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S-2

# Probability and Statistics-2

## Continuous Random Variable Notes

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## 1. § Probability Density Function (PDF):

$f(x)$ representing a continuous random variable is the probability density function (PDF) such that:

(i) $f(x) \geq 0$ as Probability $\geq 0$.

(ii) Total probability $= 1$ or $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Frequency density $= \dfrac{\text{Frequency}}{\text{Class width}}$.

Probability density $= \dfrac{\text{Frequency density}}{\text{Total frequency}}$.

Many times the data are defined across a specified interval/across specified intervals.

(iii) For a continuous random variable with PDF $f(x)$, $P(X=a)=0$

and $P(a<x<b) = P(a \leq x < b) = P(a < x \leq b) = P(a \leq x \leq b)$

(iv) The prob. of $X \in (a,b)$ is the area, $P(a<X<b) = \int_a^b f(x)\,dx$

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1. The time, T minutes, taken by people to complete a test has probability density function given by:

$$f(t) = \begin{cases} k(10t - t^2) & : 5 \leq t \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

Show that $k = \frac{3}{250}$ --[3]

S-16/73/Q5(i)

**Solution:** Total Prob $= \int_5^{10} k(10t - t^2)\,dt = k\left[5t^2 - \frac{t^3}{3}\right]_5^{10} = k\left[\left(500 - \frac{1000}{3}\right) - \left(125 - \frac{125}{3}\right)\right] = 1$

$\Rightarrow K \times \frac{250}{3} = 1 \Rightarrow K = \frac{250}{3}$ ✓

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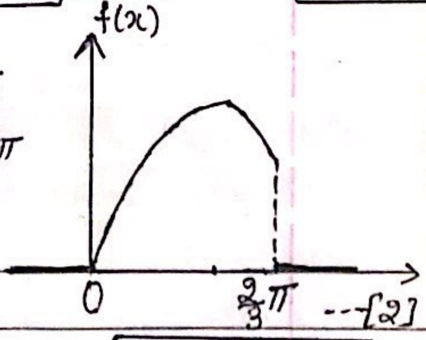
2. A random variable X has probability density function given by: $f(x) = \begin{cases} k\sin x & , 0 \leq x \leq \frac{2}{3}\pi \\ 0 & \text{otherwise} \end{cases}$

Where k is a constant.

Show that $k = 2/3$ --[2]




S-12/73/Q7

**Solution:** Area under the curve $= \int_0^{\frac{2\pi}{3}} k\sin x\,dx = 1$

$\Rightarrow k\left[-\cos x\right]_0^{\frac{2\pi}{3}} = 1 \Rightarrow k\left(-\cos\frac{2\pi}{3} + \cos 0\right) = 1$

$\Rightarrow k[0.5 + 1] = 1 \Rightarrow \frac{3}{2}k = 1 \Rightarrow \underline{k = 2/3}$ ✓

2. § **Median and other percentiles of a continuous random variable:**

(i) The median, 'm', of continuous random variable is the value for which,
(50% percentile) $P(X < m) = \int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$

(ii) The value 'r' of continuous random variable is that value that represents the say $20_{th}$ percentile, then $\int_{-\infty}^{r} f(x)\,dx = 20\%$ or $0.2$

(iii) Interquartile range $IQR = UQ - LQ$ or $= q_3 - q_1$ $\begin{cases} UQ = 75\% \text{ percentile} = q_3 \\ LQ = 25\% \text{ percentile} = q_1 \end{cases}$

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3. The random variable $X$ has probability density function given by:

$$f(x) = \begin{cases} 4x^k & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant.

(i) Show that $k = 3$ [2]

(ii) Find the upper quartile of $X$. [2]

(iii) Find the interquartile range of $X$. [2]

S-06/7/Q5

**Solution:** (i) Total sum of prob: $\int_0^1 4x^k\,dx = 1 \Rightarrow \left[\frac{4x^{k+1}}{k+1}\right]_0^1 = 1$

$$= \frac{4}{k+1}[1 - 0] = 1 \Rightarrow k+1 = 4 \Rightarrow k = 3 \checkmark$$

(ii) for upper quartile $q_3$: (75% Percentile) $\int_0^{q_3} 4x^3\,dx = 0.75$ [75%] $\left\{ \begin{array}{l} \text{for } k=3 \\ f(x) = 4x^3 \\ 0 \le x \le 1 \end{array} \right.$

$$\Rightarrow [x^4]_0^{q_3} = 0.75 \Rightarrow q_3^4 = 0.75 \Rightarrow q_3 = \sqrt[4]{0.75} = 0.931 \checkmark$$

(iii) Now lower quartile $q_1$: (25% Percentile) $\int_0^{q_1} 4x^3\,dx = 0.25 \Rightarrow [x^4]_0^{q_1} = 0.25$

$$\Rightarrow q_1^4 = 0.25 \Rightarrow q_1 = \sqrt[4]{0.25} = 0.707 \checkmark$$

$\therefore$ Interquartile range $IQR = q_3 - q_1$

$$= 0.931 - 0.707 = 0.224 \checkmark$$

4. Bottles of Lanta contains approximatly 300ml of juice. The volume of juice, millilitres, in bottle is 300+X, where X is a random variable with probability density function given by:

$$f(x) = \begin{cases} \frac{3}{4000}(100-x^2) & -10 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the probability that a randomly chosen bottle of Lanta contains more than 305 ml of juice. ---[3]

(b) Given that 25% of bottles of Lanta contains more than $(300+p)$ ml of juice, Show that: $p^3 - 300p + 1000 = 0$ ---[4]

(C) Given that $p = 3.47$, and that 50% of bottles of Lanta contains between $(300-q)$ and $(300+q)$ ml of juice, find $q$. Justify your answer. ---[2]

[M-20/62/Q5]

**Solution:** (a) $V > 305 \to 300+5 \to X=5 \to P(X>5) = \int_5^{10} \frac{3}{4000}(100-x^2)\,dx$

$= \frac{3}{4000}\left[100x - \frac{x^3}{3}\right]_5^{10} = \frac{3}{4000}\left[\left(1000 - \frac{1000}{3}\right) - \left(500 - \frac{125}{3}\right)\right] = 0.156$ ✓

(b) for $V > 300+p \to X = p \to \frac{3}{4000}\int_p^{10}(100-x^2)\,dx = 0.25$ (25%)

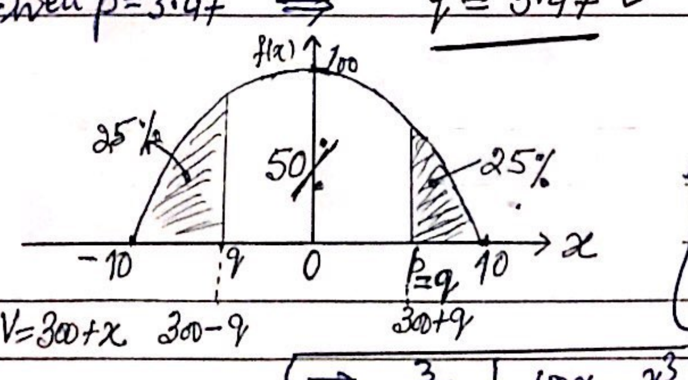
$\Rightarrow \frac{3}{4000}\left[100x - \frac{x^3}{3}\right]_p^{10} = \frac{1}{4}$

$\Rightarrow \frac{3}{4000}\left[\left(1000 - \frac{1000}{3}\right) - \left(100p - \frac{p^3}{3}\right)\right] = \frac{1}{4}$

$\Rightarrow \frac{2000}{3} - 100p + \frac{p^3}{3} = \frac{1000}{3} \Rightarrow p^3 - 300p + 1000 = 0$ ✓

(C) Curve is symmetrical about $x=0$

Given $p = 3.47 \Rightarrow q = 3.47$ ✓




$300 - q < V < 300 + q$

$\Rightarrow 300 - q < 300 + X < 300 + q$

$\Rightarrow -q < X < q$

$\Rightarrow \int_{-q}^{q} f(x)\,dx = 0.5$

$\Rightarrow \int_{-q}^{q} \frac{3}{4000}(100-x^2)\,dx = \frac{1}{2}$

$\Rightarrow \frac{3}{4000}\left[100x - \frac{x^3}{3}\right]_{-q}^{q} = \frac{3}{4000} \times 2\left(100q - \frac{q^3}{3}\right) = \frac{1}{2}$

$\Rightarrow 100q - \frac{q^3}{3} = \frac{1000}{3}$

$\Rightarrow q^3 - 300q + 1000 = 0$

Same as in Part (b)

## 3. § Expectation and Variance :

For continuous random variable X, Prob. density function (PDF) f(X).

(i) $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

(ii) $Var(X) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx - \left\{ \int_{-\infty}^{\infty} x \cdot f(x)\, dx \right\}^2$ $\quad \left\{ \sigma^2 = E(X^2) - (E(X))^2 \right.$

From $S_1$:
In Prob. distribution:
$E(X) = \sum x \cdot P(x)$
$Var(X) = \sum x^2 \cdot P(x) - (E(X))^2$

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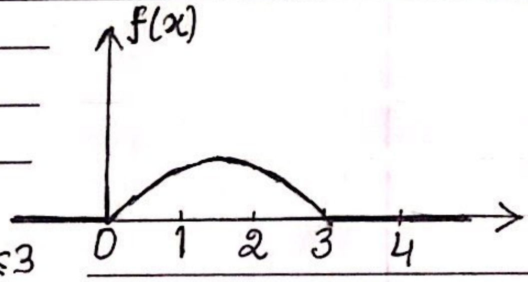
5. The diagram shows the graph of the prob. density function, f, of a random variable X, where

$$f(x) = \begin{cases} \frac{2}{9}(3x - x^2) & 0 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$



(a) State the value of E(X) and find Var(X). --[4]
(b) State the value of $P(1.5 \leq X \leq 4)$ --[1]
(c) Given that $P(1 \leq X \leq 2) = \frac{13}{27}$, find $P(X > 2)$. --[2]

SP-20/06/Q5

Solution (a) $E(X) = \underline{1.5}$ ✓ (Curve is symm. about $x = 1.5$)

$E(X^2) = \int_0^3 x^2 \cdot f(x)\, dx = \int_0^3 x^2 \frac{(3x - x^2)2}{9} dx$

$= \frac{2}{9} \int_0^3 (3x^3 - x^4)\, dx$

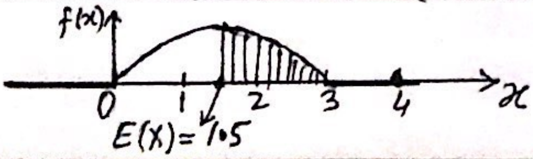
$= \frac{2}{9} \left[ \frac{3x^4}{4} - \frac{x^5}{5} \right]_0^3$

$= \frac{2}{9} \left[ \frac{243}{4} - \frac{243}{5} \right] = 2.7$

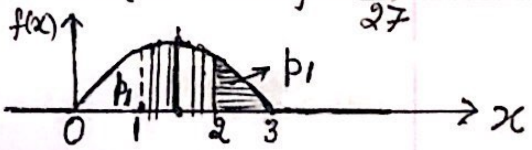
$\therefore Var(X) = E(X^2) - (E(X))^2$

$= 2.7 - (1.5)^2 = 0.45$ ✓

(b) $P(1.5 \leq X \leq 4) = 0.5 \quad \{E(X)\}$



(c) Given $P(1 \leq X \leq 2) = \frac{13}{27}$



Let $P(X > 2) = p_1$

$\Rightarrow 2p_1 + \frac{13}{27} = 1$ (Total area)

$p_1 = \frac{1}{2}\left(1 - \frac{13}{27}\right)$

$P(X > 2) = \underline{\frac{7}{27}}$ (or 0.259) ✓

6 The probability density function, f, of a random variable X, is given by

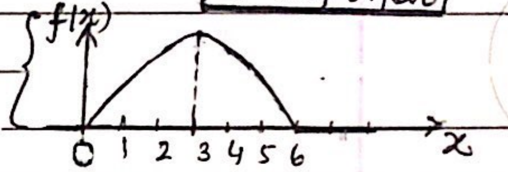
$$f(x) = \begin{cases} k(6x - x^2) & 0 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

State the value of E(X) and show that, $Var(X) = \frac{9}{5}$ ---[6]

$\boxed{S\text{-}21 \mid 61 \mid Q6}$

<u>Solution:</u> $E(X) = 3$ ✓ ( f(x) is symmetrical about $x = 3$)




$$\int_0^6 k(6x - x^2)dx = 1 \Rightarrow k\left[3x^2 - \frac{x^3}{3}\right]_0^6 = 1$$

$$\Rightarrow k\left[108 - \frac{216}{3}\right] = 1 \Rightarrow k = \frac{1}{36} ✓$$

$\left\{ E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx \right.$

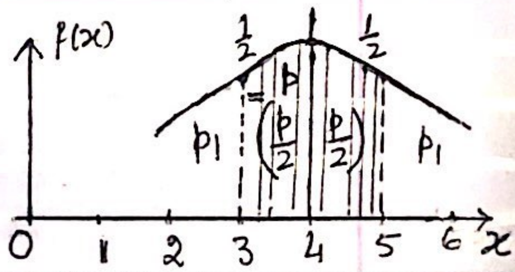
$$E(X^2) = \frac{1}{36}\int_0^6 x^2(6x - x^2)dx = \frac{1}{36}\int_0^6 (6x^3 - x^4)dx = \frac{1}{36}\left[\frac{6x^4}{4} - \frac{x^5}{5}\right]_0^6 = 10.8 ✓$$

Now $Var(X) = E(X^2) - [E(X)]^2 = 10.8 - 3^2 = 1.8$ ✓ (or $\frac{9}{5}$) ✓

7. The graph of the probability density function of a random variable X is symmetrical about the line $x = 4$

Given that $P(X < 5) = \frac{20}{27}$

find $P(3 < X < 5)$.




<u>Solution:</u> Line of Symmetry; $X = 4$.

$$P(X < 5) = \frac{20}{27} =$$

Let $P(3 < X < 5) = p$

$$\therefore P(X < 5) = p + p_1 = \frac{20}{27} \text{ ---(i)}$$

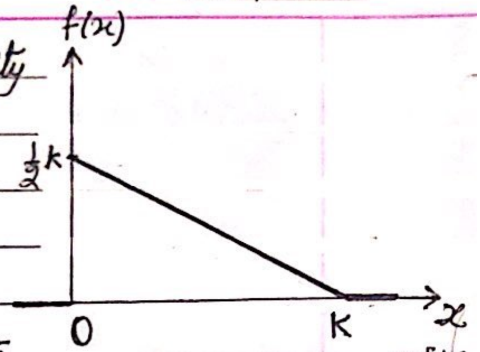
and $p + 2p_1 = 1$ ——(ii)

from (i) $2p + 2p_1 = \frac{40}{27}$ —(iii)

$$\text{(iii)} - \text{(ii)} \Rightarrow p = \frac{40}{27} - 1 = \frac{13}{27} ✓$$

$$\therefore P(3 < X < 5) = \frac{13}{27} ✓$$

8. The diagram shows the graph of the probability density function, f, of a random variable X.




(a) Find the value of the constant k. --- [2]

(b) Using this value of k, find f(x) for $0 \leq x \leq k$, and hence find E(X) ---[3]

(c) Find the value of p such that $P(p<X<1)=0.25$ --[4]

[M-21/62/Q2]

Solution (a) Area under the line, area of triangle $= \frac{1}{2} \times k \times \frac{1}{2}k = 1$

$\Rightarrow k^2 = 4 \Rightarrow k = 2$ ✓ $\longrightarrow \therefore \frac{k}{2} = 1$

(b) f(x) ; Join (2,0) and (0,1) $\Rightarrow y - = \frac{1-0}{0-2}(x-2) \Rightarrow y = -\frac{1}{2}x+1$ ✓

$\therefore f(x) = -\frac{1}{2}x+1$ ✓

$E(X) = \int_0^2 x.f(x)\,dx = \int_0^2 x\left(-\frac{1}{2}x+1\right)dx$

$\Rightarrow E(X) = \int_0^2 \left(-\frac{1}{2}x^2+x\right)dx = \left[-\frac{x^3}{6}+\frac{x^2}{2}\right]_0^2 = \frac{2}{3}$ ✓

(c) $P(p<x<1) = 0.25$

$\Rightarrow \int_p^1 f(x)\,dx = 0.25 \Rightarrow \int_p^1 \left(-\frac{1}{2}x+1\right)dx = 0.25 \Rightarrow \left[\frac{-x^2}{4}+x\right]_p^1 = 0.25$

$\Rightarrow -\frac{1}{4}+1+\frac{p^2}{4}-p = 0.25 \Rightarrow p^2-4p+2=0$

$\Rightarrow p = 2-\sqrt{2}$ ✓ (or 0.586)

9. A random variable X has probability density function given by:

$$f(x) = \begin{cases} \frac{1}{18}(9-x^2) & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) find $P(X<1.2)$ ---[3]

(b) Find E(X) ---[3]

(c) The median of X is m; Show that $m^3-27m+27=0$ --[3]

[W-21/61/Q4]

Solution (a) $P(X<1.2) = \int_0^{1.2} \frac{1}{18}(9-x^2)\,dx$

$= \frac{1}{18}\left[9x-\frac{x^3}{3}\right]_0^{1.2} = \frac{71}{125}$ ✓ (or 0.568)

(b) $E(X) = \int_0^3 x.\frac{1}{18}(9-x^2)\,dx$

$= \frac{1}{18}\int_0^3 (9x-x^3)\,dx = \frac{1}{18}\left[\frac{9x^2}{2}-\frac{x^4}{4}\right]_0^3$

$= \frac{9}{8}$ (or 1.125) ✓

(c) $\int_0^m f(x)\,dx = 0.5$

$\Rightarrow \frac{1}{18}\int_0^m (9-x^2)\,dx = 0.5$

$\Rightarrow \frac{1}{18}\left[9x-\frac{x^3}{3}\right]_0^m = 0.5$

$\Rightarrow \frac{1}{18}\left[9m-\frac{m^3}{3}\right]-0.5=0$

$\Rightarrow m^3-27m+27=0$ ✓

10. A random variable X has probability density function given by:

$$f(x) = \begin{cases} \frac{k}{x^2} & 1 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

where k and a are positive constant.

(a) Show that $k = \frac{a}{a-1}$ ---[3]

(b) Find $E(x)$ in terms of a. ---[3]

(c) Find the 60th percentile of X in terms of a. --[4]

$\boxed{S\text{-}20/62/Q6}$

Solution: (a) $\int_1^a f(x)dx = 1 \Rightarrow \int_1^a \frac{k}{x^2}dx = 1 \Rightarrow k\left[-\frac{1}{x}\right]_1^a = 1 \Rightarrow k\left[1-\frac{1}{a}\right] = 1$

$$\Rightarrow k = \frac{a}{a-1} ✓$$

(b) $E(x) = \int_1^a x \cdot f(x)dx = \int_1^a x \times \frac{a}{a-1} \times \frac{1}{x^2}dx = \frac{a}{a-1}\int_1^a \frac{1}{x}dx$

$$= \frac{a}{a-1}\left[\ln x\right]_1^a$$

$$= \frac{a}{a-1}(\ln a - \ln 1)$$

$$= \frac{a}{a-1} \cdot \ln a ✓$$

(c) 60th percentile is let 'm'.

$$\Rightarrow \int_1^m f(x)dx = 0.6 \quad (60\%)$$

$$\Rightarrow \int_1^m \frac{a}{(a-1)} \cdot \frac{1}{x^2}dx = \frac{3}{5} \Rightarrow \frac{a}{a-1}\left[-\frac{1}{x}\right]_1^m = 3/5$$

$$\Rightarrow \frac{a}{a-1}\left[1 - \frac{1}{m}\right] = \frac{3}{5} \Rightarrow \frac{1}{m} = 1 - \frac{3(a-1)}{5a}$$

$$\Rightarrow \frac{1}{m} = \frac{2a+3}{5a} \Rightarrow m = \frac{5a}{2a+3} ✓$$

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11 The length of time, $T$ minutes, that a passenger has to wait for a bus at a certain bus stop is modelled by a probability density function given by:

$$f(t) = \begin{cases} \frac{3}{4000}(20t - t^2) & 0 \le t \le 20 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch the graph of $y = f(t)$ --[1]

(b) Hence explain, without calculation, why $E(T) = 10$ --[1]

(c) Find $Var(T)$ --[3]

(d) It is given that $P(T < 10 + a) = p$ where $0 < a < 10$

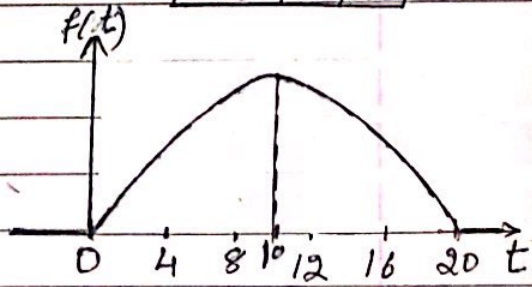
Find $P(10 - a < T < 10 + a)$ in terms of $p$. ---[2]

(e) Find $P(8 < T < 12)$ --[3]

(f) Give one reason why this model may be unrealistic. --[1]

S-20/61/Q6

Solution (a) Parabola.

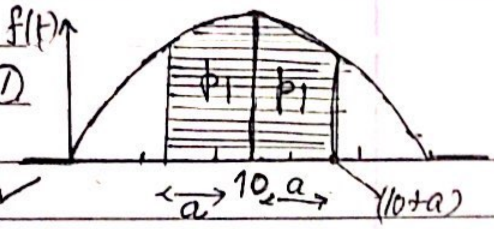


(b) $E(T) = 10$ as the graph of PDF $y = f(t)$ is symmetrical about $T = 10$.

(c) $E(T^2) = \int_0^{20} t^2 f(t)\,dt = \int_0^{20} t^2 \frac{(20t - t^2)3}{4000}\,dt$

$= \frac{3}{4000}\int_0^{20}(20t^3 - t^4)\,dt = \frac{3}{4000}\left[\frac{20t^4}{4} - \frac{t^5}{5}\right]_0^{20} = \frac{3}{4000} \times 160000 = 120$

$\therefore \underline{Var(T)} = E(T^2) - (E(T))^2 = 120 - 10^2 = \underline{20}$



(d) $P(X < 10 + a) = p = p_1 + \frac{1}{2} \Rightarrow p_1 = p - \frac{1}{2}$ --- ①

$P(10 - a < T < 10 + a) = 2p_1$

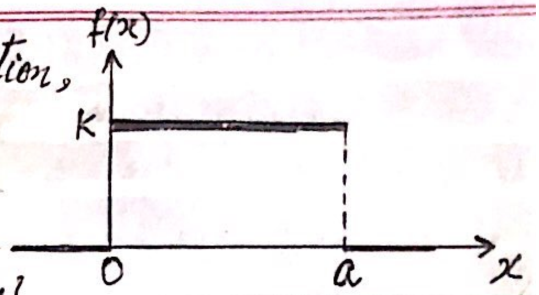
$= 2(p - \frac{1}{2}) = \underline{(2p - 1)}$

(e) $P(8 < T < 12) = \int_8^{12} f(x)\,dx = \frac{3}{4000}\int_8^{12}(20t - t^2)\,dt = \frac{3}{4000}\left[20\frac{t^2}{2} - \frac{t^3}{3}\right]_8^{12}$

$= \frac{3}{4000}\left[(1440 - 576) - (640 - \frac{512}{3})\right] = \underline{\frac{37}{125}}$ (or 0.296)

(f) Does not allow times greater than 20 minutes.

12 The diagram shows the prob. density function, $f(x)$, of a random variable $X$.
For $0 \le x \le a$, $f(x) = k$, elsewhere $f(x) = 0$.




(a) Express $k$ in term of $a$. ----[1]
(b) Given that $Var(X) = 3$, find $a$. ---[4]

W-20/61/Q4

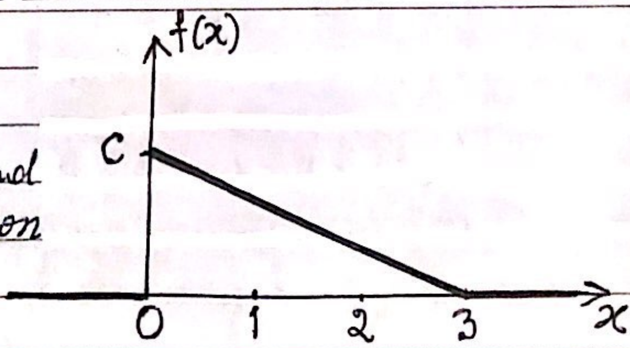
Solution: $\int_0^a f(x)\,dx = 1 \Rightarrow \int_0^a k\,dx = [kx]_0^a = 1$ $[f(x) = k]$
$\Rightarrow ka = 1 \Rightarrow k = \frac{1}{a}$ ✓

(b) $E(X) = \int_0^a x f(x)\,dx = \int_0^a x \cdot k\,dx = \int_0^a x \times \frac{1}{a}\,dx = \frac{1}{a}\left[\frac{x^2}{2}\right]_0^a = \frac{a}{2}$ ✓ ---(i)

$E(X^2) = \int_0^a x^2 \cdot f(x)\,dx = \int_0^a x^2 \times k\,dx = k\left[\frac{x^3}{3}\right]_0^a = \frac{1}{a} \times \frac{a^3}{3} = \frac{a^2}{3}$ ---(ii)

Now $Var X = E(X^2) - (E(X))^2 = 3$ (Given)
from (i) & (ii) $\frac{a^2}{3} - \left(\frac{a}{2}\right)^2 = 3 \Rightarrow \frac{a^2}{12} = 3 \Rightarrow a = 6$ ✓

13. A random variable $X$ takes values between 0 and 3 only and has prob. density function as shown in the diagram, where $c$ is a constant.




(a) Show that $c = 2/3$ ---[1]
(b) Find $P(X > 2)$. --[3]
(c) Calculate $E(X)$ --[4]

W-20/62/Q3

Solution: (a) Area under the curve, $\frac{1}{2} \times 3 \times c = 1 \Rightarrow c = 2/3$ ✓

(b) Equation of line; Gradient $= \left(\frac{2/3 - 0}{0 - 3}\right) = \frac{2}{9}$
Passes through $(3, 0)$
$y - 0 = -\frac{2}{9}(x - 3) \Rightarrow f(x) = -\frac{2}{9}x + \frac{2}{3}$

$P(X > 2) = \int_2^3 \left(-\frac{2}{9}x + \frac{2}{3}\right) dx$
$= \left[-\frac{2}{9} \cdot \frac{x^2}{2} + \frac{2x}{3}\right]_2^3 = 1 - \frac{8}{9} = \frac{1}{9}$ ✓

(c) $E(X) = \int_0^3 x \cdot f(x)\,dx$
$= \int_0^3 x\left(-\frac{2}{9}x + \frac{2}{3}\right) dx$
$= \int_0^3 \left(-\frac{2}{9}x^2 + \frac{2x}{3}\right) dx$
$= \left[-\frac{2x^3}{27} + \frac{x^2}{3}\right]_0^3 = (3 - 2) = 1$ ✓

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14.(a) The probability density function of the random variable $X$ is given by:

$$f(x)=\begin{cases} kx(4-x) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant.

(i) Show that $k=\frac{3}{16}$ ---[3]

(ii) Find $E(X)$ ---[3]

(b) The random variable $Y$ has the following properties.

- $Y$ takes values between 0 and 5 only.
- The prob. density function of $Y$ is symmetrical.

Given that $P(Y<a)=0.2$, find $P(2.5<Y<5-a)$ illustrating your method with a sketch on the axes provided. ---[3]

W-21 | 62 | Q7

**Solution:** (a)(i) $k\int_0^2 (4x-x^2)\,dx = 1$

$\Rightarrow k\left[\frac{4x^2}{2}-\frac{x^3}{3}\right]_0^2 = 1 \Rightarrow k\times\frac{16}{3}=1 \Rightarrow k=\frac{3}{16}$ ✓

(ii) $E(X)=\int_{-\infty}^{\infty} x\cdot f(x)\,dx = k\int_0^2 x(4x-x^2)\,dx = \int_0^2 k(4x^2-x^3)\,dx$

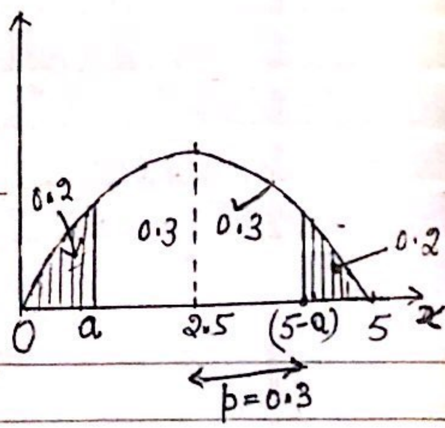
$=\frac{3}{16}\left[\frac{4x^3}{3}-\frac{x^4}{4}\right]_0^2 = \frac{3}{16}\left[\frac{32}{3}-4\right] = \frac{3}{16}\times\frac{20}{3} = \frac{5}{4}$ ✓

(b) Symmetrical frequency density graph, 0 to 5.

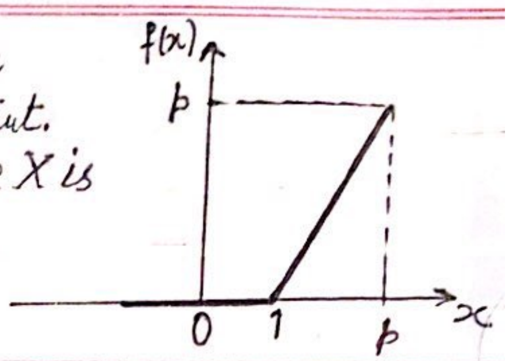
Show area 0.2 to left of $a$.

$\Rightarrow$ area to the right of $5-a = 0.2$

$\therefore P(2.5<Y<5-a) = 0.3$ ✓



15. The random variable $X$ takes values in the range $1 \leq x \leq p$, where $p$ is a constant. The graph of the prob. density function of $X$ is shown in the diagram.




(a) Show that $p = 2$. ---[2]

(b) Find $E(X)$ ---[5]

[ S-21/62 / Q3 ]

Solution: Area under the graph of PDF:

$$\frac{1}{2} p \times (p-1) = 1 \Rightarrow p^2 - p - 2 = 0$$

$$(p-2)(p-1) = 0$$

$$\Rightarrow \underline{p = 2} \checkmark, \quad p = -1 \times$$

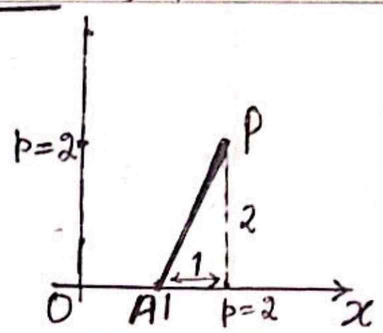
Gradient of the line $AP = 2$

Line passes through $A(1,0)$,

Equation of line $AP$: $y - 0 = 2(x-1)$

$$\Rightarrow y = 2x - 2 \quad \text{----} (i)$$

$$\text{or } f(x) = 2x - 2 \qquad 1 \leq x \leq 2$$




$$E(X) = \int_1^2 x \cdot f(x)\, dx = \int_1^2 x(2x-2)\, dx$$

$$= \int_1^2 (2x^2 - 2x)\, dx = \left[ \frac{2x^3}{3} - \frac{2 \cdot x^2}{2} \right]_1^2 = \left[ \frac{4}{3} + \frac{1}{3} \right] = \underline{\frac{5}{3}} \checkmark \; (\text{or } 1.67)$$

16 Alethia models the length of time, in minutes, by which her train is late on any day by the random variable X with prob. density function given by:

$$f(x) = \begin{cases} \frac{3}{8000}(x-20)^2 & 0 \le x \le 20 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the probability that the train is more than 10 mintes late on each of two randomly chosen days. ---[4]

(b) Find $E(X)$ ---[4]

(c) The median of X is denoted by m. Show that m satisfies the the equation: $(m-20)^3 = -4000$ and hence find m correct to 3 sf. ··[4]

(d) State one way in which Alethia's model maybe unrealistic. --[1]

S-21/63/Q6

**Solution:**

(a) $P(X > 10) = \int_{10}^{20} \frac{3}{8000}(x-20)^2 dx = \frac{3}{8000}\left[\frac{(x-20)^3}{3}\right]_{10}^{20}$

$= \frac{3}{8000}\left[0 - \left(-\frac{1000}{3}\right)\right] = \underline{\frac{1}{8}}$ (or 0.125)

P(late for two day) $= \left(\frac{1}{8}\right)^2 = \frac{1}{64}$ $= 0.0156$

(b) $E(X) = \int_0^{20} x \cdot f(x)dx = \frac{3}{8000}\int_0^{20} x(x-20)^2 dx = \frac{3}{8000}\int_0^{20}(x^3 - 40x^2 + 400x)dx$

$= \frac{3}{8000}\left[\frac{x^4}{4} - 40\frac{x^3}{3} + 400 \cdot \frac{x^2}{2}\right]_0^{20} = \frac{3}{8000}\left[\frac{160000}{4} - 40 \times \frac{8000}{3} + 400 \cdot 200\right] = \underline{5}$

(c) $m$: $P(X < m) = \int_{-\infty}^{m} f(x)dx = \frac{1}{2} \Rightarrow \int_0^m \frac{3}{8000}(x-20)^2 dx = \frac{1}{2}$

$\Rightarrow \frac{3}{8000}\left[\frac{(x-20)^3}{3}\right]_0^m = \frac{1}{2} \Rightarrow \frac{1}{8000}\left[(m-20)^3 - (-20)^3\right] = \frac{1}{2}$

$\Rightarrow (m-20)^3 = -4000$

$\Rightarrow m = 20 + \sqrt[3]{-4000}$

$= 20 - 15.874$

$\Rightarrow \underline{m = 4.13}$ (3sf)

(d) Does not allow for train > 20 minutes late

or Does not all for trains being early.

17. The length, X centimetres, of worms of a certain type is modelled by the prob. density function: $f(x)=\begin{cases}\frac{6}{125}(10-x)(x-5) & 5\leq x\leq 10\\ 0 & \text{otherwise}\end{cases}$

(a) State the value of $E(X)$ --- [1]
(b) Find $Var(X)$ --- [3]
(c) Two worms of this type are chosen at random.
Find the probability that exactly one of them has length less than 6 cm. --- [5]

$\boxed{S\text{-}20/63/Q6}$

Solution (a) $E(X)=7.5$ ✓

$\left\{ f(x)=\frac{-6}{125}\{x^2-15x+50\} \text{ is symmetrical about } x=\frac{15}{2}=7.5 \right\}$

(b) $E(X^2)=\int_{-\infty}^{\infty}x^2 \cdot f(x)dx = \frac{6}{125}\int_5^{10}x^2 \cdot (10-x)(x-5)dx$

$= \frac{6}{125}\int_5^{10}(-x^4+15x^3-50x^2)\,dx = \frac{6}{125}\left[-\frac{x^5}{5}+15\cdot\frac{x^4}{4}-50\frac{x^3}{3}\right]_5^{10} = 57.5$ ✓

Now $Var X = E(x^2)-(E(x))^2 = 57.5-(7.5)^2 = \underline{1.25}$ ✓

(c) $P(X<6)=\int_5^6 \frac{6}{125}(-x^2+15x-50)\,dx$

$= \frac{6}{125}\left[-\frac{x^3}{3}+15\cdot\frac{x^2}{2}-50x\right]_5^6 = \frac{6}{125}\left(-102+\frac{625}{6}\right)$

$p = 0.104$

now using Binomial prob. distribution

$n=2,\ p=0.104$  $B(2, 0.104)$
$q=0.896$

$({}^nC_r \cdot p^r q^{n-r})$

$P(r=1) = {}^2C_1 \cdot (0.104)^1 \cdot (0.896)^1$
$= 2\times0.104\times0.896$
$= \underline{0.186}$ ✓ (3sf)