DATE 27.01.22

S-2

Probability and Statistics-2

Continuous Random Variable Notes and Revision.

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S.2

Continuous Random Variables – Notes and Revision.

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§ **Probability Density Function (PDF):**

$f(x)$ representing a continuous random variable is the probability density function (PDF) such that:

(i) $f(x) \geq 0$ as Probability ≥ 0.

(ii) Total probability $= 1$ or $\int_{-\infty}^{\infty} f(x)dx = 1$

Frequency density $= \frac{\text{Frequency}}{\text{Class width}}$.

Probability density $= \frac{\text{Frequency density}}{\text{Total Frequency}}$.

Many times the data are defined across a specified interval/across specified intervals.

(iii) For a continuous random variable with PDF $f(x)$, $P(X=a)=0$

and $P(a<x<b) = P(a \leq x < b) = P(a < x \leq b) = P(a \leq x \leq b)$

(iv) The prob. of $X \in (a,b)$ is the area, $P(a<X<b) = \int_a^b f(x)\,dx$

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1. The time, T minutes, taken by people to complete a test has probability density function given by: $f(t) = \begin{cases} k(10t - t^2) & : 5 \leq t \leq 10 \\ 0 & \text{otherwise} \end{cases}$ where k is a constant.

Show that $k = \frac{3}{250}$ --[3]

[S-16/73/Q5(i)]

Solution: Total Prob $= \int_5^{10} k(10t - t^2)dt = k\left[5t^2 - \frac{t^3}{3}\right]_5^{10} = k\left[\left(500 - \frac{1000}{3}\right) - \left(125 - \frac{125}{3}\right)\right] = 1$

$\Rightarrow K \times \frac{250}{3} = 1 \Rightarrow K = \frac{250}{3}$ ✓

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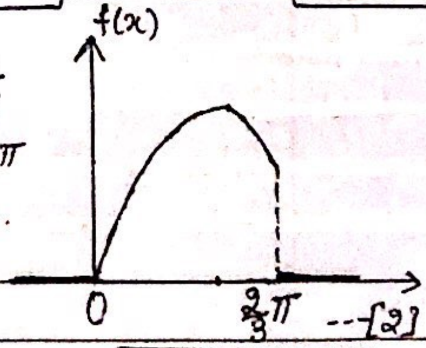
2. A random variable X has probability density function given by: $f(x) = \begin{cases} k\sin x & , 0 \leq x \leq \frac{2}{3}\pi \\ 0 & \text{otherwise} \end{cases}$

Where k is a constant.

Show that $k = 2/3$ --[2]



[S-12/73/Q7]

Solution: Area under the curve $= \int_0^{\frac{2\pi}{3}} k\sin x\,dx = 1$

$\Rightarrow k\left[-\cos x\right]_0^{\frac{2\pi}{3}} = 1 \Rightarrow k\left(-\cos\frac{2\pi}{3} + \cos 0\right) = 1$

$\Rightarrow k[0.5 + 1] = 1 \Rightarrow \frac{3}{2}k = 1 \Rightarrow k = 2/3$ ✓

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§ Median and other percentiles of a continuous random variable:

(i) The median, 'm', of continuous random variable is the value for which,
(50% percentile) $P(X < m) = \int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$

(ii) The value 'r' of continuous random variable is that value that represents the say 20_{th} percentile, then $\int_{-\infty}^{r} f(x)\,dx = 20\%$ or 0.2

(iii) Interquartile range $IQR = UQ - LQ$ or $= q_3 - q_1$ $\begin{cases} UQ = 75\% \text{ percentile} = q_3 \\ LQ = 25\% \text{ percentile} = q_1 \end{cases}$

Q5 (i) (ii) (iii) (iv)

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3. The random variable X has probability density function given by:

$$f(x) = \begin{cases} 4x^k & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(i) Show that $k = 3$ [2]
(ii) Find the upper quartile of X. [2]
(iii) Find the interquartile range of X. [2]

S-06/7/Q5

Solution: (i) Total sum of prob: $\int_0^1 4x^k\,dx = 1 \Rightarrow \left[\frac{4x^{k+1}}{k+1}\right]_0^1 = 1$

$$= \frac{4}{k+1}[1-0] = 1 \Rightarrow k+1 = 4 \Rightarrow k = 3 \checkmark$$

(ii) for upper quartile q_3: (75% Percentile) $\int_0^{q_3} 4x^3\,dx = 0.75\ [75\%]$ $\left\{ \begin{array}{l} \text{for } k=3 \\ f(x) = 4x^3 \\ 0 \leq x \leq 1 \end{array} \right.$

$$\Rightarrow [x^4]_0^{q_3} = 0.75 \Rightarrow q_3^4 = 0.75 \Rightarrow q_3 = \sqrt[4]{0.75} = 0.931 \checkmark$$

(iii) Now Lower quartile q_1: (25% Percentile) $\int_0^{q_1} 4x^3\,dx = 0.25 \Rightarrow [x^4]_0^{q_1} = 0.25$

$$\Rightarrow q_1^4 = 0.25 \Rightarrow q_1 = \sqrt[4]{0.25} = 0.707 \checkmark$$

$\therefore$ Interquartile range $IQR = q_3 - q_1$
$$= 0.931 - 0.707 = 0.224 \checkmark$$

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4. Bottles of Lanta contains approximately 300ml of juice. The volume of juice, millilitres, in bottle is 300+X, where X is a random variable with probability density function given by:

$$f(x) = \begin{cases} \frac{3}{4000}(100-x^2) & -10 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the probability that a randomly chosen bottle of Lanta contains more than 305 ml of juice. ---[3]

(b) Given that 25% of bottles of Lanta contains more than (300+p) ml of juice, Show that: $p^3 - 300p + 1000 = 0$ ---[4]

(C) Given that p = 3.47, and that 50% of bottles of Lanta contains between (300−q) and (300+q) ml of juice, find q. Justify your answer. ---[2]

M-20/62/Q5

Solution: (a) $V > 305 \rightarrow 300+5 \rightarrow X=5$, $P(X>5) = \int_5^{10} \frac{3}{4000}(100-x^2)\,dx$

$$= \frac{3}{4000}\left[100x - \frac{x^3}{3}\right]_5^{10} = \frac{3}{4000}\left[\left(1000 - \frac{1000}{3}\right) - \left(500 - \frac{125}{3}\right)\right] = 0.156$$

(b) for $V > 300+p \rightarrow X = p \rightarrow \frac{3}{4000}\int_p^{10}(100-x^2)\,dx = 0.25$ (25%)

$$\Rightarrow \frac{3}{4000}\left[100x - \frac{x^3}{3}\right]_p^{10} = \frac{1}{4}$$

$$\Rightarrow \frac{3}{4000}\left[\left(1000 - \frac{1000}{3}\right) - \left(100p - \frac{p^3}{3}\right)\right] = \frac{1}{4}$$

$$\Rightarrow \frac{2000}{3} - 100p + \frac{p^3}{3} = \frac{1000}{3} \Rightarrow p^3 - 300p + 1000 = 0$$

(C) Curve is symmetrical about $x=0$

$q = 3.47$

§ Expectation and Variance :

From S_1:
In Prob. distribution:
$E(X) = \Sigma x \cdot P(x)$
$Var(X) = \Sigma x^2 \cdot P(x) - (E(X))^2$

For continuous random variable X,
Prob. density function (PDF) f(X).

(i) $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

(ii) $Var(X) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx - \left\{ \int_{-\infty}^{\infty} x f(x)\, dx \right\}^2$ $\quad \{ \sigma^2 = E(X^2) - (E(X))^2$

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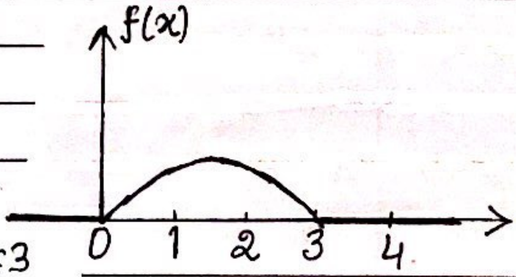
5. The diagram shows the graph of the prob. density function, f, of a random variable X, where



$$f(x) = \begin{cases} \frac{2}{9}(3x - x^2) & 0 \le x \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

(a) State the value of E(X) and find Var(X). --[4]
(b) State the value of $P(1.5 \le X \le 4)$ --[1]
(c) Given that $P(1 \le X \le 2) = \frac{13}{27}$, find $P(X > 2)$. --[2]

SP-20/06/Q5

Solution (a) $E(X) = \underline{1.5}$ ✓

$E(X^2) = \int_0^3 x^2 \cdot f(x)\, dx = \int_0^3 x^2 \frac{(3x - x^2)2}{9} dx$

$= \frac{2}{9} \int_0^3 (3x^3 - x^4)\, dx$

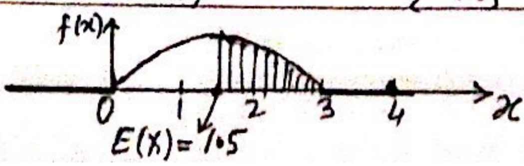
$= \frac{2}{9} \left[\frac{3x^4}{4} - \frac{x^5}{5} \right]_0^3$

$= \frac{2}{9} \left[\frac{243}{4} - \frac{243}{5} \right] = 2.7$

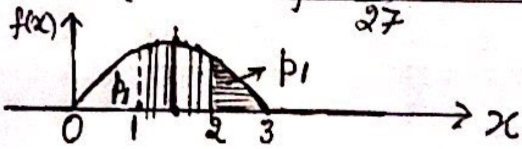
$\therefore Var(X) = E(X^2) - (E(X))^2$

$= 2.7 - (1.5)^2 = 0.45$ ✓

(b) $P(1.5 \le X \le 4) = 0.5$ $\quad \{E(X)\}$



(c) Given $P(1 \le X \le 2) = \frac{13}{27}$



Let $P(X > 2) = p_1$

$\Rightarrow 2p_1 + \frac{13}{27} = 1$ (Total area)

$p_1 = \frac{1}{2}\left(1 - \frac{13}{27}\right)$

$P(X > 2) = \underline{\frac{7}{27}}$ (or 0.259) ✓

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6 The probability density function, f, of a random variable X, is given by:

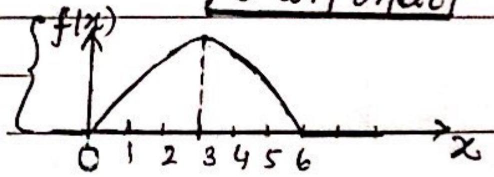
$$f(x) = \begin{cases} k(6x - x^2) & 0 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

State the value of E(X) and show that, $Var(X) = \frac{9}{5}$ ---[6]

$\boxed{S-21 / 61 / Q6}$

Solution: $E(X) = 3$ ✓



$$\int_0^6 k(6x - x^2)dx = 1 \Rightarrow k\left[3x^2 - \frac{x^3}{3}\right]_0^6 = 1$$

$$\Rightarrow k\left[108 - \frac{216}{3}\right] = 1 \Rightarrow k = \frac{1}{36} ✓$$

$$\left\{ E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx \right.$$

$$E(X^2) = \frac{1}{36}\int_0^6 x^2(6x - x^2)dx = \frac{1}{36}\int_0^6 (6x^3 - x^4)dx = \frac{1}{36}\left[\frac{6x^4}{4} - \frac{x^5}{5}\right]_0^6 = 10.8 ✓$$

Now $Var(X) = E(X^2) - [E(X)]^2 = 10.8 - 3^2 = 1.8$ ✓ (or $\frac{9}{5}$) ✓

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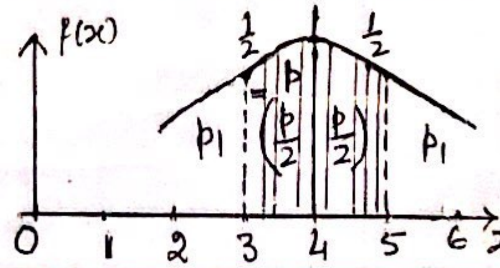
7. The graph of the probability density function of a random variable X is symmetrical about the line $x = 4$

Given that $P(X < 5) = \frac{20}{27}$

find $P(3 < X < 5)$.



Solution: Line of Symmetry: $X = 4$.

$$P(X < 5) = \frac{20}{27} =$$

Let $P(3 < X < 5) = p$

$$\therefore P(X < 5) = p + p_1 = \frac{20}{27} \quad ---(i)$$

and $$p + 2p_1 = 1 \quad ---(ii)$$

from (i) $$2p + 2p_1 = \frac{40}{27} \quad ---(iii)$$

$$(iii) - (ii) \Rightarrow p = \frac{40}{27} - 1 = \frac{13}{27} ✓$$

$$\therefore P(3 < X < 5) = \frac{13}{27} ✓$$

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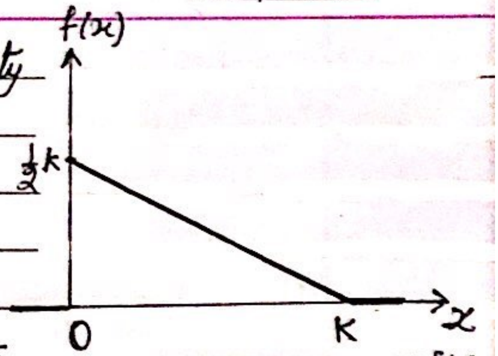
8. The diagram shows the graph of the probability density function, f, of a random variable X.



(a) Find the value of the constant k. --- [2]

(b) Using this value of k, find f(x) for $0 \le x \le k$, and hence find E(X) --- [3]

(c) Find the value of p such that $P(p<X<1)=0.25$ --- [4]

[M-21/62/Q2]

Solution (a) ~~Area~~ Area under the line, area of triangle $= \frac{1}{2} \times k \times \frac{1}{2}k = 1$

$$\Rightarrow k^2 = 4 \Rightarrow k=2 \checkmark \longrightarrow ; \frac{k}{2}=1$$

(b) $f(x)$: Joing $(2,0)$ and $(0,1)$ $\Rightarrow y- = \frac{1-0}{0-2}(x-2) \Rightarrow y = -\frac{1}{2}x+1 \checkmark$

$\therefore f(x) = -\frac{1}{2}x+1 \checkmark$

$$E(X) = \int_0^2 x.f(x)\,dx = \int_0^2 x\left(-\frac{1}{2}x+1\right)dx$$

$$\Rightarrow E(X) = \int_0^2 \left(-\frac{1}{2}x^2+x\right)dx = \left[-\frac{x^3}{6}+\frac{x^2}{2}\right]_0^2 = \frac{2}{3} \checkmark$$

(c) $P(p<x<1) = 0.25$

$$\Rightarrow \int_p^1 f(x)\,dx = 0.25 \Rightarrow \int_p^1 \left(-\frac{1}{2}x+1\right)dx = 0.25 \Rightarrow \left[\frac{-x^2}{4}+x\right]_p^1 = 0.25$$

$$\Rightarrow -\frac{1}{4}+1+\frac{p^2}{4}-p = 0.25 \Rightarrow p^2-4p+2=0$$

$$\Rightarrow p = 2-\sqrt{2} \checkmark \text{ (or } 0.586)$$

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9. A random variable X has probability density function given by:

$$f(x) = \begin{cases} \frac{1}{18}(9-x^2) & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) find $P(X<1.2)$ ---- [3]

(b) Find E(X) --- [3]

(c) The median of X is m; show that $m^3-27m+27=0$ --- [3]

[W-21/61/Q4]

Solution (a) $P(X<1.2) = \int_0^{1.2} \frac{1}{18}(9-x^2)\,dx$

$$= \frac{1}{18}\left[9x-\frac{x^3}{3}\right]_0^{1.2} = \frac{71}{125} \checkmark \text{ (or } 0.568)$$

(b) $E(X) = \int_0^3 x.\frac{1}{18}(9-x^2)\,dx$

$$= \frac{1}{18}\int_0^3 (9x-x^3)\,dx = \frac{1}{18}\left[\frac{9x^2}{2}-\frac{x^4}{4}\right]_0^3$$

$$= \frac{9}{8} \text{ (or } 1.125) \checkmark$$

(c) $\int_0^m f(x)\,dx = 0.5$

$$\Rightarrow \frac{1}{18}\int_0^m (9-x^2)\,dx = 0.5$$

$$\Rightarrow \frac{1}{18}\left[9x-\frac{x^3}{3}\right]_0^m = 0.5$$

$$\Rightarrow \frac{1}{18}\left[9m-\frac{m^3}{3}\right]-0.5=0$$

$$\Rightarrow m^3-27m+27=0 \checkmark$$

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10. A random variable X has probability density function given by:

$$f(x) = \begin{cases} \frac{k}{x^2} & 1 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

Where k and a are positive constant.

(a) Show that $k = \frac{a}{a-1}$ ---[3]

(b) Find $E(x)$ in terms of a. ---[3]

(c) Find the 60_{th} percentile of X in terms of a. --[4]

[S-20/62/Q6]

Solution: (a) $\int_1^a f(x)\,dx = 1 \Rightarrow \int_1^a \frac{k}{x^2}\,dx = 1 \Rightarrow k\left[-\frac{1}{x}\right]_1^a = 1 \Rightarrow k\left[1-\frac{1}{a}\right] = 1$

$\Rightarrow k = \frac{a}{a-1}$ ✓

(b) $E(X) = \int_1^a x \cdot f(x)\,dx = \int_1^a x \times \frac{a}{a-1} \times \frac{1}{x^2}\,dx = \frac{a}{a-1}\int_1^a \frac{1}{x}\,dx$

$= \frac{a}{a-1}\left[\ln x\right]_1^a$

$= \frac{a}{a-1}(\ln a - \ln 1)$

$= \frac{a}{a-1} \cdot \ln a$ ✓

(C) 60_{th} percentile is let 'm'-

$\Rightarrow \int_1^m f(x)\,dx = 0.6 \quad (60\%)$

$\Rightarrow \int_1^m \frac{a}{(a-1)} \cdot \frac{1}{x^2}\,dx = \frac{3}{5} \Rightarrow \frac{a}{a-1}\left[-\frac{1}{x}\right]_1^m = 3/5$

$\Rightarrow \frac{a}{a-1}\left[1-\frac{1}{m}\right] = \frac{3}{5} \Rightarrow \frac{1}{m} = 1 - \frac{3(a-1)}{5a}$

$\Rightarrow \frac{1}{m} = \frac{2a+3}{5a} \Rightarrow m = \frac{5a}{2a+3}$ ✓

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11 The length of time, T minutes, that a passenger has to wait for a bus at a certain bus stop is modelled by a probability density function given by:

$$f(t) = \begin{cases} \frac{3}{4000}(20t - t^2) & 0 \leqslant t \leqslant 20 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch the graph of $y = f(t)$ --[1]

(b) Hence explain, without calculation, why $E(T) = 10$ --[1]

(c) Find $Var(T)$ --[3]

(d) It is given that $P(T < 10 + a) = p$ where $0 < a < 10$

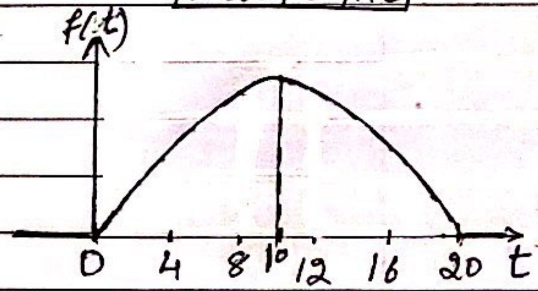
Find $P(10 - a < T < 10 + a)$ in terms of p. ---[2]

(e) Find $P(8 < T < 12)$ --[3]

(f) Give one reason why this model may be unrealistic. --[1]

S-20/61/Q6

Solution (a) Parabola.

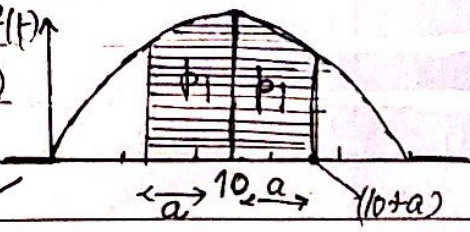


(b) $E(T) = 10$ as the graph of PDF $y = f(t)$ is symmetrical about $T = 10$.

(c) $E(T^2) = \int_0^{20} t^2 f(t)\,dt = \int_0^{20} t^2 (20t - t^2) \frac{3}{4000}\,dt$

$= \frac{3}{4000}\int_0^{20} (20t^3 - t^4)\,dt = \frac{3}{4000}\left[\frac{20t^4}{4} - \frac{t^5}{5}\right]_0^{20} = \frac{3}{4000} \times 160000 = 120$ ✓

$\therefore$ $Var(T) = E(T^2) - (E(T))^2 = 120 - 10^2 = \underline{20}$ ✓



(d) $P(X < 10 + a) = p = p_1 + \frac{1}{2} \Rightarrow p_1 = p - \frac{1}{2}$ ---①

$P(10 - a < T < 10 + a) = 2p_1$

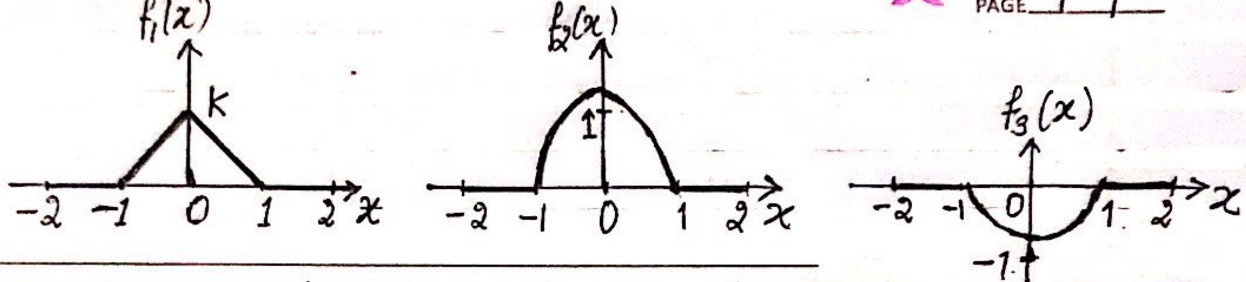
$= 2\left(p - \frac{1}{2}\right) = \underline{(2p - 1)}$ ✓

(e) $P(8 < T < 12) = \int_8^{12} f(x)\,dx = \frac{3}{4000}\int_8^{12} (20t - t^2)\,dt = \frac{3}{4000}\left[20\frac{t^2}{2} - \frac{t^3}{3}\right]_8^{12}$

$= \frac{3}{4000}\left[(1440 - 576) - \left(640 - \frac{512}{3}\right)\right] = \underline{\frac{37}{125}}$ ✓ (or 0.296)

(f) Does not allow times greater than 20 minutes.

12.
(a)



The diagram shows the graphs of three functions f_1, f_2 and f_3. The function f_1 is a probability density function.

(i) State the value of k. ---[1]

(ii) For each of the functions f_2 and f_3, state why it cannot be a probability density function. ---[2]

(b) The probability density function g is defined by:
$$g(x) = \begin{cases} 6(a^2-x^2) & -a\le x\le a \\ 0 & \text{otherwise} \end{cases}$$
where a is a constant.

Show that (i) Show that $a=\frac{1}{2}$; (ii) State the value of $E(x)$; (iii) Find $Var(X)$

[M-17/72/Q5] [3]+[1]+[2]

Solution (a) (i) f_1: Area $\Rightarrow 2\times\frac{1}{2}\times K\cdot 1 = 1 \Rightarrow K=1$ ✓

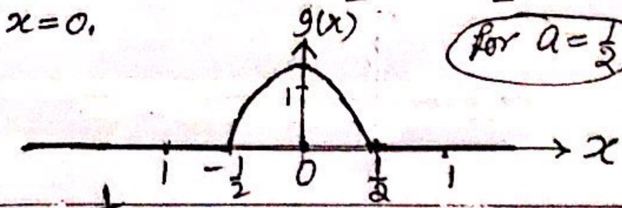
(ii) f_2: Area >1 (area $\neq 1$) ∴ f_2 is not a prob. density function.

(iii) f_3: Includes negative values of f_3, f_3 is not a P.D.F.

(b) (i) $6\int_{-a}^{a}(a^2-x^2)dx = 1 \Rightarrow 6\left[a^2x-\frac{x^3}{3}\right]_{-a}^{a} = 1 \Rightarrow 6\left(2a^3-\frac{2a^3}{3}\right)=1$
$\Rightarrow 8a^3=1 \Rightarrow a=1/2$ ✓

(ii) $E(X)= 0$ $\quad\{g(x)=6\left(\frac{1}{4}-x^2\right) \quad -\frac{1}{2}\le x\le\frac{1}{2}$

Since $g(x)$ is symmetrical about $x=0$.



(for $a=\frac{1}{2}$)

(iii) $E(X^2)=\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\cdot g(x)\,dx$
$= 6\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\left(\frac{1}{4}-x^2\right)dx = 6\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{x^2}{4}-x^4\right)dx = 6\cdot\left[\frac{x^3}{12}-\frac{x^5}{5}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$
$=0.05$

∴ $Var(X) = E(X^2) - [E(X)]^2$
$= 0.05 - 0^2 = \underline{0.05}$ ✓

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13. A random variable, W, has probability density function given by:

$$g(w) = \begin{cases} aw & 0 \leqslant w \leqslant b \\ 0 & \text{otherwise} \end{cases}$$

where a and b are constants.

Given that the median of W is 2, find a and b. --[4]

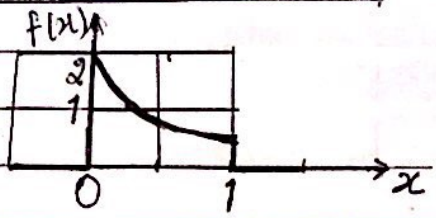
[M-16 | 72 | Q7(b)]

Solution: Given median of W is 2 $\Rightarrow \int_0^2 g(w)\,dw = \frac{1}{2} \Rightarrow \int_0^2 aw\,dw = \frac{1}{2}$

$\Rightarrow a\left[\frac{w^2}{2}\right]_0^2 = \frac{1}{2} \Rightarrow 2a = \frac{1}{2} \Rightarrow \underline{a = \frac{1}{4}}$ ✓

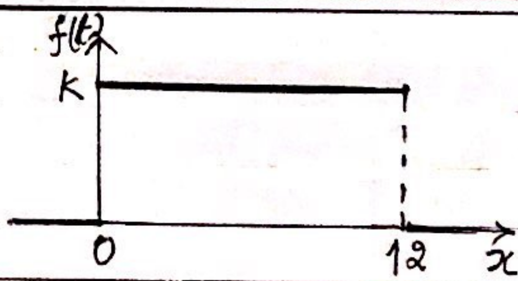
Now $\int_0^b g(w)\,dw = 1 \Rightarrow \int_0^b \frac{1}{4} w\,dw = 1 \Rightarrow \frac{1}{4} \times \left[\frac{w^2}{2}\right]_0^b = 1 \Rightarrow \frac{b^2}{8} = 1 \Rightarrow b = 2\sqrt{2}$ ✓

14. The diagram shows the graph of PDF f(x). The median of X is denoted by m. Use the diagram to explain whether $m < 0.5$, $m = 0.5$ or $m > 0.5$ [S-12 | 71 | Q4(iii)] --[2]



Solution: Area below $x = 0.5$ is greater than 0.5, hence the median $\underline{m < 0.5}$ ✓

15. Fred arrives at random times on a station platform. The time in minutes he has to wait for the next train are modelled by the continuous random variable for which the prob. density function f is shown in the figure above.



(i) State the value of k. --[1]

(ii) Explain briefly what this graph tells you about the arrival times of trains. [S-10 | 71 | Q1] --[1]

Solution (i) Area under the curve; $12K = 1 \Rightarrow \underline{K = \frac{1}{12}}$ ✓

(ii) Train arrive every 12 minutes. ✓

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