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S-2

Probability and Statistics-2

Hyothesis Testing
Notes and Revision

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S2

Hypothesis testing - Notes and Revision

§ Null hypothesis: The claim is called the null hypothesis and is denoted by 'H_0'

§ Alternate Hypothesis:

When we don't accept the null hypothesis, then we have an alternate hypothesis, denoted by 'H_1'.

The null hypothesis and the alternate hypothesis both are expressed in terms of a parameter, such as probability 'p' or a mean value 'μ' (normal distribution).

§ Rejection region (or the critical region) and the critical value:

The range of values for which we reject the null hypothesis is the critical region.

The value, at which we change accepting the null hypothesis to rejecting, is called critical value.

§ Type I error:

The type of error, where a null hypothesis is rejected despite being correct, is called a Type-I error.

or P(reject H_0/H_0 is true)

§ Type II error:

When the null hypothesis is in fact false, but is accepted. or P(Type II error) = P(accept H_0/H_0 is false)

§ Significance Level: The significance level is the probability of rejecting a claim.

The significance level is given in percentage. (generally it is 5%). "The lower the percentage significance level, the smaller the rejection region, and the more confident you can be of the result."

§ One-tailed test: when the words increased (or decreased) are used.

Two-tailed test: when the word - the parameter is different. $p \neq a$ or $\mu \neq a$

$X \sim B(n, p)$

Example 1: Jill shoots arrows at a target. Last week, 65% of her shots hit the target. This week Jill claims that she has improved. Out of her first 20 shots this week, she hits the target with 18 shots. Assuming shots are independent, test Jill's claim at the 1% significance level. --[5]

M-16/72/Q3

Solution: Let X be the number of arrows hitting the target in 20 shots.

Null hypothesis, H_0: P(Hit target) $= p = 0.65$ (65%)

Alternate hypothesis, H_1: P(hit target) : $p > 0.65$

$$X \sim B(20, 0.65)$$

$$P(X \geq 18) = P(X = 18, 19, 20)$$

$$= {}^{20}C_{18}(0.65)^{18} \cdot (0.35)^2 + {}^{20}C_{19}(0.65)^{19} \cdot (0.35) + {}^{20}C_{20}(0.65)^{20}$$

$$= 0.0121 \quad (3\ s.f)$$

($n = 20, p = 0.65, q = 0.35$; $P(X = r) = {}^{n}C_{r}\, p^r q^{n-r}$)

(${}^{20}C_{18} = {}^{20}C_2$, ${}^{20}C_{19} = {}^{20}C_1$, ${}^{20}C_{20} = 1$)

(Significance level is 1% or .01) $= 0.0121 > 0.01$

$\therefore$ There is no evidence that she has improved. (H_0 is accepted)

Note: Let us suppose if the significance level is 2% (or 0.02)

Then $0.0121 < 0.02$

($\therefore$ H_0 is rejected)

$\therefore$ H_1 is accepted.

That is Jill has improved.

$X \sim B(n,p)$

Example 2: A television channel claims that 25% of its programmes are nature programmes. Kavita thinks the percentage claims is too high. To test her hypothesis, she chooses 20 programmes at random.

(a) If Kavita carries out the hypothesis test at 10% significance level, define the random variable, state its distribution, including parameters, and define the hypothesis.

(b) If there are only two nature programmes, calculate the test statistics. What conclusion does Kavita reach?

(c) If the significance level was 5%, what conclusion would Kavita reach?

Solutions (a) Let X be the number of nature programmes (random variable)

Binomial distribution: $X \sim B(20, 0.25)$ ($B(n,p) \Rightarrow n$ and p are parameters)

Null Hypothesis: $H_0: p = 0.25$

Alternate hypothesis: $H_1: p < 0.25$

$\left\{ P(X=r) = {}^nC_r\, p^r q^{n-r} \right.$; $p = 0.25$, $q = 0.75$, $n = 20$

(b) Test statistics: $P(X \le 2) = P(X = 0, 1, 2)$

$$= (0.75)^{20} + {}^{20}C_1 \times (.25)^1 (0.75)^{19} + {}^{20}C_2 \cdot 0.25^2 \cdot (0.75)^{18}$$
$$= 0.0913$$

$0.0913 < 0.10$ (Significance level - 10% or 0.10)

True $\therefore$ Ho is rejected

The test result lead, Kavita to conclude that there is insufficient evidence to accept the television channel's claim.

(c) For 5% (or 0.05) significance level:

$P(X \le 2) = 0.0913 > 0.05$ $\therefore H_0$ is accepted (H_1 is rejected)

Kavita would now conclude that the proportion of nature programmes is likely to be 25%, as the television channel claims.

Example 3: World-wide 25% of men are colour-blind but it is believed that the condition is less widespread among a group of remote hill tribe. An anthropologist plans to test this by sending field workers to visit villages in that area. In each village 30 men are to be tested for colour-blindness. Find the rejection region for the test at the 5% level of significance, find the critical value also.

Solution: Random variable: X is the number of colour-blind men.

The distribution $X \sim B(n,p) = B(30, 0.25)$ $\{25\% = 0.25\}$

Null hypothesis $H_0: p = 0.25$

Alternate hypothesis $H_1: p < 0.25$

Significance level = 5% (0.05)

The rejection region is $X \leq k$, where

$P(X \leq k) \leq 0.05$ and $P(X \leq k+1) > 0.05$ (Significance level)

Consider

$P(X \leq 3) = P(X = 0, 1, 2, 3)$ $\quad P(X=r) = {}^{n}C_{r}\, p^{r} \cdot q^{n-r}$

$= (0.75)^{30} + {}^{30}C_1 (0.25)^1 \cdot (0.75)^{29}$ $\quad \{B(30, 0.25)$

$+ {}^{30}C_2 (0.25)^2 \cdot (0.75)^{28} + {}^{30}C_3 (0.25)^3 (.75)^{27}$ $\quad \{p = 0.25,\ q = 0.75,\ n = 30$

$\therefore P(X \leq 3) = 0.0375 \leq 0.05$ ---------- (i)

and $P(X \leq 4) = 0.0375 + {}^{30}C_4 (.25)^4 (.75)^{26} = 0.0929 > 0.05$ -- (ii)

from (i) and (ii)

The rejection region is $X \leq 3$. ✓

and critical value = 3 ✓

$X \sim B(n, p)$

Example 4: A darts player claims he can hit the bullseye 60% of the time. A fan thinks the player is better than that. Define the random variable and hypothesis. Calculate the test statistic based on the darts player hitting the bullseye 15 times in 17 darts thrown. At what integer percentage significance level is 15 the critical value?

Solution: Random variable: X be the number of times the dart player hits the bulls eye. The $X \sim B(17, 0.6)$

Null hypothesis H_0: $p = 0.6$

Alternate hypothesis H_1: $p > 0.6$

$$\left\{ \begin{array}{l} 60\% = 0.6 = p \\ n = 17 \\ q = 0.4 \end{array} \right.$$

$$P(X = r) = {}^{n}C_{r}\, p^{r}\, q^{n-r}$$

$$P(X \geq 15) = P(X = 15, 16, 17)$$

$$= {}^{17}C_{15} \cdot (0.6)^{15} \cdot (0.4)^{2} + {}^{17}C_{16} \cdot (0.6)^{16} \cdot (0.4)^{1} + (0.6)^{17}$$

$$= 0.0123$$

Given the critical value is 15.

Hence at the smallest integer significance level $1.23\% < 2\%$

$\therefore$ Significance level $= 2\%$

$B(n,p) \sim N(\mu,\sigma^2)$

Example 5: It is claimed that the proportion of people worldwide with red hair is 1.2%. Margaret thinks the proportion of people in Scotland with red hair is higher. At her school there are 850 students, 15 of the students have red hair.

(a) Test Margaret's claim at 2% significance level. ---[5]

(b) Find the critical value [3]

Solution: Random Variable: Let X be the number of red hair, and $X \sim B(850, 0.012)$

(a) $p = 0.012,\ q = 0.988$ $[1.2\% = 0.012]$

Approximating distribution from Binomial to Normal distribution $N(\mu,\sigma^2)$

$$\begin{cases} \mu = np = 850 \times 0.012 = 10.2 \\ \sigma^2 = npq = 850 \times 0.012 \times 0.988 \\ \sigma^2 = 10.0776 \end{cases}$$

$X \sim N(10.2, 10.0776)$

Null hypothesis, $H_0: \mu = 10.2$

Alternate hypothesis, $H_1: \mu > 10.2$

Now $P(X \geq 15) \sim P\left(z \geq \frac{14.5 - 10.2}{\sqrt{10.0776}}\right)$ {Continuity correction $P(X \geq 15) \to P(X \geq 14.5)$

$= P(z \geq 1.355)$

$= 1 - \phi(1.355) = 1 - 0.9123 = 0.0877$ ✓

here $0.0877 > 0.02$, [Significance Value $2\% = 0.02$]

So H_0 is accepted, or

There is insufficient evidence to support the claim that the proportion of red-haired people in Scotland is greater that 1.2%.

(b) Now let the critical value be 'x' with a continuit correction be $(x - 0.5)$.

$P\left(z \geq \left(\frac{(x-0.5)-10.2}{\sqrt{10.0776}}\right)\right) < 0.02$ (Significance Level)

$\Rightarrow P\left(z \leq \left(\frac{(x-0.5)-10.2}{\sqrt{10.0776}}\right)\right) > 0.98$

$\Rightarrow \frac{x - 10.7}{\sqrt{10.0776}} > \phi^{-1}(0.98) = 2.054$

$\Rightarrow x > 17.22$

$\therefore$ Critical Value is 18. ✓

$B(n,p) \sim N(\mu, \sigma^2)$

Example 6: A farmer finds that 30% of his sheep are deficient in a particular mineral. He changes their feed and test 80 sheep to find out if the number has decreased.

(a) Define the random variable and hypothesis. Using a suitable approximating distribution, carry out a hypothesis test at 10% significance level given that 19 of the sheep are mineral deficient.

(b) Work out the critical value.

Solutions: Random Variable: Let X is the number of sheep deficient in mineral.

Then distribution $X \sim B(80, 0.3)$ $\left[\because n = 80,\ p = 0.3 \text{ (30\% are deficient)}\right.$, $q = 0.7 \left.\right]$

Null hypothesis $H_0: p = 0.3$

Alternate hypothesis $H_1: p < 0.3$.

Approximating the distribution from Binomial to Normal distribution

$$X \sim N(\mu, \sigma^2) = N(24, 16.8) \qquad \left[\begin{array}{l}\mu = np = 80 \times 0.3 = 24 \\ \sigma^2 = npq = 80 \times 0.3 \times 0.7 \\ \quad = 16.8\end{array}\right.$$

Now $P(X \le 19) \approx P\left(Z \le \dfrac{19.5 - 24}{\sqrt{16.8}}\right)$ $\left[\begin{array}{l}\text{continuity correction;} \\ P(X \le 19) \rightarrow X = 19.5 \\ \text{for Normal distribution}\end{array}\right.$

$= P(Z \le -1.098)$

$= 1 - \phi(1.098)$

$= 0.136 > 0.10$ [$\because$ Significance level 10% (or 0.10)]

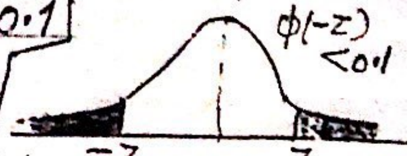
$\therefore$ H_0 is accepted.

$\therefore$ There is no evidence to suggest that the percentage of sheep mineral deficient has decreased. (H_1 is rejected).

(b) To find the critical value 'C': with continuity correction to find the largest integer C: $P(X \le C + 0.5) < 0.1$

$$\Rightarrow P\left(Z < \frac{(C+0.5) - 24}{\sqrt{16.28}}\right) < 0.1$$

$$\phi\left[\frac{(C+0.5) - 24}{\sqrt{16.28}}\right] < 0.1$$



$\phi(-z) = 1 - \phi(z) = 0.1$

$\Rightarrow \phi(z) = 0.9$

$z = \phi(0.9)$

$= 1.282$

$$\Rightarrow \frac{(C+0.5) - 24}{\sqrt{16.28}} < -1.282 \Rightarrow C < 18.245$$

$\therefore$ The critical value $= 18$ ✓

Hypothesis testing with the Poisson distribution, $Po(\lambda)$

Example 7 (a) The proportion of people having a particular medical condition is 1 in 100000. A random sample of 2500 people is obtained. The numbers of people in the sample having the condition is denoted by X.

(i) State, with a justification, a suitable approximating distribution for X, giving the values of any parameters. --[2]

(ii) Use the approximating distribution to calculate $P(X>0)$ --[2]

(iii) The percentage of people having a different medical condition is thought to be 30%. A researcher suspects that the true percentage is less than 30%. In a medical trial a random sample of 28 people was selected and 4 people were found to have this condition.
Use a binomial distribution to test the research's suspicion at the 2% significance level. [W-21|61|Q5] --[5]

Solution (a) $p = 1/100\,000$, $n = 2500 \rightarrow$ mean $\lambda = np = 2500 \times \frac{1}{100\,000} = 0.025$

(i) here $n > 50$ and $np = 0.025 < 5$.
$\therefore$ Poisson distribution. $Po(0.025)$ $\{ Po(\lambda)$

(ii) $P(X>0) = 1 - P(0) = 1 - e^{-\lambda} = \underline{(1 - e^{-0.025})}$ ✓

(b) Null hypothesis $H_0 : p = 0.3$ (30%)
Alternate hypothesis $H_1 : p < 0.3$
Now 4 out of 28 ; $p = 0.3, q = 0.7$

$P(X \leq 4) = P(X = 0, 1, 2, 3, 4)$
$= 0.7^{28} + 28 \times 0.3 \times 0.7^{27} + {}^{28}C_2 \times 0.3^2 \times 0.7^{26} + {}^{28}C_3 \times 0.3^3 \times 0.7^{25} + {}^{28}C_4 \times 0.3^4 \times 0.7^{24}$

$= 0.0474 > 0.02$ Significance level (2%)

$\therefore$ H_0 is not rejected.
or No evidence that suspicion is true.

§ Hypothesis testing with the Poisson distribution:

Example 8: The number of calls to consumer hotline can be modelled by a Poisson disaribution with mean 62 calls every 5 minutes. Salina believes that average is too low and observes the number of calls recorded during a randomly chosen 5 minute interval to be 70. Stating the null and alternate hypothesis, test Salina's belief at the 10% significance level.

Solution: Random Variable, X denotes the number calls recorded 5 minutes int-erval

Null hypothesis is H_0 mean $\lambda = 62$

Alternate hypothesis H_1: mean $\lambda > 62$ calls per 5 minutes.

$X \sim Po(62) \rightarrow$ approximated to $N(\mu, \sigma^2) = N(\lambda, \lambda) = N(62, 62)$

$$P(X \geq 70) = P\left(Z \geq \frac{69.5 - 62}{\sqrt{62}}\right)$$ { Continuity correction $X \geq 70 \rightarrow Z \rightarrow 69.5$

$= P(Z \geq 0.953)$

$= 1 - \phi(0.953) = 1 - 0.8297$

$= 0.1703$ ✓

here $0.1703 > 0.10$ (Significance level 0.1 (10%))

$\therefore$ H_0 is accepted ✓

There is insufficient evidence to support Salina's belief that the mean number of calls is greater than 62 per 5 min.

$Po(\lambda) \rightarrow N(\mu, \sigma^2) \sim N(\lambda, \lambda)$

Example 9: The numbers of customers who visit a particular shop between 9.00am and 10.00am has the distribution $Po(\lambda)$. In the past the value of λ was 5.2. Following some new advertising, the manager wishes to test whether the value of λ has increased. He chooses a random sample of 20 days and finds that the total numbers of customers who visited the shop between 9.00am and 10.00 am on those days is 125.
Use an approximating distribution to test at 2.5% significance level whether the value of λ has increased. --[6]

[S-20/63/Q3]

Solution: Random variable: X denotes the number of customers who visit a shop between 9.am and 10:00 am.

$Po(\lambda)$: $\lambda = 5.2$ each day

for 20 day → $\lambda = 5.2 \times 20 = 104$ ✓ > 15, $Po(104)$

∴ Poisson distribution approximates to normal distribution. $Po(104) \rightarrow N(\mu, \sigma^2) \sim N(104, 104)$ $\left\{ \begin{array}{l} \mu = \lambda = 104 \\ \sigma^2 = \lambda = 104 \end{array} \right.$

Null hypothesis, H_0: $\lambda = 104$
Alternate hypothesis, H_1: $\lambda > 104$

$$\text{Now } P(X \geq 125) = P\left(z > \frac{124.5 - 104}{\sqrt{104}}\right) \left\{ \begin{array}{l} \text{continuity correction} \\ x \geq 125 \rightarrow x > 124.5 \end{array} \right.$$

$$= P(z > 2.010)$$
$$= 1 - \phi(2.01)$$
$$= 1 - 0.9778 = 0.0222$$

here $0.0222 < 0.025$ (H_0 is rejected) [Significance Level $= 0.025$ (2.5%)

There is evidence that λ has increased.

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One tailed / Two tailed test.

§ **One tailed test:**

If the parameter probability, p/or mean μ is increase or decrease than it is one tailed test. ($p<a$ or $p>a$) or $\mu<a$ or $\mu>a$)

§ **Two tailed test:**

For a two tailed test the parameter p or mean is different. ($p\neq a$ or $\mu\neq a$)

§ In most two tailed testing the critical testing is shared equally between the two tails.

However in some problems the unequal region is allocated between the two tails.

Example 10: A television channel claims that 25% of its programmes are nature programmes. Xavier thinks this claim is incorrect. To test his hypothesis, he chooses 20 programmes at random and carries out a hypothesis test at 10% significance level.

(a) Define random variable, stating its parameters and write down the two hypothesis.

(b) If there are only two nature programmes, calculate the test statistic. What conclusion does Xavier reach?

Solution: Random Variable: Let X be the number of nature programmes.

(a) $n=20$, $p=0.25$ (25%) $X\sim B(20,0.25)$

Null hypothesis, H_0 : $p=0.25$

Alternate hypothesis, H_1 : $p\neq 0.25$

$\left\{ \begin{array}{l} n=20, p=0.25 \\ q=0.75 \end{array} \right.$

(b) The test statistic is:

$P(X\leq 2)=(0.75)^{20}+{}^{20}C_1\cdot(\cdot 25)^1\cdot(\cdot 75)^{19}+{}^{20}C_2\cdot(\cdot 25)^2\cdot(\cdot 75)^{18}=0.0913$

The critical value is $\frac{1}{2}\times 0.1 = 0.05$ ($\frac{1}{2}$ of 10%, as a two tailed test is required)

Now $0.0913>0.05$. Hence H_0 is accepted.

Xavier concludes that the proportion of nature programmes is likely to be 25% as claimed.

Two tailed test

Example 11: A manufacturer sells bags of 20 marbles in mixed colours. It claims that 30% of the marbles are red. Ginny thinks this is incorrect and tests the claim by opening a bag of marbles and counting how many are red.

(a) Carry out a hypothesis test at the 10% level of significance given that Ginny finds three red marbles. Find the critical value for this test.

(b) Suppose Ginny thinks the percentage should be lower than 30%. State the hypothesis. Will Ginny's conclusion change?

Solution: Random varible, X be the number of red marbles, then $X \sim B(20, 0.3)$

Null hypothesis, H_0: $p = 0.3$

Alternate hypothesis, H_1: $p \neq 0.3$

$\left\{ \begin{array}{l} n = 20 \\ p = 0.3 \ (30\% \\ q = 0.7 \end{array} \right.$

$$P(X \leq 3) = P(X = 0, 1, 2, 3)$$
$$= (0.7)^{20} + {}^{20}C_1 \times 0.3^1 \times (0.7)^{19} + {}^{20}C_2 \times 0.3^2 \times 0.7^{18} + {}^{20}C_3 \times 0.3^3 \times (0.7)^{17}$$

Significance level 10%
being two tail test
Significance level $= \frac{1}{2} \times 10\%$
$= 5\% \ (0.05)$

$\therefore P(X \leq 3) = 0.107 > 0.05$ hence H_0 is accepted.

There is sufficient evidence to accept the manufacturer that 30% of the marbles are red.

Now $P(X \leq 2) = 0.7^{20} + {}^{20}C_1 0.3^1 \times 0.7^{19} + {}^{20}C_2 \times 0.3^2 \times 0.7^{18} = 0.0355 < 0.05$

$\therefore$ H_0 is rejected at $X = 2$, but accepted at $X = 3$

$\therefore$ The critical value is $X = 2$.

(b) Now H_0, $p = 0.3$

and H_1, $p < 0.3$

One-tail test

and $P(X \leq 3) = 0.107 > 0.10$ (Significance level = 10% = 0.1)

So H_0 is accepted.

Hence Ginny's conclusion will not change.

P(Type-I error) = P(reject Ho/Ho is true)

Example 12. An archer claims that her skill at firing arrows is such that she can hit the bullseye 40% of the time.
A hypothesis test at the 5% significance level is carried out to test if the archer is as good as she claims to be.
The archer fires 12 arrows at a target.

(a) Define the random variable and its parameters.
(b) State the null and alternate hypothesis.
(c) Find the rejection region.
(d) Show that a type I error occurs if the archer hits the bullseye only once.

Solution: Random Variable: X be the number of arrows that hits the bullseye.
Then X has binomial distribution with $n=12$ and $p=0.4$, $(q=0.6)$
i.e. $X \sim B(12, 0.4)$

(b) Null hypothesis, $H_0: p = 0.4$
Alternate hypothesis, $H_1: p < 0.4$ Significance level = 0.05

(c) For rejection region, To find the smallest value of n, such that, $P(X \leq n) < 0.05$

$P(X \leq 2) = 0.6^{12} + {}^{12}C_1 \times 0.4^1 \times 0.6^{11} + {}^{4}C_2 \times 0.4^2 \times 0.6^{10}$
$= 0.0834 > 0.05$ --- (i) (Ho is accepted)

$P(X \leq 1) = 0.6^{12} + {}^{12}C_1 \times 0.4^1 \times 0.6^{11}$
$= 0.0196 < 0.05$ --- (ii) (Ho is Rejected)

from (i) and (ii) The rejection region is $X \leq 1$.

(d) If $H_0 = 0.4$ is True, then $X = 1$ lies in the rejection region. That is the archer's claim that $p = 0.4$ will be be rejected, thus a type I error occurs.

Example 13. The number of absences per week by workers at a factory has the distribution $Po(2.1)$

(a) Find the standard deviation of the number of absences per week. ---[1]

(b) Find the prob. that the number of absences in a 2-week period is at least 2. --[3]

(c) Find the probability that the number of absences in a 3-week period is more than 4 and less than 8. ---[2]

Following a change in working conditions, the management wished to test whether the mean number of absences has decreased. They found that, in a randomly chosen 3-week period, there were exactly 2 absences.

(d) Carry out the test at the 10% significance level. ---[5]

(e) State, with a reason, which of the errors, Type I or Type II, might have been made in carrying out the test in part (d). ---[2]

[W-20/ 61/ Q5]

Solution (a) $Po(2.1) \Rightarrow \lambda = 2.1$ $\quad \{Po(\lambda) \sim N(\mu, \sigma^2) = N(\lambda, \lambda)$

Hence Standard deviation, $\sigma = \sqrt{\lambda} = \sqrt{2.1} = \underline{1.45}$ ✓ $\quad \{\Rightarrow \sigma^2 = \lambda \Rightarrow S.D = \sqrt{\lambda}$

(b) Absence per week $\lambda = 2.1 \Rightarrow$ for 2-weeks period $\lambda = 2.1 \times 2 = 4.2$

$P(X \geq 2) = 1 - P(X=0, X=1) = 1 - e^{-4.2}[1 + 4.2] = \underline{0.923}$ ✓

(c) In 3-week period $\lambda = 2.1 \times 3 = 6.3$ $\quad \left\{P(X=r) = \dfrac{e^{-\lambda} \cdot \lambda^r}{r!}\right.$

$P(4 < X < 8) = P(X = 5, 6, 7)$

$$= e^{-6.3}\left[\frac{6.3^5}{5!} + \frac{6.3^6}{6!} + \frac{6.3^7}{7!}\right] = \underline{0.0455} ✓$$

(d) Null hypothesis, $H_0: \lambda = 6.3$ (3-week period)

Alternate hypothesis $H_1: \lambda < 6.3$ $\quad$ (Significance Level 10%)

$P(X \leq 2) = e^{-6.3}\left[1 + 6.3 + \frac{6.3^2}{2!}\right] = 0.0498 < 0.1$

hence there is evidence that the mean number of absences has decreased. (H_0 is rejected) ⊗

(e) ⊗ H_0 is rejected, hence Type I error possible. ✓

Example 14: Sumitra has a six-sided die. She suspect that it is biased so that it shows a six less often than it would, if it were fair. She decides to test the die by throwing it 30 times and noting the number of throws on which it shows a six

(i) It shows a six on exactly 2 throws. Use a binomial distribution to carry out the test at the 5% significance level. ---[5]

(ii) Later, Sumitra repeates the test at the 5% significance level by throwing the die 30 times again.
Find the probability of a type I error in this second test. ---[2]

S-19/71/Q3

Solution: Let X be the number of sixes, when a six-sided die is thrown 30 times. Then $X \sim B(30, \frac{1}{6})$

Null Hypothesis H_0: $P(6) = p = \frac{1}{6}$

Alternate hypothesis H_1: $P(6) = p < \frac{1}{6}$.

$\left\{ \begin{array}{l} n = 30, p = \frac{1}{6} \\ q = \frac{5}{6} \end{array} \right.$

(i) Significance level = 5% or 0.05 ✓

Test statistics is $P(X \leq 2) = P(0) + P(1) + P(2)$ $[P(X=r) = {}^{n}C_r p^r q^{n-r}]$

$$= \left(\frac{5}{6}\right)^{30} + {}^{30}C_1 \cdot \frac{1}{6} \times \left(\frac{5}{6}\right)^{29} + {}^{30}C_2 \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^{28}$$

$= 0.103 > 0.05$ the significance level

$\therefore$ No evidence that the die is biased. ($\therefore$ H_0 is accepted)

(ii) For type I error the null hypthesis is rejected: $p < 0.05$

$$p = P(0) + P(1)$$
$$= \left(\frac{5}{6}\right)^{30} + {}^{30}C_1 \cdot \frac{1}{6} \times \left(\frac{5}{6}\right)^{29} = 0.0295 < 0.05$$

$\therefore$ P(Type I error) $= 0.0295$ ✓

Note: If the significance level is given to be (let) 12% or 0.12
Then in part (i) $0.103 < 0.12$ (The significance level)
we will say H_0 is rejected.
The die is biased towards the number 6. (H_1 is accepted)

Example 15: A chocolate manufacturer claims that, only one-fifth of teenagers can tell the difference between its chocolate and the leading brand. Priya suspects that the claim is too high. For a random sample of 300 teenagers, at a 5% significance level, determine the number that would need to tell the difference for a Type I error to occur.

Solution: Random variable, X be the number of teenagers who can tell the difference between this chocolate and the leading brand. Then $X \sim B(300, 0.2)$ $\left[\frac{1}{5} = 0.2\right]$

Approximating distribution to Normal.

$$X \sim N(\mu, \sigma^2)$$

$$\text{or } X \sim N(60, 48).$$

$\left[\begin{array}{l} p = 0.2,\ q = 0.8,\ n = 300 \\ \mu = np = 300 \times 0.2 = 60 \\ \sigma^2 = npq = 300 \times 0.2 \times .8 = 48 \end{array}\right.$

Null hypothesis, H_0: $p = 0.2$ or $\mu = 60$

Alternate hypothesis, H_1: $p < 0.2$ or $\mu < 60$

$\{$ Significance level is 5% (or 0.05)

Let the critical value be a, then with continuity correction at $(a - 0.5)$

we have. $P(X \leq a - 0.5) = P\left(Z \leq \dfrac{(a - 0.5) - 60}{\sqrt{48}}\right) < 0.05$

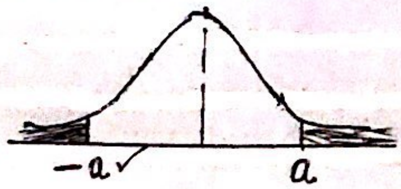
$$\Rightarrow \frac{a - 60.5}{\sqrt{48}} < -1.645$$

$$\Rightarrow a - 60.5 < -1.645 \times \sqrt{48}$$

$$a = 60.5 - 11.3968$$

$$a = 49.103 \ldots$$

$\therefore$ Type I error occurs when 49 or fewer of the 300 teenagers can tell the difference.



$P(z < -a) = 0.05$

$P(z > a) = 0.05$

$P(z < a) = 1 - 0.05 = 0.95$

$a = \phi^{-1}(0.95)$

$a = 1.648$

Example 16: At a certain hospital it was found that the probability that a patient did not arrive for an appoint was 0.2. The hospital carries out some publicity in the hope that this probability will be reduced. They wish to test whether the publicity worked.

A random sample of 30 appointments is selected and the number of patients that do not arrive is noted. This figure is used to carry out a test at the 5% significance level.

(a) Explain why the test is one-tailed and state suitable null and alternate hypothesis. --[2]

(b) Use a binomial distribution to find the critical region and find the probability of type I error. --[5]

(c) In fact 3 patient out of the 30 patients do not arrive. State the colclusion of the test, explaining your answer. --[2]

[SP/20/Q6]

Solution (a) Random variable, X denotes the number of patients not coming for appointment.
Test is one tailed as looking for decrease in number of patients noting coming for an appointment.
Null Hypothesis, H_0: P(not arrive) $p = 0.2$
Alternate hypothesis, H_1: $p < 0.2$ Significance level $= 5\% = 0.05$

(b) Binomial distribution $X \sim B(n,p)$; $n = 30$, $p = 0.2$, $q = 0.8$

$P(X \leq 2) = 0.8^{30} + 30_{C_1} 0.2^1 \times 0.8^{29} + 30_{C_2} 0.2^2 \times 0.8^{28} = 0.0442$ --(i)

$P(X \leq 3) = \text{- - - - - - - -} + 30_{C_3} 0.2^3 \times 0.8^{27} = 0.123$ --(ii)

Here $P(X \leq 2) = 0.0442 < 0.05$
but $P(X \leq 3) = 0.123 > 0.05$ $\Bigg\}$ $\therefore$ Critical region is $X \leq 2$

and the critical value is $X = 2$, P(Type I error) $= 0.0442$ ✓

(c) $P(X \leq 3) = 0.123 > 0.05$

$\therefore$ Hence No evidence that p has decreased.
(or H_0 is accepted)

$P(\text{Type II- error}) = P(\text{accept } H_0 / H_0 \text{ is false})$

Example 17: A gardener knows from experience that the height, X, of his sunflowers is normally distributed with mean 1.8 m and standard deviation 0.32 m. A friend claims that she can make sunflowers taller by singing to them daily. They decide to test the claim. One plant, selected at random, is sung to each day while it grows.

(a) State the null and alternate hypothesis.

(b) The gardener will be convinced that singing increasing sunflower height if X > 2.25 m. Calculate the probability of a Type I error.

(c) The friend says that the mean height of sunflowers sung to is 2.4 m. Assuming the standard deviation is the same, calculate the probability of a Type II error.

Solution: Random Variable, X denote the height of sunflowers.

(a) Null hypothesis, $H_0: \mu = 1.8m$. $X \sim N(\mu, \sigma^2)$

Alternate hypothesis, $H_1: \mu > 2.1m$ $X \sim N(1.8, 0.32^2)$

(b) $P(\text{Type I error}) = P(X > 2.25 / \mu = 1.8)$

$= P\left(Z > \frac{2.25 - 1.8}{0.32}\right)$

$= P(Z > 1.406)$

$= 1 - \phi(1.406) = 1 - 0.92$

$= \underline{0.0800}$ ✓

(c) $P(\text{Type II error}) = P(X \leq 2.25 / \mu = 2.4)$

$= P\left(Z \leq \frac{2.25 - 2.4}{0.32}\right)$

$= P(Z \leq -0.469)$

$= 1 - \phi(0.469)$

$= 1 - 0.6804$

$= \underline{0.320}$ ✓

Example 18 A ten sided spinner has edges numbered 1,2,3,4,5,6,7,8,9,10. Sanjeev claims that the spinner is biased so that it lands on the 10 more often than it would if were unbaised. In an experiment, the spinner landed on the 10 in 3 out of 9 spins.

(i) Test at 1% significance level whether Sanjeev's claim is justified:-[5]

(ii) Explain why a Type I error can not be made. --[1]

In fact the spinner is biased so that the probability that it will land on the 10 on any spin is 0.5.

(iii) Another test at 1% significance level, also based on 9 spins, is carried out. Calculate the probability of a Type II error. --[6]

S-18/72/Q7

Solution: Let X be the number of 10's, when a ten-sided spinner spins 9 times.

Then $X \sim B(9, 0.1)$

Null Hypothesis $H_0: P(10) = p = 0.1$

Alternate hypothesis $H_1: P(10) = p > 0.1$

(i) Significance level 1% or 0.01

$B(9, 0.1)$ $n=9$, $p=0.1$, $q=0.9$

$P(X \geq 3) = 1 - P(X = 0, 1, 2)$

$= 1 - \{(0.9)^9 + {}^9C_1 \times 0.1 \times (0.9)^8 + {}^9C_2 (0.1)^2 \cdot (0.9)^7\} = 0.05297$

$= 0.053 > 0.01$

$\therefore$ No evidence (at 1%) to reject H_0.

claim is not justified.

(ii) Type I error cannot be made as H_0 is not rejected (for Type I error H_0 is rejected)

(Here H_0 is accepted)

(iii) $P(X \geq 4) = 0.05297 - P(3)$

$= 0.05293 - {}^9C_3 (0.9)^6 (0.1)^3$

$= 0.00833 < 0.01$

$\therefore$ Critical value is 4.

$B(9, 0.5)$, $p = 0.5$, $q = 0.5$, $n = 9$

$P(X < 4) = P(X = 0, 1, 2, 3)$

$= (0.5)^9 + {}^9C_1 (.5)(.5)^8 + {}^9C_2 (.5)^2 (.5)^7 + {}^9C_3 (.5)^3 \cdot (.5)^6$

$= 0.254$

$P(\text{Type II}) = 0.254$ ✓

(null hypothesis is false but is accepted)

Example 19: In the past, the number of house sales completed per week by a building company modelled by a random variable which has the distribution $Po(0.8)$. Following a publicity campaign, the builders hope that the mean number of sales per week will increase. In order to test at the 5% significance level whether this test is the case, the total numbers of sales during the first 3 weeks after the campaign is noted. It is assumed that a Posisson model is still appropriate.

(i) Given that the total number of sales during the 3 weeks is 5, Carry out the test. --[6]

(ii) During the following 3 weeks the same test is carried out again, using the same significance level. Find the prob. of a Type I error. --[3]

(iii) Explain what is meant by a Type I error in this context. --[1]

(iv) State what further information would be required in order to find the probability of a type II error. [W-10/73/Q7] --[1]

Solution (i) For a 3 week period $\lambda = 3 \times 0.8 = 2.4$

$H_0: \lambda = 2.4$ and $H_1: \lambda > 2.4$

$P(X \geq 5) = 1 - P(X \leq 4)$

$= 1 - e^{-2.4}\left[1 + 2.4 + \frac{2.4^2}{2!} + \frac{2.4^3}{3!} + \frac{2.4^4}{4!}\right]$

$= 0.0959 > 0.05$ ∴ H_0 is accepted.

∴ There is insufficient evidence to suggest that the mean number of sales per week has increased.

(ii) $P(\text{Type I error}) = P(H_0 \text{ reject} / H_0 \text{ is true})$

Now $P(X \geq 6) = 1 - P(X \leq 5)$

$= 1 - e^{-2.4}\left[1 + 2.4 + \frac{2.4^2}{2!} + \dots + \frac{2.4^5}{5!}\right]$

$= 0.0357 < 0.05$

The smallest number $n = 6$, so that

$P(\text{type I error}) = \underline{0.0357}$ ✓

(iv) The actual mean number of sales per week (or an alternate to 2.4) is given.

Note:

$P(\text{Type II error})$

$= P(H_0 \text{ is accepted} / H_0 \text{ is false})$

(iii) A type I error would be made if we conclude that the mean number of sales per week had increased, when it had not, (or that the publicity campaign was a success, when it was not)

Example 20: A hospital patient's white blood cell count has a Poisson distribution. Before undergoing treatment the patient had a mean white blood cell count 5.2. After the treatment a random measurement of the patient's white blood cell count is made, and is used to test at the 10% significance level whether the mean white blood cell count has decreased.

(i) State what is meant by a Type I error in the context of the question, and find the probability that the test results in a Type I error. --- [4]

(ii) Given that the measured value of the white blood cell count after the treatment is 2, carry out the test. -[3]

(iii) Find the probability of a Type II error if the mean white blood cell count after the treatment is actually 4.1 [3]

S-10/71/Q7

Solution (i) Type I error occurs when we conclude that the patients white blood cell count has decreased; when in fact it has not.

we need to find the largest n for which $P(X \le n) < 0.10$

$P(X \le 0) = e^{-5.2} = 0.005516 < 0.10$ (Significance level 10%)

$P(X \le 1) = e^{-5.2}(1 + 5.2) = 0.03420 < 0.10$ (H_0 is rejected)

$P(X \le 2) = e^{-5.2}\left(1 + 5.2 + \frac{5.2^2}{2!}\right) = 0.10878 > 0.10$ (H_0 is True)

$\therefore$ Largest value for Type I error $n = 1$, $P(\text{Type I error}) = 0.0342$

(ii) $H_0: \lambda = 5.2$ and $H_1: \lambda < 5.2$

from Part I, $P(X \le 2) = 0.10878 > 0.10 \Rightarrow H_0$ is accepted.

hence there is insufficient evidence that the white blood cell count has decreased.

(iii) $P(\text{Type II error}) = P(X \geq 2 \mid \lambda = 4.1)$

$= 1 - e^{-4.1}(1 + 4.1) = 0.915$

Expectation and Variance of semple mean $\bar{x}$, (n is the number of samples)

$E(\bar{x}) = \mu$ where $\mu = E(X)$; $\underline{Var(\bar{x}(n)) = \frac{\sigma^2}{n}}$; $Var(x) = \sigma^2$

Example 21: In Europe the diameters of women's ring have mean 18.5 mm. Researchers claim that women in Jakarta have smaller fingers than women in Europe. The researchers took a random sample of 20 women in Jakarta and measured the diameters of their rings. The mean diameters was found to be 18.1 mm. Assuming that the diameters of women's ring in Jakarta have a normal distribution with standard deviation 1.1 mm, carry out a hypothesis test at 2.5% level to determine whether the researcher's claim is justified. --[5]

[S-09/71/Q1]

Solution: Random variable: X denotes the mean diameter of ring

Null hypothesis: H_0: $\mu = 18.5$ mm

Alternate hypothesis H_1: $\mu < 18.5$ mm

Significance level $= 2.5\%$ (or 0.025)

Now the mean diameter of 20 women's ring be $\bar{x}$

Then $\bar{X} \sim N\left(18.5, \frac{1.1^2}{20}\right) \left\{ N\left(\mu, \frac{\sigma^2}{n}\right) \right\}$

$$\text{Now } P(\bar{X} \leq 18.1) = P\left(Z \leq \frac{18.1 - 18.5}{\frac{1.1}{\sqrt{20}}}\right) \qquad SD = \sqrt{\frac{\sigma^2}{n}}$$

$$= P(Z \leq -1.626)$$

$$= 1 - \phi(1.626) = 1 - 0.9474$$

$$= 0.052 \checkmark$$

Now $0.052 > 0.025$ (Significance level)

so H_0 is accepted.

Hence the researcher's claim that women in Jakarta have smaller fingers (Rings with smaller diameters) is not justified.

22. The mass, in kilograms, of rocks in a certain area has mean 14.2 and standard deviation 3.1.

(i) Find the probability that the mean mass of a random sample of 50 of these rocks is less than 14.0 kg. ---[3]

(ii) Explain whether it was necessary to assume that the population of the masses of these rocks is normally distributed. ---[1]

(iii) A geologist suspects that rocks in another area have a mean mass which is less than 14.2 kg. A random sample of 100 rocks in this area has sample mean 13.5 kg. Assuming that the standard deviation for rocks in this area is 3.1 kg, test at the 2% significance level whether the geologist is correct. ---[5]

[S-18 / 71 / Q5]

Solution (i) $\mu = 14.2$, $\sigma = 3.1$, $n = 50$ $\quad \sigma^2(\bar{x}) = \frac{\sigma^2}{n}$, $SD = \frac{\sigma}{\sqrt{n}}$

$$P(\bar{x} < 14) = P\left(z < \frac{14 - 14.2}{3.1/\sqrt{50}}\right) = P(z < -0.456)$$

$$= 1 - \phi(0.456) = 0.324 \ (3\text{ sf})$$

(ii) No because n is large. (accept $n \geq 30$)

(iii) $\mu = 14.2$, $S.D = 3.1$

H_0: Null hypothesis: $\mu = 14.2$ $\quad$ SD of mean $= \frac{\sigma}{\sqrt{n}}$

H_1: Alternate hypothesis: $\mu < 14.2$

$$P(\mu < 13.5) = P\left(z < \frac{13.5 - 14.2}{3.1/\sqrt{100}}\right) = P(z < -2.258)$$

$$= 1 - \phi(2.258) = 1 - 0.9881 = 0.$$

$$= 0.0119 < 0.02 \quad \{ 2\% \text{ Significance level}$$

(H_0 is rejected)

(H_1 is accepted)

There is evidence (at 2% level) that mean mass in this area < 14.2.

Example 23: A fair spinner has five sides numbered 1, 2, 3, 4, 5. The score on one spin is denoted by X.

(a) Show that $Var(X) = 2$ --[1]

(b) Fiona has another spinner, also with five sides numbered, 1, 2, 3, 4, 5. She suspects that is baised so that the expected score is less than 3. In order to test her suspicion, she plans to spin her spinner 40 times. If the mean score is less than 2·6 she will conclude that her spinner is biased in this way.

(b) Find the probability of a Type I error. --[4]

(c) State what is meant by a Type II error in this context. --[1]

$\boxed{S\text{-}20/61/Q4}$

Solution. (a) Mean of 1, 2, 3, 4, 5 $\rightarrow \bar{x} = \frac{1+2+3+4+5}{5} = 3 \Rightarrow \bar{x} = 3$, $n = 5$

Variance $\sigma^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 = \frac{(1^2+2^2+3^2+4^2+5^2)}{5} - 3^2 = 2$ ✓

$Var(X) = 2$ ✓

(b) Random variable, X is the score on the spin → Normal distribution:

$N(\mu, \sigma^2) \sim N(3, 2)$

Null hypothesis, H_0, $\mu = 3$

Alternate hypothesis H_1, $\mu < 3$, $n = 40$

$$P(\mu < 2{\cdot}6) = P\left(z < \frac{2{\cdot}6 - 3}{\sqrt{\frac{2}{40}}}\right) \qquad \left\{ Var = \frac{\sigma^2}{n} \right.$$

$$= P(z < -1{\cdot}789)$$

$$= 1 - \phi(1{\cdot}789)$$

$$= 1 - 0{\cdot}9633$$

$$P(\text{Type I error}) = 0{\cdot}0367$$

(c) Concluding that spinner is unbiased when it is biased.

24. A shop sells 2kg bags of potatoes. A quality control inspection checks the masses of 80 randomly chosen bags. Their masses, x, are summarised as follows:

$$\Sigma x = 158.14 \quad \text{and} \quad \Sigma x^2 = 314.094$$

Assuming the masses of the bags of potatoes are normally distributed, investigate at 5% level of significance whether there is any evidence that the bags are underweight.

Solution: $\Sigma x = 158.14$ and $\Sigma x^2 = 314.094$, $n = 80$ (large)

Sample mean $\bar{x} = \frac{158.14}{80} = 1.97675$, (unknown Variance)

Unbiased estimated Variance $s^2 = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)$

$$= \frac{1}{79}\left(314.094 - \frac{158.14^2}{80}\right)$$

$$s^2 = 0.018870316$$

Null Hypothesis H_0: Population mean mass = 2 kg
Alternate Hypothesis H_1: Population mean mass < 2kg

$SD = \sqrt{\frac{s^2}{n}}$

$$P(\bar{x} \leq 1.97675) = P\left(z \leq \frac{1.97675 - 2}{\sqrt{\frac{0.018870316}{80}}}\right) = P(z \leq -1.514)$$

$$= 1 - \phi(1.514)$$
$$= 1 - 0.9350$$
$$= 0.0650$$

$0.0650 > 0.05$ (Significance level 5%)

$\therefore$ H_0 is accepted.

There is no evidence to suggest that the bags of potatoes are underweight.

25. A survey taken last year showed that the mean number of computers per household in Branley was 1·66. This year a random sample of 50 households in Branley answered a questionnaire with the following results:

Number of Computers	0	1	2	3	4	>4
Number of household	5	12	18	10	5	0

(i) Calculate the unbiased estimates for the population mean and variance of the number of computers per household in Branley this year. ---[3]

(ii) Test at the 5% significance level whether the mean number of computers per household has changed since last year. --[5]

(iii) Explain whether it is possible that a type I error may have been made in the test in part (ii) --[1]

(iv) State what is meant by a type II error in the context of the test in part (ii) and give the set of values of the test statistic that could lead to a Type II error being made. --[2]

[S-12/71/Q6]

Solution: (i) $\bar{x} = \frac{\Sigma f_i x_i}{n} = \frac{0+12+36+30+20}{50} = \frac{98}{50} = 1{\cdot}96$ ✓

$\Sigma f x^2 = 254$

Variance $s^2 = \frac{n}{(n-1)} \times \left(\frac{\Sigma f x^2}{n} - (\bar{x})^2\right)$

(Estimates σ^2 unbiased)

$= \frac{50}{49}\left(\frac{254}{50} - 1{\cdot}96^2\right)$

$s^2 = \frac{1548}{1225} = 1{\cdot}2637$

(ii) H_0: Pop Mean $= 1{\cdot}66$

H_1: Pop Mean $\neq 1{\cdot}66$

Test Statistic $\frac{1{\cdot}96 - 1{\cdot}66}{\sqrt{\frac{1{\cdot}2637}{50}}} = 1{\cdot}887$

$1{\cdot}887 < 1{\cdot}96$ [Significance level 5%, $z = 1{\cdot}96$]

No evidence that mean has changed.

(iv) mean not changed when it has.

$-1{\cdot}96 <$ test Statistic $< 1{\cdot}96$ ✓

The test statistic:

$z = \left(\frac{\bar{x} - \mu}{\sqrt{\frac{s^2}{n}}}\right)$

(iii) No, as H_0 is not rejected.

(A Type I error occurs when a true null hypothesis is rejected)

(or H_0 is rejected / H_0 is true)

Example 26: The mean weight of bags of carrots is μ kilograms. An inspector wishes to test whether $\mu = 2.0$. He weighs a random sample of 200 bags and his results are summarised as follows.

$$\Sigma x = 430 \quad , \quad \Sigma x^2 = 1290$$

Carry out the test at the 10% significance level. --[7]

[SP/20/Q7]

Solution: Randendom variable, μ is the mean weight of bags.

Null hypothesis $H_0 : \mu = 2.0$

Alternate hypothesis $H_1 : \mu \neq 2.0$

Now estimated mean $\bar{X} = \frac{\Sigma x}{n} = \frac{430}{200} = 2.15$

estimated Variance $s^2 = \frac{200}{119}\left(\frac{1290}{200} - (2.15)^2\right) = 1.8366834$

$$P(\mu < 2.15) = P\left(z < \frac{2.15 - 2}{\sqrt{\frac{1.8366834}{200}}}\right)$$

$$= P(z < 1.565)$$

$$= \phi(1.565)$$

$$= 0.9412 < 0.95$$

$\underset{(Sample)}{Var} = \frac{\sigma^2}{n}$

being two tailed test
Significance level
$= \frac{1}{2} \times 10\% = 0.05$
on the upper side $\Rightarrow p = 0.95$
or $z = 1.645$
(here $1.565 < 1.645$)

No evidence that $\mu \neq 2.0$

$$\left[est(Var\, x) = s^2 = \frac{n}{(n-1)}\left(\frac{\Sigma x^2}{n} - (\bar{x})^2\right) \right]$$

Example 27: In the past, the mean time taken by Freda for a particular daily journey was 39.2 minutes. Following the introduction of a one-way system. Freda wishes to test whether the time for the journey has decreased. She notes the times, t times, for randomly chosen journeys and summarises the results as follows:

$$n=40, \quad \Sigma t = 1504, \quad \Sigma t^2 = 57760$$

(a) Calculate unbaised estimates of the population mean and variance of the new journey time. --[3]

(b) Test, at the 5% significance level, whether the population mean time has decreased. --[5]

M-20/62/Q3

Solution:
(a) Estimated mean $\mu = \frac{\Sigma t}{n} = \frac{1504}{40} = 37.6$ ✓

Estimated variance:

$$s^2 = \frac{40}{39}\left[\frac{57760}{40} - 37.6^2\right] = 31.0154 ✓$$

(b) Significance level 5% = 0.05

Null hypothesis, H_0: $\mu = 39.2$ (population mean)

Alternate hypothesis H_1: $\mu < 39.2$

$$P(X < 37.6) = P\left(Z < \frac{37.6 - 39.2}{\sqrt{\frac{31.0154}{40}}}\right) \qquad Var\ \bar{x} = \frac{\sigma^2}{n}$$

$$= P(Z < -1.817)$$

$$= 1 - \phi(1.817)$$

$$= 1 - 9654$$

$$= 0.0346 < 0.05$$

∴ There is evidence that mean time has decreased.

(or Null hypothesis is rejected)

$$s^2 = \frac{n}{n-1}\left\{\frac{\Sigma x^2}{n} - \mu^2\right\}$$

Example 28: A market researcher is investigating the length of time that customers spend at an information desk. He plans to choose a sample of 50 customers on a particular day.

(a) He considers choosing the first 50 customers who visit the information desk.
Explain why this method is unsuitable. --[1]

The actual lengths of time, in minutes, that customers spend at the information desk may be assumed to have mean μ and variance 4.8. The researcher knows that in the past the value of μ was 6.0. He wishes to test, at the 2% significance level, whether this is still true. He chooses a random sample of 50 customers and notes how long they each spend at the information desk.

(b) State the probability of making a Type I error and explain what is meant by a Type I error in this context. --[2]

(c) Given that the mean time spent at the information desk by the 50 customers is 6.8 minutes, carry out the test. --[5]

(d) Why to use the Central Limit theorem in your answer to part (c). --[1]

S-20/63/Q7

Solution: (a) Later the customers might spend times different from first ones.

(b) P(Type I error) = 0.02
Concluding that $\mu \neq 6.0$ when actually $\mu = 6.0$, $\frac{\sigma^2}{n} = \frac{4.8}{50}$

(c) Random variable: Length of time.
Null hypothesis, $H_0: \mu = 6.0$
Alternate hypothesis, $H_1: \mu \neq 6.0$

$$P(\mu > 6) = P\left(z > \frac{6.8 - 6}{\sqrt{\frac{4.8}{50}}}\right)$$
$$= P(z > 2.582)$$

here $2.582 > 2.326$
$\therefore$ (H_0 is rejected)
or Evidence that $\mu \neq 6.0$

Two Tailed test as $\mu \neq 6.0$
Significance level $2\% \times \frac{1}{2} = 0.01$
$P(z > a) = 0.01$
$1 - \phi(a) = 0.01$
$P(z < a) = p = \phi(a) = 0.99$
$z = a = \phi^{-1}(0.99) = 2.326$ ✓

(d) Population distribution unknown.

Example 29. A shop obtains apples from a certain farm. It has been found that 5% of apples from this farm are 'Grade A'. Following a change in growing conditions at the form, the shop management plan to carry out a hypothesis test to find out whether the proportion of 'Grade A' apples has increased. The select 25 apples at random. The number of 'Grade A' apples is more than 3 they will conclude that the proportion is increased.

(a) State suitable null and alternate hypothesis for the test. --[1]

(b) Find the probability of Type I error. --[3]

In fact 2 of the 25 apples were 'Grade A'.

(c) Which of the errors, type I or type II is possible. Jusify your answer. [S-20/62/Q2] --[2]

Solution: Random variable, X is the number of 'Grade A' apples.

(a) Null hypothesis, $H_0: p = 0.05$

Alternate hypothesis $H_1: p > 0.05$

(b) $X \sim B(n,p) \sim B(25, 0.05)$; $n = 25$, $p = 0.05$, $q = 0.95$

$P(X>3) = 1 - P(X=0,1,2,3)$

$= 1 - \{ (.95)^{25} + {}^{25}C_1 (.05) \times (.95)^{24} + {}^{25}C_2 (.05)^2 . (.95)^{23}$
$+ {}^{25}C_3 (.05)^3 . (.95)^{22}$

$= 1 - .9659 = 0.0341$ ✓

$P(\text{Type I Error}) = 0.0341$

(c) Type II

will conclude proportion not increased.

(Type II error: H_0 is accept / H_0 is false)

Example 30: A national survey shows that 95% of year 12 students use social media. Arvin suspects that the percentage of year 12 students at his college who use social media is less than the national percentage. He chooses a random sample of 20 students at his college and notes the number who use social media. He then carries out a test at the 2% significance level.

(a) Find the rejection region for the test. --[4]

(b) Find the probability of a Type I error. --[1]

(c) Jimmy believes that the true percentage at Arvin's college is 70%. Assuming that Jimmy is correct, find the probability of a Type II error. [M-20/62/Q7] --[3]

Solution: Random Variable: X is number of year 12 student to use social media.

(a) Null hypothesis; H_0: $p = 0.95$ Binomial distribution $B(20, 0.95)$

Alternate hypothesis H_1: $p < 0.95$ $\left\{ n = 20, p = 0.95, q = 0.05 \right.$

Now $P(X \leq 17) = 1 - P(X = 18, 19, 20)$ {Significance level = 2% = 0.02}

$= 1 - \left\{ {}^{20}C_{18} (.95)^{18} (.05)^2 + {}^{20}C_{19} (.95)^{19} \times (.05) + (.95)^{20} \right\}$

or $P(X \leq 17) = 0.0755 > 0.02$ --- (i)

$P(X \leq 16) = 0.0755 - P(17) = 0.0755 - {}^{20}C_{17} (.95)^{17} \cdot (0.05)^3 = 0.0159$

$\Rightarrow P(X \leq 16) = 0.0159 < 0.02$ (significance level) --- (ii)

hence from (i) and (ii) Rejection region is: $X \leq 16$

(b) $P(\text{Type I error}) = 0.0159$

(c) Now using $B(20, 0.7)$; $n = 20, p = 0.7, q = 0.3$

$P(X > 16) = P(X = 17, 18, 19, 20)$

$= {}^{20}C_{17} (.7)^{17} \cdot (.3)^3 + {}^{20}C_{18} (.7)^{18} \cdot (.3)^2 + {}^{20}C_{19} (.7)^{19} \times (.3) + (.7)^{20}$

$= 0.107 < 0.02$

$\therefore P(\text{Type II error}) = 0.107$

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Example 31: In the past the yield of a certain crop, in tonnes per hectare, had mean 0.56 and standard deviation 0.08. Following the introduction of a new fertilizer, the farmer intends to test at the 2.5 significance level whether the mean yield has increased. He finds that the mean yield over 10 years is 0.61 tonnes per hectare.

(a) State two assumptions that are necessary for the test. --[2]

(b) Carry out the test. [S-20/61/Q2] --[5]

Solution (a) (i) Standard deviation unchanged, $\sigma = 0.08$

(ii) Yields are normally distributed, $N(\mu, \sigma^2)$

(b) Null hypothesis, $H_0: \mu = 0.56$

Alternate hypothesis, $H_1: \mu > 0.56$

$$P(\mu > 0.61) = P\left(Z > \frac{0.61 - 0.56}{\frac{0.08}{\sqrt{10}}}\right)$$

$$= P(z > 1.976)$$

Significance level $= 0.025$ (2.5%)

$P(z > a) = 0.025$

$\Rightarrow 1 - \phi(a) = 0.025$

$P(z < a) = p = \phi(a) = 0.975$

$\Rightarrow Z = a = \phi^{-1}(0.975)$

$= 1.96$

$1.976 > 1.96$

(H_0 is rejected)

$\therefore$ There is evidence that mean yield has increased.

or $1.976 > 1.96$