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S2

Probability and Statistics-2

Continuous Random Variable

Ex-2 Solution (Revision)

SP-20	S-20	W-20	M-20	S-22	M-22
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§ Probability density function (PDF) and Expectation and Variance :

From S_1:
In Prob. distribution :
$E(X) = \sum x \cdot P(x)$
$Var(X) = \sum x^2 \cdot P(x) - (E(x))^2$

For continuous random variable X,
Prob. density function (PDF) f(X).

(i) $f(x) \geq 0$

(ii) Total probability = 1 or $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(iii) $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

(iv) $Var(X) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx - \left\{\int_{-\infty}^{\infty} x \cdot f(x)\,dx\right\}^2$ $\quad \left\{\sigma^2 = E(X^2) - (E(X))^2\right.$

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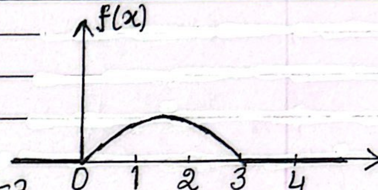
1. The diagram shows the graph of the prob. density function, f, of a random variable X, whose



$$f(x) = \begin{cases} \frac{2}{9}(3x - x^2) & 0 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

(a) State the value of E(X) and find Var(X). --[4]

(b) State the value of $P(1.5 \leq X \leq 4)$ --[1]

(c) Given that $P(1 \leq X \leq 2) = \frac{13}{27}$, find $P(X > 2)$. --[2]

SP-20/06/Q5

Solution (a) $E(X) = \underline{1.5}$ ✓ (curve is symm. about $x = 1.5$)

$E(X^2) = \int_0^3 x^2 \cdot f(x)\,dx = \int_0^3 \frac{x^2(3x - x^2)2\,dx}{9}$

$= \frac{2}{9}\int_0^3 (3x^3 - x^4)\,dx$

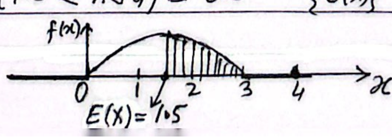
$= \frac{2}{9}\left[\frac{3x^4}{4} - \frac{x^5}{5}\right]_0^3$

$= \frac{2}{9}\left[\frac{243}{4} - \frac{243}{5}\right] = 2.7$

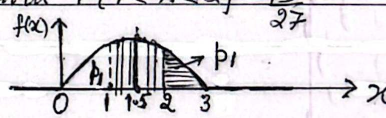
$\therefore Var(X) = E(X^2) - (E(x))^2$

$= 2.7 - (1.5)^2 = 0.45$ ✓

(b) $P(1.5 \leq X \leq 4) = 0.5$ $\{E(X)\}$



(c) Given $P(1 \leq X \leq 2) = \frac{13}{27}$



Let $P(X > 2) = p_1$

$\Rightarrow 2p_1 + \frac{13}{27} = 1$ (Total area)

$p_1 = \frac{1}{2}\left(1 - \frac{13}{27}\right)$

$P(X > 2) = \underline{\frac{7}{27}}$ (or 0.259) ✓

2. Bottles of Lanta contains approximatly 300 ml of juice. The volume of juice, millilitres, in bottle is 300+X, where X is a random variable with probability density function given by:

$$f(x) = \begin{cases} \frac{3}{4000}(100-x^2) & -10 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the probability that a randomly chosen bottle of Lanta contains more than 305 ml of juice. ---[3]

(b) Given that 25% of bottles of Lanta contains more than (300+p) ml of juice, Show that: $p^3 - 300p + 1000 = 0$ ---[4]

(C) Given that p = 3.47, and that 50% of bottles of Lanta contains between (300-q) and (300+q) ml of juice, find q. Justify your answer. ---[2]

M-20/62/Q5

Solution: (a) $V > 305 \rightarrow 300+5 \rightarrow X=5$, $P(X>5) = \int_5^{10} \frac{3}{4000}(100-x^2)\,dx$

$$= \frac{3}{4000}\left[100x - \frac{x^3}{3}\right]_5^{10} = \frac{3}{4000}\left[\left(1000 - \frac{1000}{3}\right) - \left(500 - \frac{125}{3}\right)\right] = 0.156$$

(b) for $V > 300+p \rightarrow X = p \rightarrow \frac{3}{4000}\int_p^{10}(100-x^2)\,dx = 0.25$ (25%)

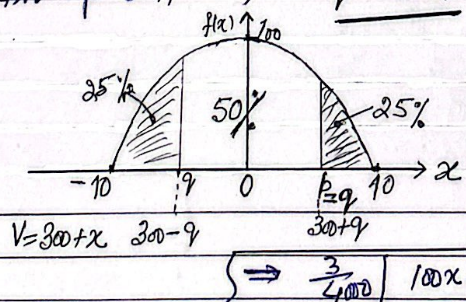
$$\Rightarrow \frac{3}{4000}\left[100x - \frac{x^3}{3}\right]_p^{10} = \frac{1}{4}$$

$$\Rightarrow \frac{3}{4000}\left[\left(1000 - \frac{1000}{3}\right) - \left(100p - \frac{p^3}{3}\right)\right] = \frac{1}{4}$$

$$\Rightarrow \frac{2000}{3} - 100p + \frac{p^3}{3} = \frac{1000}{3} \Rightarrow p^3 - 300p + 1000 = 0$$

(C) Curve is symmetrical about $x=0$

Given $p = 3.47 \Rightarrow q = 3.47$



$$\begin{cases} 300-q < V < 300+q \\ \Rightarrow 300-q < 300+X < 300+q \\ \Rightarrow -q < X < q \\ \Rightarrow \int_{-q}^{q} f(x)\,dx = 0.5 \\ \Rightarrow \int_{-q}^{q} \frac{3}{4000}(100-x^2)\,dx = \frac{1}{2} \end{cases}$$

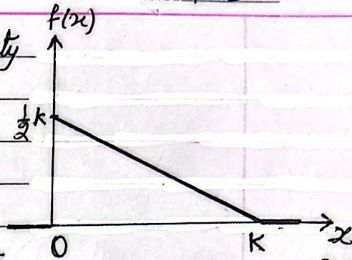
$$\Rightarrow \frac{3}{4000}\left[100x - \frac{x^3}{3}\right]_{-q}^{q} = \frac{3}{4000}\cdot 2\left(100q - \frac{q^3}{3}\right) = \frac{1}{2}$$

$$\Rightarrow 100q - \frac{q^3}{3} = \frac{1000}{3}$$

$$\Rightarrow q^3 - 300q + 1000 = 0$$

same as in Part (b)

3. The diagram shows the graph of the probability density function, f, of a random variable X.



(a) Find the value of the constant k. ---- [2]

(b) Using this value of k, find f(x) for $0 \le x \le k$, and hence find E(X) ---[3]

(c) Find the value of p such that $P(p<X<1)=0.25$ --[4]

[M-21/62/Q2]

Solution (a) Area under the line, area of triangle $= \frac{1}{2} \times k \times \frac{1}{2}k = 1$

$\Rightarrow k^2 = 4 \Rightarrow \underline{k=2}$ ✓ $\longrightarrow \therefore \frac{k}{2} = 1$

(b) f(x): Joining (2,0) and (0,1) $\Rightarrow y- = \frac{1-0}{0-2}(x-2) \Rightarrow y = -\frac{1}{2}x+1$ ✓

$\therefore f(x) = -\frac{1}{2}x+1$ ✓

$$E(X) = \int_0^2 x.f(x)\,dx = \int_0^2 x\left(-\tfrac{1}{2}x+1\right)dx$$

$$\Rightarrow E(X) = \int_0^2 \left(-\tfrac{1}{2}x^2+x\right)dx = \left[-\frac{x^3}{6}+\frac{x^2}{2}\right]_0^2 = \underline{\frac{2}{3}} \checkmark$$

(c) $P(p<x<1) = 0.25$

$$\Rightarrow \int_p^1 f(x)\,dx = 0.25 \Rightarrow \int_p^1 \left(-\tfrac{1}{2}x+1\right)dx = 0.25 \Rightarrow \left[\frac{-x^2}{4}+x\right]_p^1 = 0.25$$

$$\Rightarrow -\frac{1}{4}+1+\frac{p^2}{4}-p = 0.25 \Rightarrow p^2-4p+2=0$$

$$\Rightarrow \underline{p = 2-\sqrt{2}} \checkmark \ (\text{or } 0.586)$$

3.

4. In a game a ball is rolled down a slope and along a track until it stops. The distance, in metres, travelled by ball is modelled by the random variable X with probability density function:

$$f(x) = \begin{cases} -k(x-1)(x-3) & 1 \le x \le 3 \\ 0 & \text{otherwise,} \end{cases}$$ where k is a constant.

(a) Without calculation, explain why $E(X) = 2$ ---[1]

(b) Show that $k = 3/4$ ---[3]

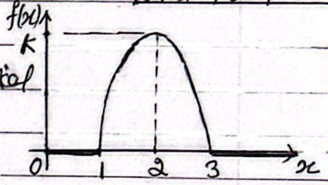
(c) Find $Var(X)$ ---[3]

One turn consists of rolling the ball 3 times and noting the largest value of X obtained. If the value is greater than 2.5, the player scores a point.

(d) Find the prob. that on a particular turn the player scores a point. ---[4]

[M-22/62/Q6]

Solution: $f(x) = -k(x-1)(x-3)$; $1 \le x \le 3$



(a) is a quad. function. The curve is symmetrical about vertex $(2, k) \Rightarrow \underline{E(X) = 2}$ ✓

(b) $\int_1^3 f(x)\,dx = 1 \Rightarrow -k\int_1^3 (x^2 - 4x + 3)\,dx = 1$

$\Rightarrow -k\left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3 = 1 \Rightarrow -k\left[0 - \frac{4}{3}\right] = 1 \Rightarrow \underline{k = \frac{3}{4}}$ ✓

(c) $Var(X) = \int_1^3 x^2 . f(x)\,dx - (E(x))^2$ --- ①

Consider: $\int_1^3 x^2 f(x)\,dx = -\frac{3}{4}\int_1^3 (x^4 - 4x^3 + 3x^2)\,dx = -\frac{3}{4}\left[\frac{x^5}{5} - x^4 + x^3\right]_1^3$

$= \frac{3}{4} \times \frac{28}{5} = \frac{21}{5}$ ✓ and $E(x) = 2$

hence from ① $Var(X) = 21/5 - 2^2 = \frac{1}{5} = \underline{0.2}$ ✓

(d) P(getting a point) = P($X > 2.5$ in atleast one roll out of 3)

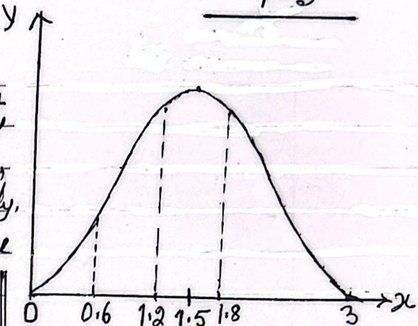
$= 1 - [\text{not } X > 2.5]^3$

$\underset{\text{in one roll}}{P(X > 2.5)} = \int_{2.5}^3 f(x)\,dx = \int_{2.5}^3 -\frac{3}{4}(x^2 - 4x + 3)\,dx = -\frac{3}{4}\left[\frac{x^3}{3} - 2x^2 + 3x\right]_{2.5}^3 = \frac{5}{32}$

P($X > 2.5$ in atleast 1 out of 3] $= 1 - [P(X > 2.5 \text{ is false}]^3 = 1 - (1 - 5/32)^3$

$= 0.399$ (3 S.f)

5. The diagram shows the graph of the probability density function, f, of a random variable X, that takes values between $x=0$ and $x=3$ only. The graph is symmetrical about the line $x = 1.5$.



(a) It is given that $P(X<0.6) = a$ and $P(0.6 < X < 1.2) = b$; Find $P(0.6 < X < 1.8)$ in terms of a and b. ---[2]

(b) It is now given that the equation of the probability density function of X is:

$$f(x) = \begin{cases} kx^2(3-x)^2 & 0 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$ where k is a constant.

(i) Show that $k = 10/81$. ---[3]

(ii) Find $Var(X)$ ---[3]

[M-23/62/Q3]

Solution (a) $P(0.6 < X < 1.2) = b$ and $P(X < 0.6) = a$

$$P(0.6 < X < 1.8) = P(X<1.8) - P(X<0.6)$$
$$= \{1-(a+b)\} - a$$
$$= \underline{1-2a-b} \checkmark$$

$$\left[\begin{aligned} P(X<1.8) &= 1-P(X>1.8) \\ &= 1-P(X<1.2) \\ &= 1-\{P(X<0.6) + P(0.6<X<1.2)\} \\ P(X<1.8) &= 1-(a+b) \end{aligned} \right.$$

(b) $$f(x) = \begin{cases} k\,x^2(3-x)^2 & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(i) $\int_0^3 f(x)\,dx = 1 \Rightarrow \int_0^3 k(9x^2-6x^3+x^4)\,dx = 1$

$$\left[\begin{aligned} x^2(3-x)^2 &= x^2(9-6x+x^2) \\ &= (9x^2-6x^3+x^4) \end{aligned} \right.$$

$$= k\left[\frac{9x^3}{3} - 6\frac{x^4}{4} + \frac{x^5}{5}\right]_0^3 = 1 \Rightarrow k\left[\frac{81}{10}\right] = 1 \Rightarrow k = \frac{10}{81} \checkmark$$

(ii) $E(X) = 1.5$ ① (as the graph is symmetrical about $X = 1.5$)

Now $\int_0^3 x^2 f(x)\,dx = \int_0^3 \frac{10}{81} x^2(9x^2-6x^3+x^4)\,dx$

$$\left[Var(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - (E(X))^2\right] \text{ --- ②}$$

$$= \frac{10}{81}\int_0^3 (9x^4 - 6x^5 + x^6)\,dx = \frac{10}{81}\left[\frac{9x^5}{5} - x^6 + \frac{x^7}{7}\right]_0^3 = \frac{18}{7} \text{ --- ③}$$

from ① and ③ in ②: $Var\,X = \frac{18}{7} - (1.5)^2 = \frac{9}{28} \checkmark$ (or 0.321)

6. The length of time, T minutes, that a passenger has to wait for a bus at a certain bus stop is modelled by a probability density function given by:

$$f(t) = \begin{cases} \frac{3}{4000}(20t - t^2) & 0 \leq t \leq 20 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch the graph of $y = f(t)$ --- [1]

(b) Hence explain, without calculation, why $E(T) = 10$ --- [1]

(c) Find $Var(T)$ --- [3]

(d) It is given that $P(T < 10 + a) = p$ where $0 < a < 10$

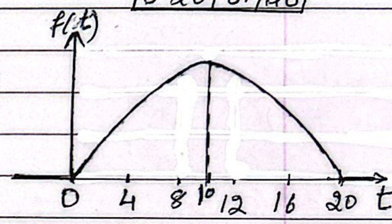
Find $P(10 - a < T < 10 + a)$ in terms of p. --- [2]

(e) Find $P(8 < T < 12)$ --- [3]

(f) Give one reason why this model may be unrealistic. --- [1]

S-20/61/Q6

Solution (a) Parabola.

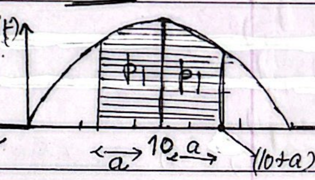


(b) $E(T) = 10$ as the graph of PDF $y = f(t)$ is symmetrical about $T = 10$.

(c) $E(T^2) = \int_0^{20} t^2 f(t)\,dt = \int_0^{20} t^2 (20t - t^2)\frac{3}{4000}\,dt$

$= \frac{3}{4000}\int_0^{20} (20t^3 - t^4)\,dt = \frac{3}{4000}\left[\frac{20t^4}{4} - \frac{t^5}{5}\right]_0^{20} = \frac{3}{4000} \times 160000 = 120$ ✓

$\therefore$ $Var(T) = E(T^2) - (E(T))^2 = 120 - 10^2 = \underline{20}$ ✓



(d) $P(X < 10 + a) = p = p_1 + \frac{1}{2} \Rightarrow p_1 = p - \frac{1}{2}$ --- ①

$P(10 - a \leq T < 10 + a) = 2p_1$

$= 2(p - \frac{1}{2}) = \underline{(2p - 1)}$ ✓

(e) $P(8 < T < 12) = \int_8^{12} f(x)\,dx = \frac{3}{4000}\int_8^{12} (20t - t^2)\,dt = \frac{3}{4000}\left[20\frac{t^2}{2} - \frac{t^3}{3}\right]_8^{12}$

$= \frac{3}{4000}\left[\left(1440 - 576\right) - \left(640 - \frac{512}{3}\right)\right] = \underline{\frac{37}{125}}$ ✓ (or 0.296)

(f) Does not allow times greater than 20 minutes.

7. A random variable X has probability density function given by:

$$f(x) = \begin{cases} \frac{k}{x^2} & 1 \leq x \leq a \\ 0 & \text{otherwise.} \end{cases}$$

where k and a are positive constant.

(a) Show that $k = \frac{a}{a-1}$ ---[3]

(b) Find $E(x)$ in terms of a, ---[3]

(c) Find the 60th percentile of X in terms of a. --[4]

[S-20/62/Q6]

Solution: (a) $\int_1^a f(x)\,dx = 1 \Rightarrow \int_1^a \frac{k}{x^2}\,dx = 1 \Rightarrow k\left[-\frac{1}{x}\right]_1^a = 1 \Rightarrow k\left[1-\frac{1}{a}\right] = 1$

$\Rightarrow k = \frac{a}{a-1}$ ✓

(b) $E(X) = \int_1^a x \cdot f(x)\,dx = \int_1^a x \times \frac{a}{a-1} \times \frac{1}{x^2}\,dx = \frac{a}{a-1}\int_1^a \frac{1}{x}\,dx$

$= \frac{a}{a-1}\left[\ln x\right]_1^a$

$= \frac{a}{a-1}(\ln a - \ln 1)$

$= \frac{a}{a-1} \cdot \ln a$ ✓

(c) 60th percentile is let 'm'.

$\Rightarrow \int_1^m f(x)\,dx = 0.6 \quad (60\%)$

$\Rightarrow \int_1^m \frac{a}{(a-1)} \cdot \frac{1}{x^2}\,dx = \frac{3}{5} \Rightarrow \frac{a}{a-1}\left[-\frac{1}{x}\right]_1^m = 3/5$

$\Rightarrow \frac{a}{a-1}\left[1-\frac{1}{m}\right] = \frac{3}{5} \Rightarrow \frac{1}{m} = 1 - \frac{3(a-1)}{5a}$

$\Rightarrow \frac{1}{m} = \frac{2a+3}{5a} \Rightarrow m = \frac{5a}{2a+3}$ ✓

8. The length, X centimetres, of worms of a certaintype is modelled by the prob. density function: $f(x)=\begin{cases}\frac{6}{125}(10-x)(x-5) & 5\leq x\leq 10\\ 0 & \text{otherwise}\end{cases}$

(a) State the value of $E(X)$ --- [1]

(b) Find $Var(X)$ --- [3]

(c) Two worms of this type are chosen at random. Find the probability that exactly one of them has length less than 6 cm. --- [5]

S-20/63/Q6

Solution (a) $E(X)=7.5$ ✓

$\left\{ f(x)=\frac{-6}{125}\left\{x^2-15x+50\right\} \text{ is symmetrical about } x=\frac{15}{2}=7.5 \right]$

(b) $E(X^2)=\int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \frac{6}{125}\int_5^{10} x^2 \cdot (10-x)(x-5)\,dx$

$= \frac{6}{125}\int_5^{10}(-x^4+15x^3-50x^2)\,dx = \frac{6}{125}\left[-\frac{x^5}{5}+15\cdot\frac{x^4}{4}-50\frac{x^3}{3}\right]_5^{10} = 57.5$ ✓

Now $Var X = E(X^2)-(E(X))^2 = 57.5-(7.5)^2 = \underline{1.25}$ ✓

(c) $P(X<6)=\int_5^6 \frac{6}{125}(-x^2+15x-50)\,dx$

$= \frac{6}{125}\left[-\frac{x^3}{3}+15\cdot\frac{x^2}{2}-50x\right]_5^6 = \frac{6}{125}\left(-102+\frac{625}{6}\right)$

$p = 0.104$

now using Binomial prob. distribution

$n=2,\ p=0.104$ $B(2, 0.104)$

$q=0.896$

$({}^nC_r \cdot p^r q^{n-r})$

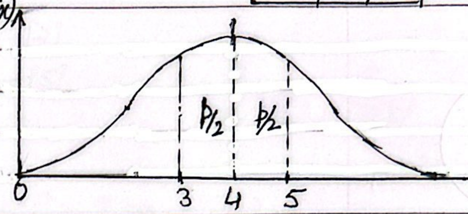
$P(r=1) = {}^2C_1 \cdot (0.104)^1 \cdot (0.896)^1$

$= 2\times 0.104 \times 0.896$

$= \underline{0.186}$ ✓ (3sf)

9. The graph of the probability density function of a random variable X is symmetrical about the line $x=4$
Given that $P(X<5)=\frac{20}{27}$; Find $P(3<X<5)$ ---[2]
[S-21/61/Q3]

Solution: Let $P(3<X<5)=p$



$P(X<5)=\frac{1}{2}+\frac{p}{2}=\frac{20}{27}$ Given

$\Rightarrow \frac{p}{2}=\frac{20}{27}-\frac{1}{2}=\frac{13}{2\times 27}$

$\Rightarrow p=\frac{13}{27}$ ✓

$\Rightarrow P(3<X<5)=\frac{13}{27}$ ✓

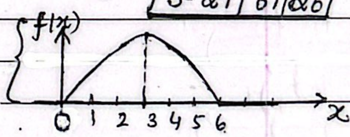
10. The probability density function, f, of a random variable X, is given by:

$$f(x)=\begin{cases} k(6x-x^2) & 0\leq x\leq 6 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

State the value of $E(X)$ and show that, $Var(X)=\frac{9}{5}$ ---[6]
[S-21/61/Q6]

Solution: $E(X)=3$ ✓ ($f(x)$ is symmetrical about $x=3$)



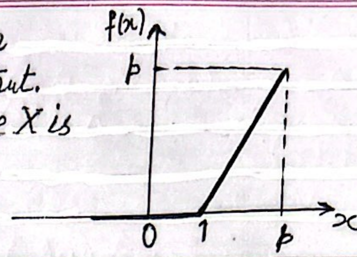
$\int_0^6 k(6x-x^2)\,dx=1 \Rightarrow k\left[3x^2-\frac{x^3}{3}\right]_0^6=1$

$\Rightarrow k\left[108-\frac{216}{3}\right]=1 \Rightarrow k=\frac{1}{36}$ ✓ $\quad \{E(X^2)=\int_{-\infty}^{\infty}x^2\cdot f(x)\,dx$

$E(X^2)=\frac{1}{36}\int_0^6 x^2(6x-x^2)\,dx=\frac{1}{36}\int_0^6(6x^3-x^4)\,dx=\frac{1}{36}\left[\frac{6x^4}{4}-\frac{x^5}{5}\right]_0^6=10.8$ ✓

Now $Var(X)=E(X^2)-[E(X)]^2=10.8-3^2=1.8$ ✓ (or $\frac{9}{5}$) ✓

11. The random variable X takes values in the range $1 \leq x \leq p$, where p is a constant. The graph of the prob. density function of X is shown in the diagram.



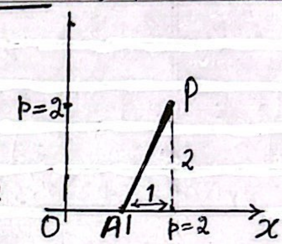
(a) Show that $p = 2$. ---[2]

(b) Find $E(X)$ ---[5]

[S-21/62/Q3]

<u>Solution:</u> Area under the graph of PDF:

$$\frac{1}{2} p \times (p-1) = 1 \Rightarrow p^2 - p - 2 = 0$$
$$(p-2)(p-1) = 0$$
$$\Rightarrow \underline{p = 2} \checkmark, \quad p = -1 \times$$



Gradient of the line $AP = 2$

line passes through $A(1,0)$,

Equation of line AP: $y - 0 = 2(x-1)$

$$\Rightarrow y = 2x - 2 \text{ ---- (i)}$$
or $f(x) = 2x - 2 \quad 1 \leq x \leq 2$

$$E(X) = \int_1^2 x \cdot f(x)\, dx = \int_1^2 x(2x-2)\, dx$$
$$= \int_1^2 (2x^2 - 2x)\, dx = \left[\frac{2x^3}{3} - \frac{2 \cdot x^2}{2}\right]_1^2 = \left[\frac{4}{3} + \frac{1}{3}\right] = \underline{\frac{5}{3}} \checkmark \left(\text{or } 1.67\right)$$

12. Alethia models the length of time, in minutes, by which her train is late on any day by the random variable X with prob. density function given by:

$$f(x) = \begin{cases} \frac{3}{8000}(x-20)^2 & 0 \le x \le 20 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the probability that the train is more than 10 mintes late on each of two randomly chosen days. ---[4]

(b) Find $E(X)$ ---[4]

(c) The median of X is denoted by m. Show that m satisfies the the equation: $(m-20)^3 = -4000$ and hence find m correct to 3 sf. --[4]

(d) State one way in which Alethia's model maybe unrealistic. --[1]

S-21/63/Q6

Solution:

(a) $P(X>10) = \int_{10}^{20} \frac{3}{8000}(x-20)^2\,dx = \frac{3}{8000}\left[\frac{(x-20)^3}{3}\right]_{10}^{20}$

$= \frac{3}{8000}\left[0-\left(-\frac{1000}{3}\right)\right] = \frac{1}{8}$ (or 0.125)

P(late for two day) $= \left(\frac{1}{8}\right)^2 = \frac{1}{64} = 0.0156$

(b) $E(X) = \int_0^{20} x.f(x)\,dx = \frac{3}{8000}\int_0^{20} x(x-20)^2\,dx = \frac{3}{8000}\int_0^{20}(x^3-40x^2+400x)\,dx$

$= \frac{3}{8000}\left[\frac{x^4}{4} - \frac{40x^3}{3} + 400.\frac{x^2}{2}\right]_0^{20} = \frac{3}{8000}\left[\frac{160000}{4} - \frac{40\times 8000}{3} + 400.200\right] = 5$

(c) m: $P(X<m) = \int_{-\infty}^{m} f(x)\,dx = \frac{1}{2} \Rightarrow \int_0^m \frac{3}{8000}(x-20)^2\,dx = \frac{1}{2}$

$\Rightarrow \frac{3}{8000}\left[\frac{(x-20)^3}{3}\right]_0^m = \frac{1}{2} \Rightarrow \frac{1}{8000}\left[(m-20)^3 - (-20)^3\right] = \frac{1}{2}$

$\Rightarrow (m-20)^3 = -4000$

$\Rightarrow m = 20 + \sqrt[3]{-4000}$

$= 20 - 15.874$

$\Rightarrow m = 4.13$ (3sf)

(d) Does not allow for train > 20 minutes late

or Does not all for trains being early.

13. A random variable X has prob. density function f. The graph of $f(x)$ is a straight line segment parallel to the x-axis from $x=0$ to $x=a$, where a is a positive constant.

(a) State, in terms of a, the median of X. --[1]

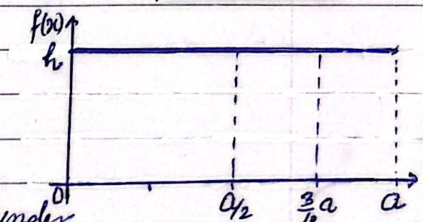
(b) Find $P(X > \frac{3}{4}a)$ --[1]

(c) Show that $Var(X) = \frac{1}{12}a^2$ --[5]

(d) Given that $P(X<b) = p$, where $0 < b < \frac{1}{2}a$, find $P(\frac{1}{3}b < X < a - \frac{1}{3}b)$ in terms of p. --[2]

[S-22/61/Q6]

Solution (a) Median of $X = a/2$



(b) $P(X > \frac{3}{4}a) = \frac{1}{4}$

(c) $y = f(x) \Rightarrow f(x) \times a = 1$ (Total area under the graph)

$\Rightarrow f(x) = \frac{1}{a}$

$E(X) = a/2$ —— ①

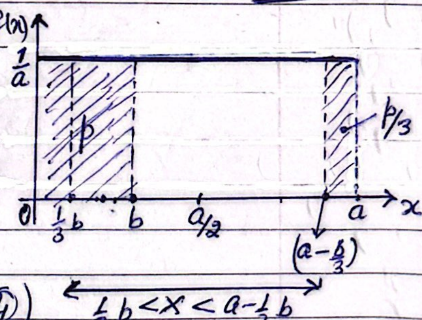
Consider $\int_0^a x^2 \cdot f(x)\,dx = \int_0^a \frac{1}{a} x^2\,dx = \frac{1}{a}\left[\frac{x^3}{3}\right]_0^a = \frac{1}{a} \times \frac{a^3}{3} = \frac{a^2}{3}$ --②

$\therefore Var\,X = \int_0^a x^2 \cdot f(x)\,dx - (E(X))^2 = \frac{a^2}{3} - \left(\frac{a}{2}\right)^2 = \frac{a^2}{3} - \frac{a^2}{4} = \frac{a^2}{12}$ ✓

(d) $P(X < b) = p \Rightarrow P(X < \frac{1}{3}b) = \frac{1}{3}p$ --- ③

$P(X > a - \frac{1}{3}b) = p/3$

$\Rightarrow P(X < a - \frac{1}{3}b) = 1 - \frac{1}{3}p$ --- ④



$P(\frac{1}{3}b < X < a - \frac{1}{3}b)$

$= P(X < a - \frac{1}{3}b) - P(X < \frac{1}{3}b)$

$= (1 - \frac{1}{3}p) - \frac{1}{3}p$ (from ③ & ④)

$= 1 - \frac{2}{3}p$ ✓

14 A random variable X has prob. density function given by:

$$f(x) = \begin{cases} \frac{3}{16}(4x - x^2) & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $E(X) = 11/4$ ---[3]

(b) Find $Var(X)$. ---[3]

(c) Given that the median of X is m, Find $P(m < X < 3)$ ---[3]

S-22/62/Q5

Solution:
(a)

$$E(X) = \int_2^4 x \cdot f(x)\,dx = \frac{3}{16}\int_2^4 x(4x - x^2)\,dx = \frac{3}{16}\int_2^4 (4x^2 - x^3)\,dx$$

$$= \frac{3}{16}\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_2^4 = \frac{3}{16}\left[\left(\frac{256}{3} - 64\right) - \left(\frac{32}{3} - 4\right)\right]$$

$$= \frac{11}{4} \quad \text{--- (1)}$$

(b) $Var(X) = \int_2^4 x^2 f(x)\,dx - [E(X)]^2$ --- (2)

Consider

$$\int_2^4 x^2 f(x)\,dx = \frac{3}{16}\int_2^4 x^2(4x - x^2)\,dx = \frac{3}{16}\int_2^4 (4x^3 - x^4)\,dx$$

$$= \frac{3}{16}\left[x^4 - \frac{x^5}{5}\right]_2^4 = \frac{39}{5} \quad \text{--- (3)}$$

from (1) & (3) in (2)

$$Var(X) = \frac{39}{5} - \left(\frac{11}{4}\right)^2 = \frac{19}{80} = 0.238 \quad (3\ s.f)$$

(c)

$$P(m < X < 3) = P(X < 3) - P(X < m) \quad \left\{\begin{array}{l} P(X < m) = \frac{1}{2} \\ m \text{ is median.} \end{array}\right.$$

$$= \int_2^3 f(x)\,dx - \frac{1}{2} \quad \text{--- (4)}$$

Consider:

$$\int_2^3 f(x)\,dx = \frac{3}{16}\int_2^3 (4x - x^2\,dx = \frac{3}{16}\left[2x^2 - \frac{x^3}{3}\right]_2^3$$

$$= \frac{3}{16}\left[(18 - 9) - (8 - 8/3)\right] = \frac{11}{16} \quad \text{--- (5)}$$

from (4) and (5)

$$P(m < X < 3) = \frac{11}{16} - \frac{1}{2}$$

$$= \frac{3}{16} = 0.1875$$

15 The random variables X and W have prob. density functions f and g defined as: $f(x) = \begin{cases} p(a^2-x^2) & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$

where a, p, and q are constants $g(W) = \begin{cases} q(a^2-w^2) & -a \le W \le a \\ 0 & \text{otherwise} \end{cases}$

(a) (i) Write down the value of $P(X \ge 0)$ ---[1]

(ii) Write down the value of $P(W \ge 0)$ ---[1]

(iii) Write down an expression for q in terms of p only. ---[1]

(b) Given that $E(X) = 3$, find the value of a. ---[6]

$\boxed{S\text{-}22/63/Q7}$

Solution: (a) (i) $P(X \ge 0) = 1$

(ii) $P(W \ge 0) = \frac{1}{2}$ (The function $g(w)$ is symmetrical about $w = a$)

(iii) $q = \frac{1}{2}p$

$\left\{ \int_0^a f(x)dx = 1 \Rightarrow p\int_0^a (a^2-x^2)dx = 1 \right.$

$\Rightarrow p\int_0^a (a^2-w^2)dw = 1$

$\Rightarrow \int_0^a (a^2-w^2)dw = \frac{1}{p}$ --- ①

$\left\{ \text{and} \int_0^a f(w)dw = \frac{1}{2} \Rightarrow q\int_0^a (a^2-w^2)dw = \frac{1}{2} \right.$

$\Rightarrow \int_0^a (a^2-w^2)dw = \frac{1}{2q}$ --- ②

(from ① & ② $\frac{1}{2q} = \frac{1}{p} \Rightarrow q = \frac{1}{2}p$ ✓)

(b) $\int_0^a f(x)dx = 1 \Rightarrow p\int_0^a (a^2-x^2)dx = 1 \Rightarrow p\left[a^2x - \frac{x^3}{3}\right]_0^a = 1 \Rightarrow p\left[a^3 - \frac{a^3}{3}\right] = 1$

$\Rightarrow p \times \frac{2}{3}a^3 = 1 \Rightarrow p = 3/2a^3$ ---- ③

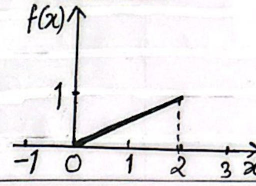
Now $E(X) = 3 \Rightarrow \int_0^a x.f(x)dx = 3 \Rightarrow p\int_0^a x(a^2-x^2)dx = 3$

$\Rightarrow p\int_0^a (a^2x - x^3)dx = 3 \Rightarrow p\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a = 3$

$\Rightarrow p\left[\frac{a^4}{2} - \frac{a^2}{4}\right] = 3 \Rightarrow p.\frac{a^4}{4} = 3$

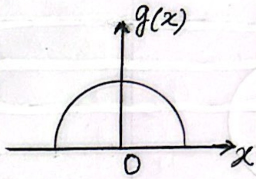
$\Rightarrow \frac{3}{2a^3} \times \frac{a^4}{4} = 3$ [$\because$ from ③ $p = 3/2a^3$]

$\Rightarrow \underline{a = 8}$ ✓

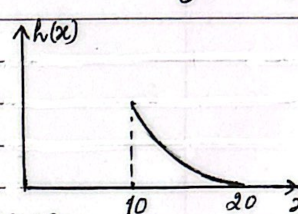


16.(a) The graph of a function f is a straight line segment from $(0,0)$ to $(2,1)$. ---[2]
Show that f could be probability density function

(b) The graph of a function g is a semicircle, centre $(0,0)$, entirely above x-axis. Given that g is a probability density function, find the radius of the semicircle. ---[2]



(c) The time, X minutes, taken by a large number of students to complete a test has probability density function $h(x)$, as shown in the diagram.



(i) Without calculation, use the diagram to explain how you can tell that the median time is less than 15 minutes. ---[1]

It is now given that: $h(x) = \begin{cases} \frac{40}{x^2} - \frac{1}{10}, & 10 \le x \le 20 \\ 0 & \text{otherwise} \end{cases}$

(ii) Find the mean time ---[3]

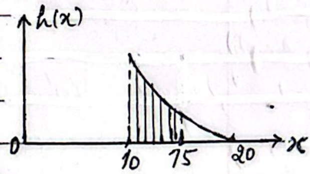
[S-23/61/Q2]

Solution (a) Area under the line $= \frac{1}{2} \times 2 \times 1 = 1$; which the correct area under a pdf. and $f(x) \ge 0$.

(b) Area under the semicircle $= \frac{1}{2}\pi r^2 = 1$ [for $g(x)$ is a pdf]

$\Rightarrow r^2 = \frac{2}{\pi} \Rightarrow r = \sqrt{\frac{2}{\pi}}$ or 0.798 (3sf)

(c)(i) Area to the left of 15 is greater than 0.5

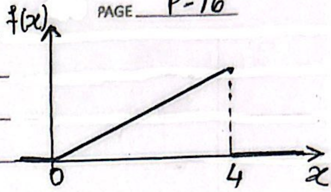


(ii) Mean time $E(X) = \int_{10}^{20} x \cdot h(x)\,dx = \int_{10}^{20} x\left(\frac{40}{x^2} - \frac{1}{10}\right) dx$

$= \int_{10}^{20} \left(\frac{40}{x} - \frac{1}{10}x\right) dx = \left[40 \ln x - \frac{x^2}{20}\right]_{10}^{20}$

$= \{(40\ln 20 - 20) - (40\ln 10 - 5)\}$ $= 40\ln 2 - 15 = 12.7$ (3sf)

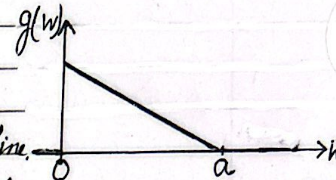
17. (a) The diagram shows the graph of the probability density function, f, of a random variable X, which takes values between 0 and 4 only. Between these two values the graph is a straight line.



(i) Show that $f(x) = kx$ for $0 \le x \le 4$, where k is a constant to be determined --- [2]

(ii) Hence, or otherwise, find $E(X)$ --- [3]

(b) The diagram shows the graph of the probability density function, g, of a random variable W, which takes values between 0 and a only $a > 0$. Between these two values the graph is a straight line. Given that the median of W is 1, find the value of a. --- [3]



S-23/62/Q7

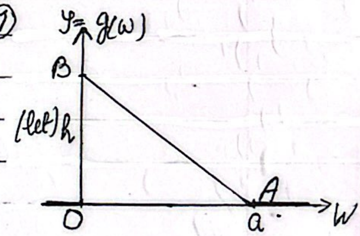
Solution (a)(i) Area under the line $f(x) = kx \Rightarrow \int_0^4 kx\,dx = 1$

$$\Rightarrow \left[k\frac{x^2}{2}\right]_0^4 = 1 \Rightarrow k\cdot\frac{4^2}{2} = 1 \Rightarrow 8k = 1 \Rightarrow k = \frac{1}{8}$$

(ii) $E(x) = \int_0^4 x\cdot f(x) = \int_0^4 x \times \frac{1}{8}x\,dx$ $\quad [f(x) = \frac{1}{8}x$ for $k = \frac{1}{8}]$

$$= \int_0^4 \frac{1}{8} x^2\,dx = \frac{1}{8}\left[\frac{x^3}{3}\right]_0^4 = \frac{1}{24} \times 64 = \frac{8}{3} = 2.67$$

(b) Given median is 1 $\Rightarrow \int_0^1 g(w)\,dw = \frac{1}{2}$ --- (1)



Let $OB = h$, $\frac{1}{2}ah = 1$ (Area under pdf)

$\Rightarrow h = 2/a$

Hence $A(a, 0)$, $B(0, \frac{2}{a}) \Rightarrow$

$\Rightarrow$ gradient of line $AB = \frac{\frac{2}{a} - 0}{0 - a} = -\frac{2}{a^2}$

$\therefore$ Equation of line AB is: $y - 0 = -\frac{2}{a^2}(w - a) \Rightarrow y = \frac{2}{a} - \frac{2}{a^2}w$

$\Rightarrow g(w) = \frac{2}{a} - \frac{2}{a^2}w$ --- (2)

from (1) & (2)

$$\int_0^1 \left(\frac{2}{a} - \frac{2}{a^2}w\right) dw = \frac{1}{2} \Rightarrow \left[\frac{2}{a}w - \frac{1}{a^2}w^2\right]_0^1 = \frac{1}{2} \Rightarrow \left(\frac{2}{a} - \frac{1}{a^2}\right) = \frac{1}{2}$$

$$\Rightarrow \frac{2a - 1}{a^2} = \frac{1}{2} \Rightarrow a^2 = 4a - 2 \Rightarrow a^2 - 4a + 2 = 0 \Rightarrow a = \frac{4 \pm 2\sqrt{2}}{2} \quad [a > 0]$$

$$\Rightarrow a = 2 + \sqrt{2} = 3.41$$

18. A random variable X has a prob. density function f, where

$$f(x) = \begin{cases} \frac{3}{2}(1-x^2) & 0 \le x \le 1 \\ 0 & \text{otherwise;} \end{cases}$$

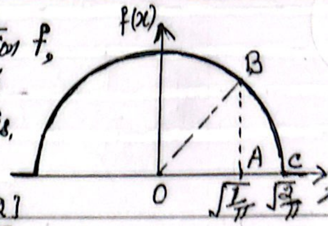
Find $E(X)$ ---[3]

[S-23/63/Q1]

Solution: $E(X) = \int_0^1 x f(x)\,dx = \int_0^1 x \cdot \frac{3}{2}(1-x^2)\,dx$

$= \frac{3}{2}\int_0^1 (x - x^3)\,dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{8}$ ✓

19. A random variable X has a prob. density function f, where the graph of $y = f(x)$ is a semicircle with centre $(0,0)$ and radius $\sqrt{\frac{2}{\pi}}$, entirely above x-axis. Elsewhere $f(x) = 0$.



(a) Verify that f can be prob. density function. ---[2]

A and B are the points where the line $x = \sqrt{\frac{1}{\pi}}$ meets the x-axis and the semicircle respectively.

(b) Show that angle AOB is $\frac{1}{4}\pi$ radians and hence find $P\left(X > \sqrt{\frac{1}{\pi}}\right)$ --- [6]

[S-23/63/Q7]

Solution (a) Area under the semicircle $= \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \cdot \left(\sqrt{\frac{2}{\pi}}\right)^2 = \frac{1}{2}\pi \times \frac{2}{\pi} = 1$ ✓

hence f is pdf (and $f(x) \ge 0$).

(b) Let angle AOB $= \theta$, $\cos\theta = \frac{OA}{OB} = \sqrt{\frac{1}{\pi}} \Big/ \sqrt{\frac{2}{\pi}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$ ✓ $\left\{OB = r = \sqrt{\frac{2}{\pi}}\right.$

Area of sector BOC $= \frac{1}{2}r^2\theta = \frac{1}{2} \times \frac{2}{\pi} \times \frac{\pi}{4} = \frac{1}{4}$ ✓ ---①

Area of triangle AOB $= \frac{1}{2}\,OA \times OB \cdot \sin\theta = \frac{1}{2} \times \sqrt{\frac{1}{\pi}} \times \sqrt{\frac{2}{\pi}}\,\sin\frac{\pi}{4} = \frac{1}{2\pi} = 0.1592$ ✓ ---②

$P\left(X > \sqrt{\frac{1}{\pi}}\right) =$ area under arc BC $= \frac{1}{4} - 0.1592$ {from ① & ②}

$= 0.0908$ (3sf)

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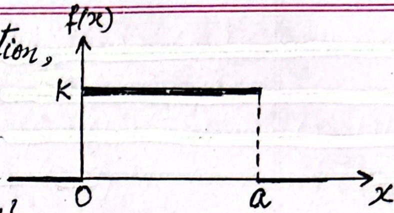
20. The diagram shows the prob. density function, $f(x)$, of a random variable X.



For $0 \le x \le a$, $f(x) = k$, elsewhere $f(x) = 0$.

(a) Express k in term of a. ---- [1]

(b) Given that $Var(X) = 3$, find a. --- [4]

W-20/61/Q4

Solution: $\int_0^a f(x)\,dx = 1 \Rightarrow \int_0^a k\,dx = [kx]_0^a = 1 \quad [f(x) = k]$

$\Rightarrow ka = 1 \Rightarrow k = \frac{1}{a}$ ✓

(b) $E(X) = \int_0^a x\,f(x)\,dx = \int_0^a x \times k\,dx = \int_0^a x \times \frac{1}{a}\,dx = \frac{1}{a}\left[\frac{x^2}{2}\right]_0^a = \frac{a}{2}$ ✓ ---(i)

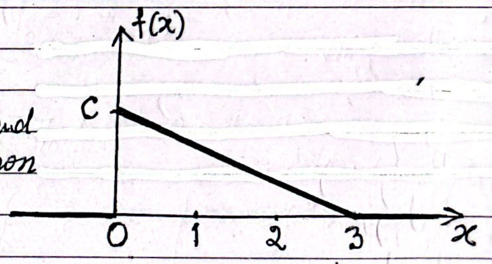
$E(X^2) = \int_0^a x^2 \cdot f(x)\,dx = \int_0^a x^2 \times k\,dx = k\left[\frac{x^3}{3}\right]_0^a = \frac{1}{a} \times \frac{a^3}{3} = \frac{a^2}{3}$ ---(ii)

Now $Var X = E(X^2) - (E(X))^2 = 3$ (Given)

from (i) & (ii) $\quad \frac{a^2}{3} - \left(\frac{a}{2}\right)^2 = 3 \Rightarrow \frac{a^2}{12} = 3 \Rightarrow a = 6$ ✓

---

21. A random variable X takes values between 0 and 3 only and has prob. density function as shown in the diagram, where c is a constant.



(a) Show that $c = 2/3$ ---[1]

(b) Find $P(X > 2)$. --[3]

(c) Calculate $E(X)$ --[4]

W-20/62/Q3

Solution: (a) Area under the curve; $\frac{1}{2} \times 3 \times C = 1 \Rightarrow C = 2/3$ ✓

(b) Equation of line; Gradient $= \left(\frac{2/3 - 0}{0 - 3}\right) = \frac{-2}{9}$

Passes through $(3,0)$

$y - 0 = -\frac{2}{9}(x-3) \Rightarrow f(x) = -\frac{2}{9}x + \frac{2}{3}$

$P(X>2) = \int_2^3 \left(-\frac{2}{9}x + \frac{2}{3}\right) dx$

$= \left[-\frac{2}{9} \cdot \frac{x^2}{2} + \frac{2x}{3}\right]_2^3 = 1 - \frac{8}{9} = \frac{1}{9}$ ✓

(c) $E(X) = \int_0^3 x \cdot f(x)\,dx$

$= \int_0^3 x\left(-\frac{2}{9}x + \frac{2}{3}\right) dx$

$= \int_0^3 \left(-\frac{2}{9}x^2 + \frac{2x}{3}\right) dx$

$= \left[-\frac{2x^3}{27} + \frac{x^2}{3}\right]_0^3 = (3-2) = 1$ ✓

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22. A random variable X has probability density function given by:

$$f(x) = \begin{cases} \frac{1}{18}(9-x^2) & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) find $P(X < 1.2)$ ---[3]

(b) Find $E(X)$ ---[3]

(c) The median of X is m; show that $m^3 - 27m + 27 = 0$ --[3]

[W-21/61/Q4]

Solution (a) $P(X<1.2) = \int_0^{1.2} \frac{1}{18}(9-x^2)\,dx$

$= \frac{1}{18}\left[9x - \frac{x^3}{3}\right]_0^{1.2} = \frac{71}{125}$ (or 0.568)

(b) $E(X) = \int_0^3 x \cdot \frac{1}{18}(9-x^2)\,dx$

$= \frac{1}{18}\int_0^3 (9x - x^3)\,dx = \frac{1}{18}\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3$

$= \frac{9}{8}$ (or 1.125)

(c)

$\int_0^m f(x)\,dx = 0.5$

$\Rightarrow \frac{1}{18}\int_0^m (9-x^2)\,dx = 0.5$

$\Rightarrow \frac{1}{18}\left[9x - \frac{x^3}{3}\right]_0^m = 0.5$

$\Rightarrow \frac{1}{18}\left[9m - \frac{m^3}{3}\right] - 0.5 = 0$

$\Rightarrow m^3 - 27m + 27 = 0$

23.(a) The probability density function of the random variable X is given by:

$$f(x) = \begin{cases} kx(4-x) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(i) Show that $k = \frac{3}{16}$ ---[3]

(ii) Find $E(X)$ ---[3]

(b) The random variable Y has the following properties.

- Y takes values between 0 and 5 only.
- The prob. density function of Y is symmetrical.

Given that $P(Y < a) = 0.2$, find $P(2.5 < Y < 5-a)$ illustrating your method with a sketch on the axes provided. ---[3]

W-21/62/Q7

<u>Solution</u>: (a)(i) $k \int_0^2 (4x - x^2)\,dx = 1$

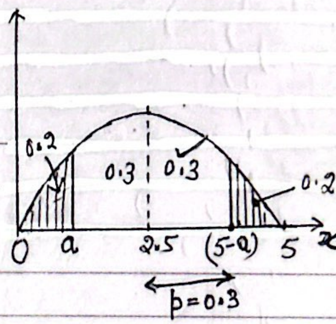
$\Rightarrow k\left[\frac{4x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1 \Rightarrow k \times \frac{16}{3} = 1 \Rightarrow k = \frac{3}{16}$ ✓

(ii) $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = k\int_0^2 x(4x - x^2)\,dx = \int_0^2 k(4x^2 - x^3)\,dx$

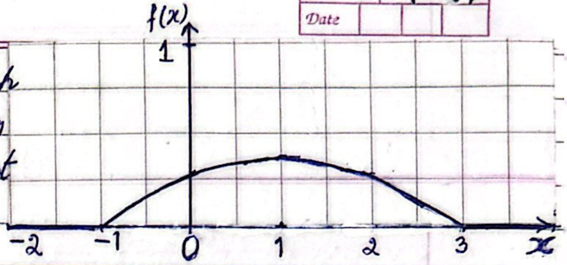
$= \frac{3}{16}\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{16}\left[\frac{32}{3} - 4\right] = \frac{3}{16} \times \frac{20}{3} = \frac{5}{4}$ ✓

(b) Symmetrical frequency density graph, 0 to 5.
Show area 0.2 to left of a,
$\Rightarrow$ area to the right of $5-a = 0.2$

$\therefore P(2.5 < Y < 5-a) = 0.3$ ✓



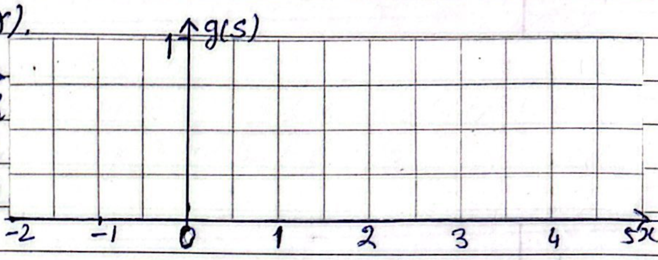
24. The diagram shows the graph of the prob. density function of a random variable X, that takes values between −1 and 3 only. It is given that



the graph is symmetrical about the line $x = 1$. Between $x = -1$ and $x = 3$, the graph a quadratic curve.

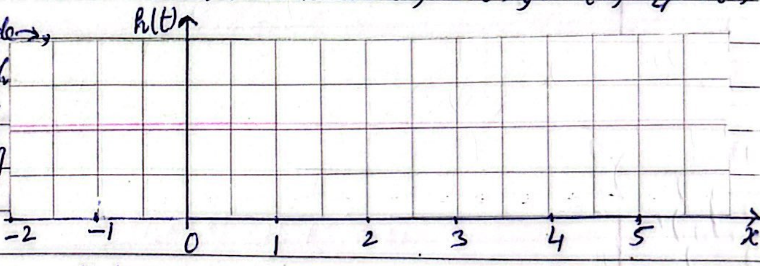
The random variable S is such that $E(S) = 2E(X)$ and $Var(S) = Var(X)$.

(a) On the grid aside → sketch a quad. graph for the prob. density function of S. ---[1]



The random variable T is such that $E(T) = E(X)$; $Var(T) = \frac{1}{4} Var(X)$

(b) On the grid aside →, Sketch a quad. graph for the prob. density function of T. --[2]



It is now given that: $f(x) = \begin{cases} \frac{3}{32}(3 + 2x - x^2) & ; -1 \le x \le 3 \\ 0 & \text{otherwise.} \end{cases}$

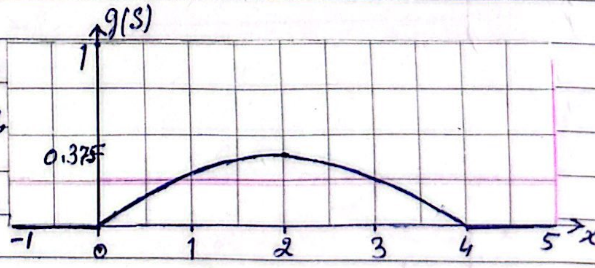
(c) Given that $P(1-a < X < 1+a) = 0.5$ show that $a^3 - 12a + 8 = 0$ --[3]

(d) Hence Verify that $0.69 < a < 0.70$ --[1]

[W-22/61/Q6]

Solution: (a) $E(S) = 2E(X)$ and $Var(S) = Var(X)$

Curve of similar shape, $x = 0$ to $x = 4$; with highest point at, $(2, 0.375)$



(continued →)

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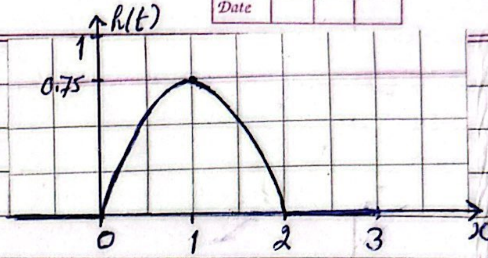
(Continued →)

24(b) $E(T) = E(X)$; $Var(T) = \frac{1}{4} Var(X)$

Curve of similar shape,
from $x=0$ to $x=2$.
highest point at $x=1$,
Highest point $(1, 0.75)$



(c) $f(x) = \frac{3}{32}(3+2x-x^2)$: $-1 \leq x \leq 3$

and $P(1-a < X < 1+a) = \frac{1}{2}$

$$\Rightarrow \frac{3}{32}\int_{1-a}^{1+a}(3+2x-x^2) = \frac{1}{2} \Rightarrow \frac{3}{32}\left[3x + x^2 - \frac{x^3}{3}\right]_{1-a}^{1+a} = \frac{1}{2}$$

$$\Rightarrow \frac{3}{32}\left[\left\{3(1+a)+(1+a)^2 - \frac{(1+a)^3}{3}\right\} - \left\{3(1-a)+(1-a)^2 - \frac{(1-a)^3}{3}\right\}\right] = \frac{1}{2}$$

$$\Rightarrow a^3 - 12a + 8 = 0$$

(d) Consider $f(a) = a^3 - 12a + 8$

$f(0.69) = (0.69)^3 - 12 \times 0.69 + 8 = 0.049$ (2 sf) > 0

and $f(0.70) = (0.70)^3 - 12 \times 0.70 + 8 = -0.057$ (2 sf) < 0

Hence $0.69 < a < 0.70$ ✓

Note: Part (a): (i) $Var(S) = Var(X)$ ⇒ Shape of the function $g(s)$ will be same and spread will be same.

and $E(S) = 2E(X) = 2 \times 1 = 2$
⇒ Shift of the mean from 1 to 2.

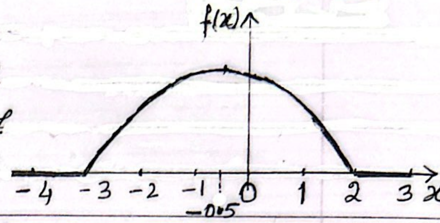
(b) (i) $E(T) = E(X)$ ⇒ mean of $h(t)$ will remain at $x=1$ (same)

(ii) $Var(T) = \frac{1}{4} Var(X)$ ⇒ S.D of T $\sigma_T = \frac{1}{2}\sigma_X$

⇒ The graph will be more steep and less spread
(Spread will chage from $4 \rightarrow \frac{1}{2} \times 4 = 2$ unit.

but height will be $0.375 \times 2 = 0.75$
(area remains the same)

25. The diagram shows the graph of the prob. density function, f, of a random variable X which takes values between −3 and 2 only.



(a) Given that the graph is symmetrical about the line $x = -0.5$ and that $P(X<0) = p$, find $P(-1<X<0)$ in terms of p. ---[2]

(b) It is now given that the prob. density function show in the diagram is given by:

$$f(x) = \begin{cases} a - b(x^2+x); & -3 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

where a and b are positive constants.

(i) Show that $30a - 55b = 6$ ---[3]

(ii) By substituting a suitable value of x into $f(x)$, find another equation relating a and b and hence determine the values of a and b. ---[3]

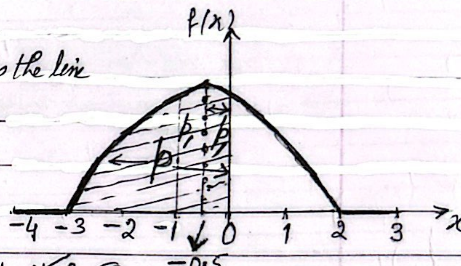
W-22/62/Q7

Solution: (a) $P(X<0) = p$

and $P(X<-0.5) = \frac{1}{2}$ as $x = 0.5$ is the line of symmetry

Let $P(-.5<X<0) = p_1$

$\Rightarrow p_1 = p - \frac{1}{2}$ ---- ①



$\Rightarrow P(-1<X<0) = 2p_1$

$= 2\left(p - \frac{1}{2}\right) = 2p - 1$ ✓ fr ①

(b) (i) $\int_{-3}^{2} f(x) = 1 \Rightarrow \int_{-3}^{2} (a - b(x^2+x))dx = 1 \Rightarrow \left[ax - b\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\right]_{-3}^{2} = 1$

$\Rightarrow \left(2a - b \times \frac{8}{3} - 2b\right) - \left(-3a + 9b - \frac{9}{2}b\right) = 1 \Rightarrow 5a - \frac{55}{6}b = 1$

$\Rightarrow 30a - 55b = 6$ ----- ②

(ii) $f(x) = a - b(x^2+2)$

$f(2) = 0 \Rightarrow a - 6b = 0$ ---- ③

Solving ② and ③ $\Rightarrow$ $a = \frac{36}{125}$ (or 0.288) ✓

and $b = \frac{6}{125}$ (or 0.048) ✓