —— ② ✓

Example 9. The point P has position vector $3i-2j+k$. The line 'l' has equation $r = 4i+2j+5k+\mu(i+2j+3k)$ [S-18/31/Q10(i)]
Find the length of perpendicular from P to l, ---[5]
(see fig 1)

Solution:
$\vec{OP} = 3i-2j+k$ --- ③
point of line 'l' A, $\vec{OA} = (4i+2j+5k)$ --- ④
direction of line l, $\vec{v} = i+2j+3k$ & $|\vec{v}| = \sqrt{14}$
from ③ & ④ $\vec{AP} = \vec{OP} - \vec{OA} = -i-4j-4k$
from ① $AN = \frac{\vec{AP}\cdot\vec{v}}{|\vec{v}|} = \left|\frac{-1-8-12}{\sqrt{14}}\right| = \frac{+21}{\sqrt{14}}$

$|\vec{AP}| = \sqrt{33}$

from ② $PN = \sqrt{AP^2 - AN^2} = \sqrt{33 - \frac{441}{14}}$

$= \sqrt{1.5} = 1.22$ ✓

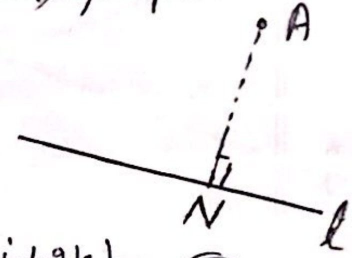
# Vectors in 3D

To find the the foot of perpendicular from a point to a line.

**Example 10.**

The position vector of a point A; $\vec{OA} = i+2j+4k$.
The line 'l' has equation, $r = 9i-j+8k+\mu(3i-j+2k)$ --- [5]
find the position vector of the foot of perpendicular from A to 'l'. Hence find the position vector of the reflection of A in l.
[S-17/32/Q9(i)]

**Solution:** draw AN perp to l, hence 'N' is the foot of perp.




$$\vec{OA} = i+2j+4k = \begin{pmatrix}1\\2\\4\end{pmatrix} \quad \text{--- (1)}$$

Equation of line l is $\vec{r} = 9i-j+8k+\mu(3i-j+2k)$ --- (2)

or any point of line 'l' $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}9+3\mu\\-1-\mu\\8+2\mu\end{pmatrix}$ --- (3)

from (3) & (1)

$$\therefore \vec{AN} = \vec{ON} - \vec{OA} = \begin{pmatrix}9+3\mu\\-1-\mu\\+8+2\mu\end{pmatrix} - \begin{pmatrix}1\\2\\4\end{pmatrix} = \begin{pmatrix}8+3\mu\\-3-\mu\\4+2\mu\end{pmatrix} \quad \text{--- (4)}$$

from (2) direction of line l $\vec{v} = \begin{pmatrix}3\\-1\\2\end{pmatrix}$

as AN is perp to l, $\vec{AN}\cdot\vec{v} = 0$

$$\begin{pmatrix}+8+3\mu\\-3-\mu\\4+2\mu\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\2\end{pmatrix} = 0$$

$\Rightarrow 3(8+3\mu) - 1(-3-\mu) + 2(4+2\mu) = 0 \Rightarrow 35+14\mu = 0$

$\Rightarrow \mu = -5/2$

from (3) for $\mu = -\frac{5}{2}$; $\vec{ON} = \begin{pmatrix}3/2\\3/2\\3\end{pmatrix} = \left(\frac{3}{2}i + \frac{3}{2}j + 3k\right)$ ✓
Req. foot of perp =

Now let Reflection A in l is $B(x_1, y_1, z_1)$; Mid point of AB is N

$\left(\frac{1+x_1}{2}, \frac{2+y_1}{2}, \frac{4+z_1}{2}\right) \equiv \left(\frac{3}{2}, \frac{3}{2}, 3\right) \Rightarrow B(2,1,2) \Rightarrow \vec{OB} = 4i+j+2k$ ✓

Example 11. The line $l$ has equation:

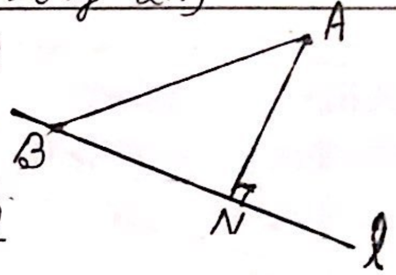
$$r = 4i - 9j + 9k + \lambda(-2i + j - 2k).$$

The point A has position vector $3i + 8j + 5k$

Show that the length of perpendicular from A to $l$ is 15. ---[5]

[W-14/31/Q10(i)]

Solution. Equation of line '$l$' $r = 4i - 9j + 9k + \lambda(-2i + j - 2k)$ —— ①




Direction of line, $\vec{v} = -2i + j - 2k$ —— ②

and Given $\vec{OA} = 3i + 8j + 5k$ —— ③

Position vector of B on $l$.

from ① $\vec{OB} = 4i - 9j + 9k$ —— ④

Let N be any point of line $l$,

$\vec{ON} = (4-2\lambda)i + (-9+\lambda) + (9-2\lambda)k$ —— ⑤

Let AN is perpendicular to $l$, Required distance $AN = \sqrt{AB^2 - BN^2}$ —— ⑥

$\vec{BA} = \vec{OA} - \vec{OB} = -i + 17j - 4k$ —— ⑦

Now the projection of AB on $l$ $= BN = \dfrac{\vec{BA} \cdot \vec{v}}{|\vec{v}|}$ —— ⑧

$$= \frac{(-i + 17j - 4k)\cdot(-2i + j - 2k)}{|-2i + j - 2k|}$$

$$BN = \left|\frac{+2 + 17 + 8}{3}\right| = 9 ✓ —— ⑨$$

from ⑧ $AN = \sqrt{AB^2 - BN^2}$

from ⑨ & ⑩

$= \sqrt{306 - 9^2}$

$= \sqrt{225} = 15.$ ✓

from ⑦

$|\vec{BA}| = \sqrt{1 + 289 + 16}$

$= \sqrt{306}$

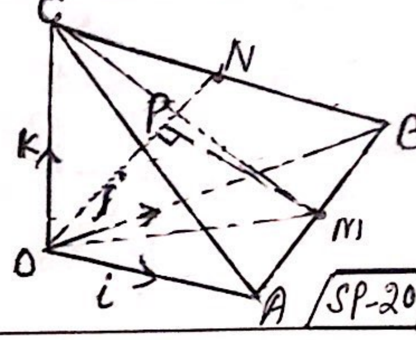
$AB^2 = 306$ ✓ —— ⑩

Example 12. In the diagram, OABC is a pyramid in which $OA = 2$ units, $OB = 4$ units and $OC = 2$ units. The edge OC is vertical, the base OAB is horizontal and angle $AOB = 90°$. Unit vectors $i, j, k$ are parallel to OA, OB and OC, respectively. The mid points of AB and BC are M and N respectively.

(a) Express the vectors $\vec{ON}$ and $\vec{CM}$ in terms of $i, j$ and $k$. --- [3]

(b) Calculate the angle between the directions of $\vec{ON}$ and $\vec{CM}$ --[3]

(c) Show that the length of perpendicular from M to ON is $\frac{3}{5}\sqrt{5}$ --[4]




SP-20/03/Q8

Solution: $\vec{OA} = 2i$; $\vec{OB} = 4j$, $\vec{OC} = 2k$

(a) $\vec{ON} = \frac{\vec{OC} + \vec{OB}}{2} = \frac{1}{2}(2k + 4j) = (2j + k)$ ✓

$\vec{CM} = \vec{OM} - \vec{OC} = \frac{1}{2}(2i + 4j) - 2k = i + 2j - 2k$ ✓

(b) angle between $\vec{ON}$ & $\vec{CM}$, $\cos\theta = \frac{\vec{ON} \cdot \vec{CN}}{|\vec{ON}|\,|\vec{CN}|} = \frac{(2j + k)\cdot(i + 2j - 2k)}{\sqrt{5}\sqrt{9}}$

$= \frac{4 - 2}{3\sqrt{5}} = \frac{2}{3\sqrt{5}}$

$\cos\theta = \frac{2\sqrt{5}}{15}$

$\theta = \cos^{-1}\frac{2\sqrt{5}}{15} = \cos^{-1} 0.2981 = 72.7°$ ✓

(c) Let MP is perp to ON $\Rightarrow MP = \sqrt{OM^2 - OP^2}$

Projection of $\vec{OM}$ on $\vec{ON} = OP = \frac{\vec{OM} \cdot \vec{ON}}{|\vec{ON}|} = \frac{(i + 2j)\cdot(2j + k)}{\sqrt{4+1}}$

$\therefore OP = \frac{4}{\sqrt{5}}$ ✓

In rt triangle MPO, $MP = \sqrt{OM^2 - OP^2}$

$= \sqrt{5 - \frac{16}{5}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$ ✓

$\therefore$ Required perp distance $= \frac{3\sqrt{5}}{5}$

(P3)

# Vectors in 3D (Line)

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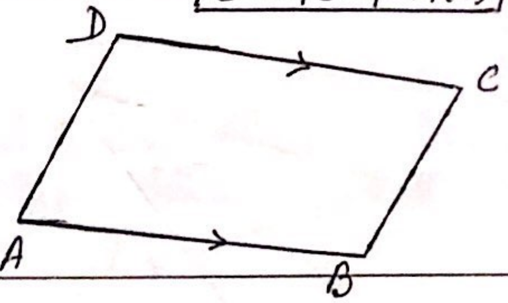
**Example 13:** The points A, B and C have position vectors $\vec{OA} = i + 2j + 3k$, $\vec{OB} = 4j + k$ and $\vec{OC} = 2i + 5j - k$. A fourth point D is such that the quadrilateral ABCD is a parallelogram. ---[5]
Find the position vector of D and verify that the parallelogram is a rhombus. [S-16/32/Q9(i)]




**Solution:** Given

$\vec{OA} = i + 2j + 3k$

$\vec{OB} = 4j + k$

$\vec{OC} = 2i + 5j - k$

Let $\vec{OD} = (xi + yj + zk)$

for ABCD is a parallelogram.

$\therefore \quad \vec{AB} = \vec{DC}$

$\Rightarrow \quad -i + 2j - 2k = (2-x)i + (5-y)j + (-1-z)k$

$$\Rightarrow \left\{ \begin{array}{l} 2-x = -1 \Rightarrow x = 3 \\ 5-y = 2 \Rightarrow y = 3 \\ -1-z = -2 \Rightarrow z = 1 \end{array} \right.$$

$\therefore$ D(3, 3, 1)

or $\vec{OD} = 3i + 3j + k$

Now $|\vec{AB}| = |-i + 2j - 2k| = \sqrt{9} = 3$

and $|\vec{BC}| = |2i + j - 2k| = \sqrt{9} = 3$

$\therefore$ adjacent sides AB = BC of the parallelogram, $\therefore$ ABCD is a Rhombus