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S-2

Probability and Statistics-2

Linear combination of random variables.

Notes

Contents:

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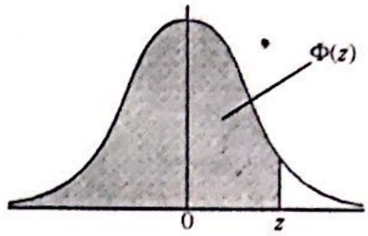
THE NORMAL DISTRIBUTION FUNCTION

If Z has a normal distribution with mean 0 and variance 1 then, for each value of z, the table gives the value of $\Phi(z)$, where

$$\Phi(z) = P(Z \le z).$$

For negative values of z use $\Phi(-z) = 1 - \Phi(z)$.



z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
															ADD				
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	4	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0

Critical values for the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that

$$P(Z \le z) = p.$$

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Linear combination of random variables.

§ Expectation and variance:

§ 1(a) For a random variable X and constant b:

(i) $E(X+b) = E(X)+b$

(ii) $Var(X+b) = Var(X)$

(b) For a random variable X and a constant a:

(i) $E(aX) = a \cdot E(X)$

(ii) $Var(aX) = a^2 \cdot Var(X)$

(c) For a random variable X and constants a and b:

(i) $E(aX+b) = a \cdot E(X)+b$

(ii) $Var(aX+b) = a^2 \cdot Var(X)$

Example 1: The random variable X has the distribution $B(400, 0.01)$

Find (i) $E(4x+2)$ (ii) $Var(4X+2)$ ---[3]

[S-21/62/Q2]

Solution: $B(400, 0.01) \Rightarrow n=400,\ p=0.01,\ q=0.99 \Rightarrow E(x) = np = 400 \times 0.01 = 4$

[Binomial distribution $B(n,p)$] and $Var X = npq = 400 \times 0.01 \times 0.99 = 3.96$

(i) $E(4X+2) = 4E(X)+2 = 4\times4+2 = \underline{16}$ ✓

(ii) $Var(4X+2) = 4^2 \cdot Var(X) = 16\times3.96 = \underline{63.36}$ ✓

Example 2: The random variable X has expectation 2.4, and Variance 0.8.
Find two pairs of values for the constants a and b such that:
$E(aX+b) = 32$ and $Var(aX+b) = 20$.

Solution: Given $E(X) = 2.4$ and $Var(X) = 0.8$

Now $E(aX+b) = 32 \Rightarrow aE(X)+b) = 32 \Rightarrow 2.4a+b = 32$ --- ①

and $Var(ax+b) = 20 \Rightarrow a^2 \cdot Var(X) = 20 \Rightarrow 0.8a^2 = 20 \Rightarrow a^2 = \frac{20}{0.8} = 25$

$\Rightarrow a = \pm 5$ -- ②

for $a=5$ in ① $2.4\times5+b = 32 \Rightarrow b = 20$ ✓

And for $a=-5$ in ① $2.4\cdot(-5)+b = 32 \Rightarrow b = 44$ ✓

$\therefore$ Required values of a and b are:

$\underline{a=5, b=20}$ ✓ & $\underline{a=-5, b=44}$ ✓

§ 2. Expectation and Variance of sum and difference of independent random variables:

(a) For two independent variables X and Y.

(i) $E(X+Y) = E(X) + E(Y)$

(ii) $Var(X+Y) = Var(X) + Var(Y)$.

(b) For two independent random variables, X and Y, and constants a and b.

(i) $E(aX+bY) = a \cdot E(X) + b \cdot E(Y)$

(ii) $Var(aX+bY) = a^2 Var(X) + b^2 Var(Y)$.

(iii) $E(aX-bY) = a \cdot E(X) - b E(Y)$

(iv) $Var(aX-bY) = a^2 Var(X) + b^2 Var(Y)$.

Example 3: Two independent random variable X and Y have standard deviations 3 and 6 respectively. Calculate the standard deviation of $(4X-5Y)$. ---[3]

S-15/72/Q1

Solution: $Var\,X = \sigma_x^2 = 3^2 = 9$ and $Var\,Y = \sigma_y^2 = 6^2 = 36$

$\therefore Var(4X-5Y) = 4^2 \cdot Var(X) + 5^2 \cdot Var\,Y)$

$= 16 \times 9 + 25 \times 36 = 1044$

$\therefore$ standard deviation of $(4X-5Y) = \sqrt{1044} = 32.3$ ✓

Example 4: An examination consists of a written paper and a practical test. The written paper marks (M) have mean 54.8 and standard deviation 16.0. The practical test marks (P) are independent of the written paper marks and have mean 82.4 and standard deviation 4.8. The final marks is found by adding 75% of M and 25% of P. Find the mean and standard deviation of the final marks for the examination. --[3]

S-12/71/Q2

Solution: $E(M) = 54.8$; $Var(M) = 16^2$ and $E(P) = 82.4$; $Var(P) = 4.8^2$ [$Var = \sigma^2$]

$X \sim (75\%$ of M & 25% of $P)$

$E(X) = (0.75 \times 54.8 + 0.25 \times 82.4) = 61.7$ ✓

$Var(X) = 0.75^2 \times 16^2 + 0.25^2 \times 4.8^2 = 145.44 \Rightarrow S.D = \sqrt{145.44} = 12.1$ ✓ (3 s.f)

Example 5: A fair six-sided die is thrown 20 times and the number of sixes, X is recorded. Another fair six-sided die is thrown 20 times and the number of odd-numbered scores, Y, is recorded. Find the mean and standard deviation of $X+Y$. ---[5]

[M-16/72/Q1]

Solution: $P(\text{six}) = \frac{1}{6}$; $n = 20$; Using Binomial distribution, $E(X) = np = 20 \times \frac{1}{6} = \frac{10}{3}$ ✓

$Var(X) = npq = 20 \times \frac{1}{6} \times \frac{5}{6} = \frac{25}{9}$ ✓

$P(\text{odd}) = 3/6 = \frac{1}{2}$; $E(Y) = np = 20 \times \frac{1}{2} = 10$

$Var(Y) = npq = 20 \times \frac{1}{2} \times \frac{1}{2} = 5$

The variables X and Y are independent;

Hence $E(X+Y) = E(X) + E(Y) = \frac{10}{3} + 10 = \frac{40}{3} = 13.3$ ✓ (3 s.f)

and $Var(X+Y) = Var(X) + Var(Y) = \frac{25}{9} + 5 = \frac{70}{9}$

$\therefore$ Standard deviation $\sigma_{(X+Y)} = \sqrt{Var(X+Y)} = \sqrt{\frac{70}{9}} = 2.79$ ✓

(*) **Example 6:** The mean and variance of the random variable X are 5.8 and 3.1, respectively. The random variable S is the sum of three independent values of X. The independent variable T is defined by; $T = (3X+2)$

(i) Find the Variance of S ---[1]

(ii) Find the variance of T. ---[1]

(iii) Find the mean and variance of $S-T$. ---[3]

[S-13/73/Q1]

Solution: $E(X) = 5.8$ and $Var(X) = 3.1$

$S = X_1 + X_2 + X_3$ such that three are independent values of X,

(i) $\therefore Var\,S = Var(X_1 + X_2 + X_3) = Var(X_1) + Var(X_2) + Var(X_3) = 3 \times 3.1 = 9.3$ ✓ ---①

(ii) $T = (3X+2) \Rightarrow Var\,T = Var(3X+2) = 3^2 Var(X) = 9 \times 3.1 = 27.9$ ---②

(iii) $E(S-T) = E(S) - E(T)$

$= 3 \times 5.8 - (3 \times 5.8 + 2)$

$= -2$ ✓

$\left\{ \begin{array}{l} E(S) = E(X_1 + X_2 + X_3) = 3 \times 5.8 \\ E(T) = E(3X+2) = 3 \times 5.8 + 2 \end{array} \right.$

and $Var(S-T) = Var\,S + Var\,T$ (S & T are independent variables)

$= 9.3 + 27.9 = 37.2$ ✓ (from (i) & (ii))

(*) Note: $Var\,S = Var(X_1 + X_2 + X_3) \neq Var(3X)$

Three independent values of X

as $Var(3X) = 3^2 \times 3.1 = 27.9$
and
$Var(S) = 9.3$

Ex. 7. The independent variables X and Y are such that: $X \sim B(10, 0.8)$ and $Y \sim P_0(3)$, Find:
(i) $E(7X+5Y-2)$, (ii) $Var(4X-3Y+3)$; (iii) $P(2X-Y=18)$ [2]+[4]+[4]

[S-15/71/Q7]

Solution: $X \sim B(10, 0.8) \Rightarrow n=10, p=0.8, q=0.2, E(X)=np=10\times0.8=8$ ✓
and $Var(X)=npq=10\times0.8\times0.2=1.6$ ✓

and $Y \sim P_0(3)$ is Poisson distribution $\rightarrow \lambda=\sigma^2=3$ ✓
$E(Y)=3$ and $Var(Y)=3$ ✓ (X & Y are independent)

(i) $E(7X+5Y-2)=7E(X)+5E(Y)-2$
$=7\times8+5\times3-2=\underline{69}$ ✓

(ii) $Var(4X-3Y+3)=4^2 Var(X)+3^2 Var\,Y=16\times1.6+9\times3=\underline{52.6}$ ✓

(iii) $2X-Y=18 \Rightarrow X=10$ and $Y=2$ or $X=9$ & $Y=0$

$P(2X-Y=18) = P(X=10)\cdot P(Y=2)+P(X=9)\cdot P(Y=0)$

$= {}^{10}C_{10}(0.8)^{10}\times\frac{e^{-3}\times3^2}{2!}+{}^{10}C_9\,0.8^9\times0.2^1\times e^{-3}$

$=\underline{0.0374}$ ✓

$B(n,p)$
$\left\{ P(r)={}^{n}C_r\,p^r q^{n-r} \right.$
for Poisson.
$\left\{ P(r)=\frac{e^{-\lambda}\cdot\lambda^r}{r!} \right.$ ✓ $\lambda=3$

Example 8: Accidents at two factories occur randomly and independently. On average the number of accidents per month are 3.1 at factory A, and 1.7 at factor B. Find the probability that the total number of accidents in two factories during 2-month period is more than 3.

[S-21/61/Q1] --- [4]

Solution: At factor 'A' $P_0(3.1)$ and factor B. $P_0(1.7)$.
hence in two-months period for Total $\lambda=2(\lambda_1+\lambda_2)$
$\lambda=2(3.1+1.7)=9.6$

$P(X>3) = 1-P(r=0,1,2,3)$
$=1-e^{-9.6}\left(1+9.6+\frac{9.6^2}{2!}+\frac{9.6^3}{3!}\right)$

$\left\{ P(X=r)=\frac{e^{-\lambda}\cdot\lambda^r}{r!} \right.$

$=\underline{0.986}$ (3 s.f)

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3. § For Normal Distribution: $N(\mu, \sigma^2)$

(i) If a continuous random variable X has normal distribution, then $aX+b$, where a and b are constants, also has a normal distribution.

(ii) If continuous random variables X and Y have independent normal distribution, then $aX+bY$, where a and b are constants, has a normal distribution.

Example 9. The lifetime, in hours, of Longlive bulbs and Enerlow light bulbs have the independent distribution $N(1020, 45^2)$ and $N(2800, 52^2)$ respectively.

(a) Find the probability that the total of the life times of five randomly chosen Longlive bulbs is less than 5200 hours. ---[4]

(b) Find the probability that the life time of a randomly chosen Enerlow bulb is atleast three times that of a randomly chosen Longlive bulb. --[6]

SP-20/06/Q4

Solution: Given Longlive bulb, $N(1020, 45^2)$

(a) For five randomly chosen Longlive bulbs, $N(1020\times5, 45^2\times5)$

$$\Rightarrow X \sim (5100, 10125) \; \{N(\mu,\sigma^2)$$

$$P(X<5200) = P\left(Z < \frac{5200-5100}{\sqrt{10125}}\right) = P(Z<0.994) = \phi(0.994)$$
$$= \underline{0.840} \checkmark (3sf)$$

(b) $E: N(2800, 52^2)$ and $L: N(1020, 45^2)$

for $E-3L$: $E(E-3L) = E(E) - 3E(L) = 2800 - 3\times1020 = -260 \checkmark$

and $Var(E-3L) = VarE + 3^2 Var(L) = 52^2 + 9\cdot45^2 = 20929 \checkmark$

Hence $P(E-3L \geq 0) = P\left(Z > \frac{0-(-260)}{\sqrt{20929}}\right)$

$$= P(Z > 1.797)$$
$$= 1 - \phi(1.797)$$
$$= 1 - 0.9639$$
$$= \underline{0.0361} \checkmark$$

10. The volume, in litres, of juice in large and small bottles have the distributions $N(5.10, 0.0102)$ and $N(2.51, 0.0036)$ respectively.

(a) Find the probability that the total volume of juice in 3 randomly chosen large bottles and 4 randomly chosen small bottles is less than 25.5 litres. ---[5]

(b) Find the probability that the volume of juice in a randomly chosen large bottle is at least twice the volume of juice in a randomly chosen small bottle. --[5]

[M-21/62/Q5]

Solution: Large bottle $L \sim N(5.10, 0.0102)$ and small bottle $S \sim N(2.51, 0.0036)$.

(a) 3 Large and 4 small randomly chosen bottles.

$E(L_1+L_2+L_3+S_1+S_2+S_3+S_4) = 3\times5.1 + 4\times2.51 = 25.34$ ✓

$Var(L_1+L_2+L_3+S_1+S_2+S_3+S_4) = 3\times0.0102 + 4\times0.0036 = 0.045$ ✓

$$P(X < 25.5) = P\left(z < \frac{25.5-25.34}{\sqrt{0.045}}\right)$$

$$= P(z < 0.754) = \phi(0.754) = \underline{0.775}\ (3sf)$$ ✓

(b) Now $E(L-2S) = 5.10 - 2\times2.51 = 0.08$ ✓

$Var(L-2S) = 0.0102 + 2^2\times0.0036 = 0.0246$ ✓

$$P((L-2S) > 0) = P\left(z > \frac{0-0.08}{\sqrt{0.0246}}\right)$$

$$= P(z > -0.510) = P(z < 0.510)$$

$$= \phi(0.510)$$

$$= \underline{0.695}$$ ✓ (3 sf)

11 The masses, in kilograms, of large sacks of flour and small sack of flour have the independent distributions $N(40, 1.5^2)$ and $N(12, 0.7^2)$ respectively.

(a) Find the probability that the total mass of 6 randomly chosen large sacks of flour is more than 245 kg. ---[4]

(b) Find the probability that the mass of a randomly chosen large sack of flour is less than 4 times the mass of a randomly chosen small sack of flour. ---[6]

[S-20/61/Q3]

Solution: 6 randomly chosen large bottles $L \sim N(6\times 40, 6\times 1.5^2) \sim N(240, 13.5)$

(a) $P(L > 245) = P\left(z > \frac{245-240}{\sqrt{13.5}}\right) = P(z > 1.361)$

$= 1 - \phi(1.361)$

$= 1 - 0.9133 = \underline{0.0867}$ ✓ (3 sf)

(b) $E(L-4S) = 40 - 4\times 12 = -8$ ✓

$Var(L-4S) = 1.5^2 + 4^2 \times 0.7^2 = 10.09$

$L < 4S \Rightarrow L - 4S < 0$

$P(L-4S < 0) = P\left(z < \frac{0-(-8)}{\sqrt{10.09}}\right) = P(z < 2.519)$

$= \phi(2.519) = \underline{0.994}$ ✓ (3sf)

12 The masses, in kilograms, of female and male animals of certain species have the distributions $N(102, 27^2)$ and $N(170, 55^2)$ respectively. ---[6]

Find the probability that a randomly chosen female has a mass that is less than half the mass of a randomly chosen male. [W-20/61/Q3]

Solution: $E(F - \frac{1}{2}M) = 102 - \frac{1}{2}\times 170 = 17$ ✓; $Var(F - \frac{1}{2}M) = 27^2 + (\frac{1}{2})^2 \times 55^2 = 1485.25$ ✓

$P((F - \frac{1}{2}M) < 0) = P\left(z < \frac{0-17}{\sqrt{1485.25}}\right) = P(z < -0.4411)$

$= 1 - \phi(0.4411)$

$= 1 - 0.670 = \underline{0.330}$ ✓ (3sf)

13. Before a certain type of book is published it is checked for errors, which are then corrected. For costing purposes each error is classified as either minor or major. The minor and major errors in a book are modelled by the independent distributions $N(380, 140)$ and $N(210, 80)$ respectively. You should assume that no continuity corrections are needed. A book of this type is chosen at random.

(a) Find the probability that the number of minor errors is at least 200 more than the number of major errors. ---[5]

The cost of correcting a minor error and a major error are 20 cents and 50 cents respectively.

⊗ (b) Find the probability that the total cost of correcting the errors in the book is less than \$190. ---[5]

[W-20/62/Q7]

Solution: Minor error $M_1 \sim N(380, 140)$ and major error $M_2 \sim N(210, 80)$.

(a) $E(M_1 - M_2) = 380 - 210 = 170$ ✓ and $Var(M_1 - M_2) = 140 + 80 = 220$ ✓

$$P\left((M_1 - M_2) \geqslant 200\right) = P\left(Z \geqslant \frac{200 - 170}{\sqrt{220}}\right) = P(Z \geqslant 2.023)$$

$$= 1 - \phi(2.023) = 1 - 0.9784 = \underline{0.0216} \text{ ✓ (3sf)}$$

(b) $E(\text{total cost}) = 380 \times 20 + 210 \times 50 = 18100$ cent

$Var(\text{total cost}) = 140 \times 20^2 + 80 \times 50^2 = 256000$ cents

{\$190 = 19000 cents}

$$P(\text{Total cost} < 19000) = P\left(Z < \frac{19000 - 18100}{\sqrt{256000}}\right)$$

$$= P(Z < 1.778)$$

$$= \phi(1.778)$$

$$= \underline{0.962} \text{ ✓ (3s.f)}$$

14. The volumes, in millilitres, of large and small cups of tea are modelled by the distributions $N(200,30)$ and $N(110,20)$ respectively.

(a) Find the probability that the total volume of a randomly chosen large cup of tea and a randomly chosen small cup of tea is less than 300 ml. ---[4]

(b) Find the probability that the volume of a randomly chosen large cup of tea is more than twice the volume of a randomly chosen small cup of tea. --[6]

M-20/62/Q6

Solution (a) Large cup $L \sim N(200, 30)$, Small cup $S \sim N(110, 20)$

$E(L+S) = 200 + 110 = 310$; $Var(L+S) = 30 + 20 = 50$

$$P((L+S) < 300) = P\left(z < \frac{300-310}{\sqrt{50}}\right)$$

$$= P(z < -1.414)$$

$$= \phi(-1.414)$$

$$= 1 - \phi(1.414) = 1 - 0.9214$$

$$= \underline{0.0786}$$

(b) Now for $P(L-2S) > 0$, $E(L-2X) = 200 - 2\times110 = -20$

$Var(L-2X) = 30 + 2^2 \times 20 = 110$

Now $N(-20, 110)$

$$P((L-2S) > 0) = P\left(z > \frac{0-(-20)}{\sqrt{110}}\right)$$

$$= P(z > 1.907)$$

$$= 1 - \phi(1.907)$$

$$= 1 - 0.9717$$

$$= \underline{0.0283} \text{ (3sf)}$$

15. The masses, in kilograms of large and small sacks of floor have the distribution $N(55, 3^2)$ and $N(27, 2.5^2)$ respectively.

(a) Some sacks are loaded on a boat. The maximum load of flour that the boat can carry safly is 340kg. Find the probability that the boat can carry 3 randomly chosen large sacks of flour and 6 randomly chosen small sacks of floor. ---[5]

(b) Find the prob. that the mass of a randomly chosen large sack of flour is greater than the total mass of two randomly chosen small sacks of flour. ---[5]

[S-21/61/Q7]

Solution: (a) $E(\text{Total mass}) = E(3\,\text{large} + 6\,\text{small}) = 3\times55 + 6\times27 = 327$ ✓

$Var(\text{Total mass}) = 3\times3^2 + 6\times2.5^2 = 64.5$ ✓

$P(\text{Total mass} < 340) = P\left(Z < \frac{340-327}{\sqrt{64.5}}\right) = P(Z < 1.619)$

$= \phi(1.619) = \underline{0.947}$ ✓ (3s.f)

(b) $E(L - S_1 - S_2) = 55 - 2\times27 = 1$

$Var(L - S_1 - S_2) = 3^2 + 2\times2.5^2 = 21.5$

$P((L - S_1 - S_2) > 0) = P\left(Z > \frac{0-1}{\sqrt{21.5}}\right) = P(Z > -0.216)$

$= P(Z < 0.216) = \phi(0.216) = \underline{0.586}$ ✓

16. The independent random variables X and Y have the distributions $N(9.2, 12.1)$ and $N(3.0, 8.6)$ respectively. Find $P(X > 3Y)$. ---[5]

[M-19/72/Q2]

Solution: $E(X - 3Y) = E(X) - 3E(Y) = 9.2 - 3\times3 = 0.2$ ✓

and $Var(X - 3Y) = Var(X) + 3^2 \times Var(Y) = 12.1 + 3^2\times8.6 = 89.5$ ✓

$P((X - 3Y) > 0) = P\left(Z > \frac{0-0.2}{\sqrt{89.5}}\right) = P(Z > -0.021) = P(Z < 0.021)$

$= \phi(0.021) = \underline{0.508}$ ✓ (3s.f)

§ 4. Linear combination of Poisson distributions:

(i) If X and Y have independent Poisson distributions, Then $X+Y$ has a Poisson distribution.
If $X \sim Po(\lambda)$ and $Y \sim Po(\mu)$, then $X+Y \sim Po(\lambda+\mu)$.

(ii) If R is the sum of 'n' independent values of X, then $R \sim Po(n \times \lambda)$

Example 17: The independent variables X and Y have the distributions $Po(2)$ and $Po(3)$ respectively.

(i) Given that $X+Y=5$, find the probability '$X=1$ and $Y=4$' --- [4]

(ii) Given that $P(X=r) = \frac{2}{3} P(X=0)$, show that $3 \times 2^{r-1} = r!$ and verify that $r=4$ satisfies this equation. --[2]

$\boxed{S\text{-}13/73/Q4}$

Solution: $X \sim Po(2)$ and $Y \sim Po(3)$ are independent $\Rightarrow X+Y \sim Po(2+3) = Po(5)$

(i) Now $P(X+Y=5) = \frac{e^{-5} \cdot 5^5}{5!} = 0.17546$ $\left\{ \begin{array}{l} \text{for } Po(5) \\ \lambda = 5 \\ r = 5 \end{array} \right.$ $\left[P(r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!} \right.$

Now $P(X=1 \text{ and } Y=4) = 2e^{-2} \times \frac{e^{-3} \cdot 3^4}{4!} = 0.045481$ $\left| \begin{array}{l} P(X=1) = \frac{e^{-2} \cdot 2^1}{1!} \quad (\lambda=2) \\ P(Y=4) = \frac{e^{-3} \cdot 3^4}{4!} \quad (\lambda=3) \end{array} \right.$

Hence $P\left(X=1 \text{ and } Y=4 / X+Y=5\right) = \frac{P(X+Y=5 \cap (X=1 \& Y=4))}{P(X+Y=5)}$

$$= \frac{0.045481}{0.17546} = 0.259$$

(ii) Given $P(X=r) = \frac{2}{3} P(X=0)$ $\qquad X \sim Po(2) \Rightarrow \lambda = 2$

$$\Rightarrow \frac{e^{-\lambda} \cdot \lambda^r}{r!} = \frac{2}{3} e^{-\lambda} \Rightarrow \lambda^r = \frac{2}{3} \times r!$$

$$\Rightarrow 2^r = \frac{2}{3} \times r! \qquad (\lambda = 2)$$

$$\Rightarrow 3 \cdot 2^{r-1} = r! \quad \text{---} \ (1)$$

Put $r=4$ in (1)

$$\left. \begin{array}{r} 3 \times 2^{4-1} = 4! \\ 3 \times 2^3 = 4! \\ 24 = 24 \end{array} \right\} \Rightarrow r = 4 \text{ satisfies equation (1).}$$

Example 18: A random variable X has distribution $Po(1.6)$

(i) The random variable R is the sum of three independent values of X. Find $P(R<4)$ ---[3]

(ii) The random variable S is the sum of n independent values of X. It is given that $P(S=4) = \frac{16}{3} \times P(S=2)$ find n. ---[4]

(iii) The random variable T is the sum of 40 independent values of X. Find $P(T>75)$ ---[4]

W-12/72/Q7

Solution: $X \sim Po(1.6) \Rightarrow \lambda = 1.6$

(i) $R \sim X_1+X_2+X_3 \Rightarrow R = Po(3\times 1.6) = Po(4.8)$

Now $P(R<4) = P(0,1,2,3)$ $\quad \left[P(x) = \frac{e^{-\lambda}.\lambda^x}{x!}\right.$

$= e^{-4.8}\left[1+4.8+\frac{4.8^2}{2!}+\frac{4.8^3}{3!}\right]$ $(\lambda = 4.8)$

$= 0.294$

(ii) $S = X_1+X_2+ \cdots x_n \Rightarrow S \sim Po(1.6n)$

Given $P(S=4) = \frac{16}{3}.P(S=2)$ $\quad \left\{ P(S=4) = e^{-1.6n}.\frac{(1.6n)^4}{4!} \right.$ $\quad P(S=2) = e^{-1.6n}.\frac{(1.6n)^2}{2!}$

$\Rightarrow e^{-1.6n}.\frac{(1.6n)^4}{4!} = \frac{16}{3}\times e^{-1.6n}\times\frac{(1.6n)^2}{2!}$

$\Rightarrow (1.6n)^2 = \frac{16}{3\times 2}\times 4! \Rightarrow (1.6n)^2 = 64 \Rightarrow 1.6n = 8$

$\Rightarrow n = \frac{8}{1.6} = \underline{5}$

(iii) $T = X_1+X_2+\cdots+X_{40}$

$\Rightarrow T \sim Po(40\times 1.6) = Po(64)$: Now $\lambda = 64 > 15$

Hence the $Po(64)$ approximates to Normal distribution $N(64,64)$

$\{\mu = 64, \sigma^2 = 64$

$P(T>75) = P\left(Z > \frac{75.5-64}{\sqrt{64}}\right)$ $\quad \left\{ \begin{array}{l} \underline{\text{continuity correction}} \\ T>75 \Rightarrow T\geq 76 \\ T \sim 75.5 \end{array} \right.$

$= P(Z > 1.4375)$

$= 1 - \phi(1.4375)$

$= \underline{0.0753}$ (or 0.0754)

19. The number of lions seen per day during a standard safari has the distribution $Po(0.8)$. The number of lions seen per day during an off-road safari has the distribution $Po(2.7)$. The two distributions are independent.

(i) Susan goes on a standard safari for one day. Find the probability that she sees at least 2 lions. ---[2]

(ii) Deena goes on a standard safari for 3 days and then on an off-road safari for 2 days. Find the probability that she sees a total of fewer than 5 lions. ---[3]

(iii) Kaled goes on a standard safari for n days, where n is an integer. He wants to ensure that his chance of not seeing any lions is less than 10%. Find the smallest possible value of n. --[3]

S-12/73/Q4

Solution: Standard safari $\sim Po(0.8) \to \lambda_1 = 0.8$ & offroad safari $Po(2.7) \to \lambda_2 = 2.7$

(i) $P(X \geq 2) = 1 - P(X=0,1) = 1 - e^{-0.8}(1+0.8)$ $\left\{ P(x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!} \right.$

$= \underline{0.191}$ ✓

(ii) $\lambda = 3\times0.8 + 2\times2.7 = 7.8$

$P(X<5) = P(X=0,1,2,3,4)$

$= e^{-7.8}\left[1 + 7.8 + \frac{7.8^2}{2!} + \frac{7.8^3}{3!} + \frac{7.8^4}{4!}\right] = \underline{0.112}$ ✓

(iii) Now using $\lambda = 0.8n$,

Given $P(X=0) < 0.10$ (10%)

$\Rightarrow e^{-0.8n} < 0.10$

$\Rightarrow \ln e^{-0.8n} < \ln 0.10$

$\Rightarrow -0.8n < \ln 0.10$

$\Rightarrow n > \frac{\ln 0.10}{-0.8} = 2.878..$

Hence the smallest value of $\underline{n = 3}$ ✓