DATE 24.01.2022

S-2

Probability and Statistics-2

Linear combination of random variables.

Notes and Revision

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(S-2) Linear combinations of random variable.

§ Expectation and variance:

1. For a random variable X and constant b:
   (i) $E(X+b) = E(X)+b$ (ii) $Var(X+b) = Var(X)$

2. For a random variable X and constants a and b.
   (i) $E(aX+b) = a \cdot E(X)+b$ (ii) $Var(aX+b) = a^2 \cdot Var(X)$

3. For two independent random variables X and Y:
   (i) $E(X+Y) = E(X)+E(Y)$ (ii) $Var(X+Y) = Var(X)+Var(Y)$

4. For two independent random variables X and Y, and constants a and b.
   (i) $E(aX+bY) = a \cdot E(X)+b \cdot E(Y)$ and $Var(aX+bY) = a^2 \cdot Var(X)+b^2 \cdot Var(Y)$

   (ii) $E(aX-bY) = a \cdot E(X) - b \cdot E(Y)$ and $Var(aX-bY) = a^2 \cdot Var(X) + b^2 \cdot Var(Y)$

5. If continuous random variables X and Y have independent normal distributions, then $aX+bY$, where a and b are constants, has a normal distribution.

6. If X and Y have independent Poisson distribution, then $X+Y$ has a Poisson distribution.
   $X \sim Po(\lambda)$ and $Y \sim Po(\mu) \Rightarrow X+Y \sim Po(\lambda+\mu)$

7. Two independent observation of X, X_1 & X_2.
   $E(X_1+X_2) = E(X_1)+E(X_2)$; $Var(X_1+X_2) = Var X_1 + Var X_2$.

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1. The independent random variables X and Y have standard deviations 3 and 6 respectively. Calculate the standard deviation of $(4X-5Y)$. [S-15/72/Q1] --[3]

Solution: $Var X = 3^2 = 9$, $Var Y = 6^2 = 36$

$\therefore Var(4X-3Y) = 4^2 \cdot Var(X) + 5^2 \cdot Var(Y) = 16\times 9 + 25\times 36 = 1044$

$\therefore$ Standard deviation $= \sqrt{1044} = 32.3$ (or $6\sqrt{29}$) ✓

2. An examination consists of a written paper and a practical test. The written paper marks (M) have mean 54.8 and standard deviation 16.0. The practical test marks (P) are independent of the written paper marks and have mean 82.4 and standard deviation 4.8. The final marks is found by adding 75% of M to 25% of P. Find the mean and standard deviation of the final marks for the examination. --[3]
[S-12/71/Q2]

Solution: $E(M) = 54.8$, $Var(M) = 16^2$ and $E(P) = 82.4$, $Var(P) = 4.8^2$ $\{Var = \sigma^2\}$

$X \sim$ (75% of M & 25% of P)

$E(X) = (0.75\times 54.8 + 0.25\times 82.4) = \underline{61.7}$ ✓

$Var(X) = 0.75^2\times 16^2 + 0.25^2\times 4.8^2 = 145.44 \Rightarrow S.D = \sqrt{145.44} = \underline{12.1}$ (3sf)

3. A fair six sided die is thrown 20 times and the number of sixes, X is recorded. Another fair six-sided die is thrown 20 times and the number of odd-numbered scores, Y, is recorded. Find the mean and standard deviation of X+Y. [M-16/72/Q1] --[5]

$S = \{1, 2, 3, 4, 5, 6\}$

Solution: $P(Six) = \frac{1}{6}$, $n = 20$, using Binomial distribution, $E(X) = np = 20\times\frac{1}{6} = \frac{10}{3}$ ✓

Variance $(X) = npq = 20\times\frac{1}{6}\times\frac{5}{6} = \frac{25}{9}$ ✓

$P(odd) = \frac{3}{6} = \frac{1}{2}$; $E(Y) = np = 20\times\frac{1}{2} = 10$ ✓

$Var(Y) = npq = 20\times\frac{1}{2}\times\frac{1}{2} = 5$ ✓

$\therefore E(X+Y) = E(X) + E(Y) = \frac{10}{3} + 10 = 40/3 = 13.3$ (3sf) ✓

and $Var(X+Y) = Var(X) + Var(Y) = \frac{25}{9} + 5 = \frac{70}{9}$

$\therefore$ Standard devition of $(X+Y) = \sqrt{\frac{70}{9}} = \underline{\frac{\sqrt{70}}{3}}$ (or 2.79 (3s.f))

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4. The mean and variance of the random variable X are 5.8 and 3.1 respectively. The random variable S is the sum of three independent values of X. The independent random variable T is defined by: $T = 3X+2$.

(i) Find the variance of S. (ii) Find the variance of T. [1]+[1]

(iii) Find the mean and variance of S−T. [S-13/73/Q7] --[3]

Solution (i) $S = X_1 + X_2 + X_3$ three independent values of X.

$\therefore Var(S) = P(X_1+X_2+X_3) = 3\times 3.1 = \underline{9.3}$ ✓ ($\because Var X = 3.1$) --(i)

(ii) $T = (3X+2) \Rightarrow Var(T) = 3^2 \times 3.1 = \underline{27.9}$ ✓ ------(ii)

(iii) $E(S-T) = E(S) - E(T)$

$= 3\times 5.8 - (3\times 5.8 + 2) = \underline{-2}$ ✓

and $Var(S-T) = Var S + Var T$

$= \underline{9.3 + 27.9}$ ✓ from (i) and (ii)

5. The independent variables X and Y are such that:

$X \sim B(10, 0.8)$ and $Y \sim Po(3)$ Find:

(i) $E(7X+5Y-2)$, (ii) $Var(4X-3Y+3)$, (iii) $P(2X-Y=18)$ [2]+[4]+[4]

[S-15/71/Q7]

Solution: $n = 10, p = 0.8, q = 0.2$; $Y: \lambda = 3 \Rightarrow E(Y) = 3, Var(Y) = 3$

$E(X) = np = 10\times 0.8 = 8$, $Var(X) = npq = 10\times 0.8\times 0.2 = 1.6$

(i) $E(7X+5Y-2) = 7E(x) + 5.E(Y) - 2 = 7\times 8 + 5\times 3 - 2 = \underline{69}$ ✓

(ii) $Var(4X-3Y+3) = 4^2 . Var(X) + 3^2 . Var(Y) = 16\times 1.6 + 9\times 3 = \underline{52.6}$ ✓

(iii) $2X - Y = 18 \Rightarrow X = 10, Y = 2$ or $X = 9, Y = 0$

$P(2X-Y=18) = P(10)\times P(2) + P(9)\times P(0)$

$= {}^{10}C_{10} \cdot (0.8)^{10} \times \left(e^{-3}\times \frac{3^2}{2!}\right) + {}^{10}C_9\, 0.8^9 \times 0.2 \times (e^{-3})$

$= \underline{0.03745}$ ✓

$B(n,p)$

$P(X=r) = {}^nC_r\, p^r q^{n-r}$

$P(Y) = e^{-\lambda} \cdot \frac{\lambda^r}{r!} : P_o(\lambda)$

$\lambda = 3$

6 The lifetimes, in hours, of Longlive bulbs and Enerlow light bulbs have the independent distributions $N(1020, 45^2)$ and $N(2800, 52^2)$ respectively.

(a) Find the probability that the total of the lifetimes of five randomly chosen Longlive bulbs is less than 5200 hours. --[4]

(b) Find the probability that the lifetime of a randomly chosen Enerlow bulb is at least three times that of a randomly chosen Longlive bulb. --[6]

[SP-20/06/Q4]

Solution: Given Longlive bulb $X \sim N(1020, 45^2)$

(a) for five randomly chosen Longlive bulb, $N(1020\times5,\ 45^2\times5)$

or $X \sim N(5100, 10125)$ $\{N(\mu, \sigma^2)$

$$P(X < 5200) = P\left(Z < \frac{5200 - 5100}{\sqrt{10125}}\right)$$

$$= P(Z < 0.994)$$

$$= \phi(0.994) = \underline{0.840} \ (3sf)$$

(b) $E: N(2800, 2704)$ and $L: N(1020, 2025)$

for $E - 3L$; $E(E-3L) = E(L) - 3E(L) = 2800 - 3\times1020$

$\therefore E(E-3L) = -260$

and $Var(E-3L) = Var(E) + 3^2\ Var(L) = 2704 + 9\times2025$

$\therefore Var(E-3L) = 20929$

$$P(E-3L > 0) = P\left(z > \frac{0-(-260)}{\sqrt{20929}}\right)$$

$$= P(z > 1.797)$$

$$= 1 - \phi(1.797)$$

$$= 1 - 0.9639$$

$$= \underline{0.0361}$$

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7. The volume, in litres, of juice in large and small bottles have the distributions $N(5.10, 0.0102)$ and $N(2.51, 0.0036)$ respectively.

(a) Find the probability that the total volume of juice in 3 randomly chosen large bottles and 4 randomly chosen small bottles is less than 25.5 litres. ---[5]

(b) Find the probability that the volume of juice in a randomly chosen large bottle is at least twice the volume of juice in a randomly chosen small bottle. ---[5]

[M-21/62/Q5]

Solution: large bottle $L \sim N(5.10, 0.0102)$ and small bottle $S \sim N(2.51, 0.0036)$

(a) 3 large and 4 small randomly chosen bottles.

$E(L_1+L_2+L_3+S_1+S_2+S_3+S_4) = 3\times5.1 + 4\times2.51 = 25.34$ ✓

$Var(L_1+L_2+L_3+S_1+S_2+S_3+S_4) = 3\times0.0102 + 4\times0.0036 = 0.045$ ✓

$$P(X < 25.5) = P\left(z < \frac{25.5-25.34}{\sqrt{0.045}}\right)$$

$$= P(z < 0.754) = \phi(0.754) = \underline{0.775} \text{ (3sf)}$$ ✓

(b) Now $E(L-2S) = 5.10 - 2\times2.51 = 0.08$ ✓

$Var(L-2S) = 0.0102 + 2^2\times0.0036 = 0.0246$ ✓

$$P((L-2S) > 0) = P\left(z > \frac{0-0.08}{\sqrt{0.0246}}\right)$$

$$= P(z > -0.510) = P(z < 0.510)$$

$$= \phi(0.510)$$

$$= \underline{0.695} \text{ (3 sf)}$$ ✓

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8. The masses, in kilograms, of large sacks of flour and small sack of flour have the independent distribution $N(40, 1.5^2)$ and $N(12, 0.7^2)$ respectively.

(a) Find the probability that the total mass of 6 randomly chosen large sacks of flour is more than 245 kg. ---[4]

(b) Find the probability that the mass of a randomly chosen large sack of flour is less the 4 times the mass of a randomly chosen small sack of flour. ---[6]

[S-20/61/Q3]

Solution: 6 randomly chosen large bottles $L \sim N(6\times40, 6\times1.5^2) = N(240, 13.5)$

(a) $P(L > 245) = P\left(Z > \frac{245-240}{\sqrt{13.5}}\right) = P(Z > 1.361)$

$= 1 - \phi(1.361)$

$= 1 - 0.9133 = \underline{0.0867}$ ✓ (3 sf)

(b) $E(L-4S) = 40 - 4\times12 = -8$ ✓

$Var(L-4S) = 1.5^2 + 4^2 \times 0.7^2 = 10.09$

$P(Z < (L-4S) = P\left(Z < \frac{0-(-8)}{\sqrt{10.09}}\right) = P(Z < 2.519)$

$= \phi(2.519) = \underline{0.994}$ ✓ (3sf)

9. The masses, in kilograms, of female and male animals of certain species have the distributions $N(102, 27^2)$ and $N(170, 55^2)$ respectively. ---[6]
Find the probability that a randomly chosen female has a mass that is less than half the mass of a randomly chosen male. [W-20/61/Q3]

Solution: $E(F - \frac{1}{2}M) = 102 - \frac{1}{2}\times170 = 17$ ✓; $Var(F - \frac{1}{2}M) = 27^2 + (\frac{1}{2})^2 \times 55^2 = 1485.25$ ✓

$P((F - \frac{1}{2}M) < 0) = P\left(Z < \frac{0-17}{\sqrt{1485.25}}\right) = P(Z < -0.4411)$

$= 1 - \phi(0.4411)$

$= 1 - 0.670 = \underline{0.330}$ ✓ (3sf)

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10. Before a certain type of book is published it is checked for errors, which are then corrected. For costing purposes each error is classified as either minor or major. The minor and major errors in a book are modelled by the independent distribution $N(380, 140)$ and $N(210, 80)$ respectively. You should assume that no continuity corrections are needed.
A book of this type is chosen at random.
(a) Find the probability that the number of minor errors is at least 200 more than the number of major errors. ---[5]
The cost of correcting a minor error and a major error are 20 cents and 50 cents respectively.
(b) Find the probability that the total cost of correcting the errors in the book is less than \$190. ---[5]
[W-20/62/Q7]

Solution: Minor error $M_1 \sim N(380, 140)$ and major error $M_2 \sim N(210, 80)$.
(a) $E(M_1 - M_2) = 380 - 210 = 170$ and $Var(M_1 - M_2) = 140 + 80 = 220$

$$P\left((M_1 - M_2) \geq 200\right) = P\left(z \geq \frac{200-170}{\sqrt{220}}\right) = P(z \geq 2.023)$$

$$= 1 - \phi(2.023) = 1 - 0.9784 = \underline{0.0216} \text{ (3sf)}$$

(b) $E(\text{total cost}) = 380 \times 20 + 210 \times 50 = 18100$ cent
$Var(\text{total cost}) = 140 \times 20^2 + 80 \times 50^2 = 256000$ cents
{\$190 = 19000 cents}

$$P(\text{Total cost} < 19000) = P\left(z < \frac{19000 - 18100}{\sqrt{256000}}\right)$$

$$= P(z < 1.778)$$

$$= \phi(1.778)$$

$$= \underline{0.962} \text{ (3 s.f)}$$

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11. Accidents at two factories occur randomly and independenly. On average the number of accident per month are 3·1 at factory A and 1·7 at factory B. Find the probability that the total number of accidents in two factories during a 2-month period is more than 3. --[4]
[S-21|61|Q1]

Solution: $Po(3\cdot1)$ and $Po(1\cdot7) \Rightarrow$ for Total in 2 month $\rightarrow \lambda = 2(3\cdot1+1\cdot7) = 9\cdot6$

$P(X>3) = 1-P(X=0,1,2,3) = 1-e^{-9.6}\left(1+9.6+\frac{9.6^2}{2!}+\frac{9.6^3}{3!}\right) = \underline{0\cdot986}$ (3sf)

12. The masses, in kilograms, of large and small sacks of flour have the distributions $N(55, 3^2)$ and $N(27, 2.5^2)$ respectively.

(a) Some sacks are loaded on a boat. The maximum load of flour that the boat can carry safly is 340 kg. Find the probability that the boad can carry 3 randomly chosen large sacks of flour and 6 randomly chosen small sacks of flour. --[5]

(b) Find the probability that the mass of a randomly chosen large sack of flour is greater than the total mass of two randomly chosen small sacks of flour. [S-21|61|Q7] --[5]

Solution: $E(\text{Total mass}) = 3\times55+6\times27 = 327$ and $Var(\text{Total mass}) = 3\times3^2+6\times2.5^2 = 64.5$

$$P(\text{Total mass}<340) = P\left(Z<\frac{340-327}{\sqrt{64.5}}\right) = P(Z<1\cdot619)$$

$$= \Phi(1\cdot619) = \underline{0\cdot947} \quad (3sf)$$

(b) $E(L-S_1-S_2) = 55-2\times27 = 1$

$Var(L-S_1-S_2) = 3^2+2\times2.5^2 = 21\cdot5$

$$P((L-S_1-S_2)>0) = P\left(Z>\frac{0-1}{\sqrt{21\cdot5}}\right)$$

$$= P(Z>-0\cdot216)$$

$$= P(Z<0\cdot216)$$

$$= \Phi(0\cdot216)$$

$$= \underline{0\cdot586} \quad (3s.f)$$

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13 The random variable X has the distribution $B(400, 0.01)$

(a) Find $Var(4X+2)$ ---[3]

(b) (i) State an appropriate approximating distribution for X, giving the values of parameters. Justify your choice of approximating distribution. --[2]

(ii) Use your approximating distribution to find $P(2 \leq X \leq 5)$ --[2]

[S-21/62/Q2]

Solution (a) $B(400, 0.01) \rightarrow n = 400, p = 0.01, q = 0.99 \Rightarrow Var(X) = npq = 400 \times 0.01 \times 0.99 = 3.96$ ✓

hence $Var(4x+2) = 4^2 \times Var X = 16 \times 3.96 = \underline{63.36}$ ✓

b(i) $np = 400 \times 0.01 = 4 < 5$ and $n = 400 > 50$

$\therefore$ Poisson distribution $Po(4)$: $\lambda = 4$ $\left\{ P(X=\lambda) = e^{-\lambda} \times \frac{\lambda^\lambda}{\lambda!} \right.$

(ii) $P(2 \leq X \leq 5) = e^{-4}\left[\frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}\right] = 0.694$ ✓ (3 s.f)

14. Wendy's journey to work consists of three parts; walking to train station, riding on the train and then walking to the office. The times, in minutes, for the three parts of her journey are indendent and have distributions $N(15.0, 1.1^2)$, $N(32.0, 3.5^2)$ and $N(8.6, 1.2^2)$ respectively.

(a) Find the mean and variance of the total time for Wendy's journey. ---[2]

If Wendy's journey takes more than 60 minutes, she is late for work.

(b) Find the probability that, on a randomly chosen day, Wendy will be late for work. --[3]

(c) Find the probability that the mean of Wendy's journey time over 15 randomly chosen days will be less than 54.5 minute. --[3]

[S-21/62/Q4]

Solution: (a) Mean. $E(\text{Total time}) = 15 + 32 + 8.6 = \underline{55.6}$ ✓

$Var(\text{Total time}) = 1.1^2 + 3.5^2 + 1.2^2 = \underline{14.9}$ ✓

(b) $P(T > 60) = P\left(z > \frac{60 - 55.6}{\sqrt{14.9}}\right)$

$= P(z > 1.140)$

$= 1 - \phi(1.140)$

$= 1 - 0.873$

$= \underline{0.127}$ ✓ (3sf)

(c) for 15 randomly chosen day:

$P(\text{Time} < 54.5) = P\left(z < \frac{54.5 - 55.6}{\sqrt{\frac{14.9}{15}}}\right)$

$= P(z < -1.104)$

$= 1 - \phi(1.104)$

$= 1 - 0.865$

$= \underline{0.135}$ ✓ (3sf)

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15. The number of goals scored by a team in a match is independent of other matches, and is denoted by the random variable X, which has a Poisson distribution with mean 1.36. A supporter offers to make a donation of \$5 to the team for each goal that they score in the next 10 matches. Find the expectation and standard deviation of the amount that the supporter will pay. [S-21/63/Q1] --[5]

Solution: $Po(1.36)$ for one match $\Rightarrow$ for 10 matches $\lambda = 10 \times 1.36 = 13.6$

$E(\text{amount}) = \$5 \times 13.6 = \$68$

$Var(\text{amount}) = 5^2 \times 13.6 = \$340 \Rightarrow$ Standard deviation $= \sqrt{340} = \$18.44$ (3sf)

16. The heights of a certain variety of plant are normally distributed with mean 110 cm and variance 1050 cm². Two plants of this variety are chosen at random. Find the probability that the height of one of these plants is atleast 1.5 times higher the height of the other. ---[7]
[S-19/72/Q4]

Solution: Given $X \sim N(110, 1050) \rightarrow N(\mu, \sigma^2)$

Now for $(1.5X_1 - X_2) > 0$ or

$E(1.5X_1 - X_2) = 1.5 \times 110 - 110 = 55$

$Var(1.5X_1 - X_2) = 1.5^2 \times 1050 + 1050 = 3412.5$

$$P((1.5X_1 - X_2) > 0) = P\left(z > \frac{0-55}{\sqrt{3412.5}}\right)$$

$$= P(z > -0.942)$$

$$= 1 - P(z < 0.942)$$

$$= 1 - \phi(0.942)$$

$$= 1 - 0.827$$

$$= 0.173$$

or May be otherwise $\left\{ (1.5X_2 - X_1) > 0 \right\}$ $\therefore$ Req prob $= 0.173 \times 2$

$= 0.346$

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17. The volumes, in millilitres, of large and small cups of tea are modelled by the distributions $N(200, 30)$ and $N(110, 20)$ respectively.

(a) Find the probability that the total volume of a randomly chosen large cup of tea and a randomly chosen small cup of tea is less than 300 ml. ---[4]

(b) Find the probability that the volume of a randomly chosen large cup of tea is more than twice the volume of a randomly chosen small cup of tea. ---[6]

[M-20/62/Q6]

Solution (a) Large cup $L \sim N(200, 30)$, Small cup $S \sim N(110, 20)$

$E(L+S) = 200 + 110 = 310$; $Var(L+S) = 30 + 20 = 50$

$$P((L+S) < 300) = P\left(z < \frac{300-310}{\sqrt{50}}\right)$$
$$= P(z < -1.414)$$
$$= \phi(-1.414)$$
$$= 1 - \phi(1.414) = 1 - 0.9214$$
$$= \underline{0.0786}$$

(b) Now for $P(L-2S) > 0$, $E(L-2X) = 200 - 2\times110 = -20$

$Var(L-2X) = 30 + 2^2\times20 = 110$

Now $N(-20, 110)$

$$P((L-2S) > 0) = P\left(z > \frac{0-(-20)}{\sqrt{110}}\right)$$
$$= P(z > 1.907)$$
$$= 1 - \phi(1.907)$$
$$= 1 - 0.9717$$
$$= \underline{0.0283} \text{ (3sf)}$$

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18. Each day at the gym, Sarah completes three runs. The distances, in metres, that she completes in the three runs have independent distributions $W \sim N(1520, 450)$, $X \sim (2250, 720)$ and $Y \sim N(3860, 1050)$.
Find the probability that, on a particular day, Y is less than the total of W and X. --[5]

[S-20|63| Q2]

Solution: $E(Y-(W+X)) = 3860-(1520+2250) = 90$

$Var(Y-W-X) = 1050+450+720 = 2220$

$P(Y-(W+X)<0) = P\left(Z < \frac{0-90}{\sqrt{2220}}\right) = P(Z<-1.910)$

$= \phi(-1.910) = 1-\phi(1.910) = \underline{0.0281}$

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19. The random variable A has the distribution $Po(1.5)$. A_1 and A_2 are independent values of A.

(a) Find $P(A_1+A_2<2)$ ---[3]

(b) Given that $A_1+A_2<2$, find $P(A_1=1)$ --[4]

(c) Give reason why A_1-A_2 cannot be a Poisson distribution --[1]

[S-20|63|Q4]

Solution: $A \sim Po(1.5) \Rightarrow A_1+A_2 \sim Po(2\times1.5) = P(3)$, $\lambda=3$

(a) $P(A_1+A_2<2) = P(A_1+A_2=0 \text{ or } 1) = e^{-3}(1+3) = \underline{0.199}$

(b) $P(A=1 \text{ and } A_1+A_2<2) = P(A_1=1)\times P(A_2=0)$

$= (e^{-1.5}\times1.5)\times(e^{-1.5}) = 0.0747$

$\therefore P(A_1=1 / A_1+A_2<2) = \frac{P(A_1=1 \text{ and } A_1+A_2<0)}{P(A_1+A_2<2)}$

$= \frac{(1.5\times e^{-1.5})\times e^{-1.5}}{0.199} = \frac{0.0747}{0.199} =$

$= \underline{0.375}$ (3 sf)

(c) Takes negative values.

$(E(A_1-A_2) = 1.5-1.5=0$ but $\neq$ $Var(A_1-A_2) = 1.5+1.5=3)$

20: At an internet café, the charge for using a computer is 5 cents per minute. The number of minutes for which people use a computer has mean 23 and standard deviation 8.

(i) Find, in cents, the mean and standard deviation of the amount people pay when using a computer. --[2]

(ii) Each day, 15 people use computer independently. Find, in cents, the mean and standard deviation of the total amount paid by 15 people. --[3]

[S-19/71/Q1]

Solution: (i) Mean $= 23\times5 = \underline{115}$ and $SD = 8\times5 = \underline{40}$.

(ii) Mean for 15 people $= 15\times115 = 1725$ and $Var = 15\times40^2 = 24000$

$SD = \sqrt{24000} = \underline{155}$ (cents) (3 sf)

21. A factory supplies boxes of children's bricks. Each box contains 10 randomly chosen large bricks and 20 randomly chosen small bricks. The mass, in grams, of large and small bricks have the distributions $N(60, 1.2)$ and $N(30, 0.7)$ respectively. The mass of an empty box is 8g. Find the probability that the total weight of a box and its contents is less than 1200g. --[5]

[S-19/73/Q4]

Solution: Total $E(X) = 10\times60 + 20\times30 + 8 = 1208$

$Var(total) = 10\times1.2 + 20\times0.7 + 0 = 26$

$$P(\text{Total} < 1200) = P\left(z < \frac{1200-1208}{\sqrt{26}}\right) = P(z < -1.569)$$

$$= 1 - \phi(1.569)$$

$$= 1 - 0.9417$$

$$= 0.0583$$

22. The masses, in grams, of large boxes of chocolates and small boxes of chocholates have the distributions $N(325, 6.1)$ and $N(167, 5.6)$ respectively.

(i) Find the prob. that the total mass of 10 randomly chosen large boxes of chocholates is less than 3240g. --[4]

(ii) Find the prob. that the mass of a randomly chosen large box of chocolates is more than twice the mass of a randomly chosen small box of chocolates. ---[5]

W-19/72/Q5

Solution (i) $E(10\text{ Large}) = 10\times 325 = 3250g$, and $Var = 10\times 6.1 = 61$

$P(10\text{Large} < 3240) = P\left(z < \frac{3240-3250}{\sqrt{61}}\right) = P(z < -1.280)$

$= 1 - \phi(1.280) = 1 - 0.900 = 0.100$ ✓

(ii) $E(L-2S) = 325 - 2\times 167 = -9$, $V(L-2S) = 6.1 + 2^2 \times 5.6 = 28.5$

$P((L-2S) > 0) = P\left(z > \frac{0-(-9)}{\sqrt{28.5}}\right) = P(z > 1.689)$

$= 1 - \phi(1.689) = 1 - 0.9541 = 0.0459$ ✓

23. A random variable X has mean 2.4 and variance 3.1

(i) The random variable Y is the sum of four independent values of X. Find the mean and variance of Y. --[2]

(ii) The random variable Z is defined by $Z = 4X - 3$. Find the mean and variance of Z. --[2]

W-19/73/Q1

Solution (i) $Y = X_1 + X_2 + X_3 + X_4 \Rightarrow E(X) = 4\times 2.4 = 9.6$ ✓

$Var\,Y = Var\,X_1 + Var\,X_2 + Var\,X_3 + Var\,X_4 = 4\times 3.1 = 12.4$ ✓

(ii) $Z = 4X - 3 \Rightarrow E(Z) = 4\times 2.4 - 3 = 6.6$ ✓

$Var(Z) = 4^2 \times 3.1 + 0 = 49.6$ ✓

24. The independent random variables X and Y have the distributions $N(9.2, 12.1)$ and $N(3.0, 8.6)$ respectively. Find $P(X>3Y)$. --[5]

[M-19/72/Q2]

Solution: $E(X-3Y) = 9.2 - \ \times 3.0 = 0.2$ ✓ and $Var(X-3Y) = 12.1 + 3^2 \times 8.6 = 89.5$ ✓

$$P((X-3Y)>0) = P\left(Z > \frac{0-0.2}{\sqrt{89.5}}\right) = P(Z > -0.021) = P(Z < 0.021)$$

$$= \phi(0.021) = \underline{0.508} ✓ \quad (3\text{sf})$$

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25. The heights of plants of type A have mean 1.2 m and standard deviation 0.03 m. A random sample of 5 plants of type A is selected. The sum of the heights of these 5 plants is denoted by H_A m.

(i) Find the mean and variance of H_A. --[3]

The heights of plants of type B have mean 0.6 m and standard deviation 0.02 m. A random sample of 5 plants of type B is selected. The sum of heights of these 5 plants is denoted by H_B m.

(ii) Find the mean and variance of $H_A - 2H_B$. --[4].

[M-18/72/Q3]

Solution (i) $E(HA) = 5 \times 1.2 = \underline{6}$ ✓ and $Var(HA) = 5 \times 0.03^2 = \underline{0.0045}$ ✓

(ii) $E(HA - 2HB) = 5 \times 1.2 - 2\cdot(5 \times 0.6) = \underline{0}$ ✓

$Var(HA - 2HB) = Var(HA) + 2^2 \cdot Var(HB)$

$= 0.0045 + 4\cdot(5 \times 0.02^2) = \underline{0.0125}$ ✓ (3 sf)

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26. The random variable X has the distribution $N(3, 1.2)$. The random variable A is defined by $A = 2X$. The random variable B is defined by $X = X_1 + X_2$, where X_1 and X_2 are independent random values of X. Describe fully the distribution of A and the distribution of B. --[3]

[S-18/72/Q2]

Solution: A: $A = 2X \Rightarrow E(X) = 2 \times 3 = 6$ ✓ and $Var A = 2^2 \times 1.2 = 4.8 \Rightarrow \underline{A \sim N(6, 4.8)}$ ✓

B: $X = X_1 + X_2 \Rightarrow E(B) = 3 + 3 = 6$ ✓ and $Var B = 1.2 + 1.2 = 2.4$

$\therefore \underline{B \sim N(6, 2.4)}$

27. The volume, in millilitres, of a small cup of coffee has the distribution $N(103.4, 10.2)$. The volume of a large cup of coffee is 1.5 times the volume of a small cup of coffee.

(i) Find the mean and standard deviation of the volume of a large cup of coffee. --- [3]

(ii) Find the probability that the total volume of a randomly chosen small cup of coffee and a randomly chosen large cup of coffee is greater than 250 ml.

$\boxed{S\text{-}18 \,/\, 71 \,/\, Q4}$ -- [4]

Solution (i) For Large cup. $E(L) = 1.5 \times 103.4 = \underline{155.1}$ ✓

$Var = 1.5^2 \times 10.2 = 22.95 \Rightarrow S.D = \sqrt{22.95} = \underline{4.79}$ ✓

(ii) $E(S+L) = 103.4 + 155.1 = 258.5$ ✓

$Var(S+L) = Var(S) + Var(L) = 10.2 + 22.95 = 33.15$ ✓

$$P((S+L) > 250) = P\left(z > \frac{250 - 258.5}{\sqrt{33.15}}\right) = P(z > -1.476)$$

$$= P(z < 1.476) = \phi(1.476) = \underline{0.930} \checkmark \quad (3sf)$$

28. The times, in minutes, taken to complete the two parts of a task are normally distributed with means 4.5 and 4.3 respectively and standard deviations 1.1 and 0.7 respectively.

(i) Find the probability that the total time taken for the task is less than 8.5 minutes. --- [4]

(ii) Find the probability that the time taken for the first part of the task is more than twice the time taken for the second part. -- [5]

$\boxed{S\text{-}18 \,/\, 73 \,/\, Q6}$

Solution (i) $E(T) = 4.5 + 2.3 = 6.8$

$Var(T) = 1.1^2 + 0.7^2 = 1.7$

$$P(T < 8.5) = P\left(z < \frac{8.5 - 6.8}{\sqrt{1.7}}\right)$$

$= P(z < 1.304)$

$= \phi(1.304)$

$= \underline{0.904}$ ✓ (3sf)

(ii) $E(D) = 4.5 - 2 \times 2.3 = -0.1$ ✓ $\{ D = (T_1 - 2T_2) \}$

$Var(D) = 1.1^2 + 2^2 \times 0.7^2 = 3.17$

$$P(D > 0) = P\left(z > \frac{0 - (-0.1)}{\sqrt{3.17}}\right)$$

$= P(z > 0.056)$

$= 1 - \phi(0.056)$

$= 1 - 0.522$

$= \underline{0.478}$ ✓ (3sf)

29. The times, in months, taken by a builder to build two types of house, P and Q, are represented by the independent variables $T_1 \sim N(2.2, 0.4^2)$ and $T_2 \sim N(2.8, 0.5^2)$ respectively.

(i) Find the probability that the total time taken to build one house of each type is less than 6 months. ---[4]

(ii) Find the probability that the time taken to build a type Q house is more that 1.2 times the time to taken to build a type P house. ---[5]

[W-18/71/Q5]

Solution (i) $E(T_1+T_2) = 2.2+2.8 = 5$ and $Var(T_1+T_2) = 0.4^2+0.5^2 = 0.41$

$$P((T_1+T_2) < 6) = P\left(Z < \frac{6-5}{\sqrt{0.41}}\right) = P(Z < 1.562) = \phi(1.562) = \underline{0.941} \checkmark$$

(ii) $E(T_2 - 1.2T_1) = 2.8 - 1.2 \times 2.2 = 0.16 \checkmark$ and $Var(T_2 - 1.2T_1) = 0.5^2 + 1.2^2 \times 0.4^2 = 0.4804 \checkmark$

$$P((T_2 - 1.2T_1) > 0) = P\left(Z > \frac{0-0.16}{\sqrt{0.4804}}\right)$$

$$= P(Z > -0.231) = P(Z < 0.231) = \phi(0.231) = \underline{0.591} \checkmark$$

30. Sugar and flour for making cakes are measured in cups. The mass, in grams, of one cup of sugar has the distribution $N(250, 10)$. The mass, in grams, of one cup of flour has the independent distribution $N(160, 9)$. Each cake contains 2 cups of sugar and 5 cups of flour. Find the probability that the total mass of sugar and flour in one cake exceeds 1310 grams. ---[5]

[W-18/72/Q3]

Solution: $E(\text{Total}) = 2 \times 250 + 5 \times 160 = 1300.$

$Var(T) = 2 \times 10 + 5 \times 9 = 65$

$$P(T > 1310) = P\left(Z > \frac{1310-1300}{\sqrt{65}}\right) = P(Z > 1.240)$$

$$= 1 - \phi(1.240)$$

$$= 1 - 0.8925$$

$$= \underline{0.1075} \checkmark$$