DATE 26.10.2023

S-2

Probability and Statistics-2

Linear combination of random variables.

Solution Ex-2 (Revision)

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S.2 Linear combinations of random variable.

§ Expectation and variance:

1. For a random variable X and constant b:
   (i) $E(X+b) = E(X)+b$ (ii) $Var(X+b) = Var(X)$

2. For a random variable X and constants a and b.
   (i) $E(aX+b) = a \cdot E(X)+b$ (ii) $Var(aX+b) = a^2 \cdot Var(X)$

3. For two independent random variables X and Y:
   (i) $E(X+Y) = E(X)+E(Y)$ (ii) $Var(X+Y) = Var(X)+Var(Y)$

4. For two independent random variables X and Y, and constants a and b.
   (i) $E(aX+bY) = a \cdot E(X)+b \cdot E(Y)$ and $Var(aX+bY) = a^2 \cdot Var(X)+b^2 \cdot Var(Y)$

   (ii) $E(aX-bY) = a \cdot E(X) - b \cdot E(Y)$ and $Var(aX-bY) = a^2 \cdot Var(X) + b^2 \cdot Var(Y)$

5. If continuous random variables X and Y have independent normal distributions, then $aX+bY$, where a and b are constants, has a normal distribution.

6. If X and Y have independent Poisson distribution, then $X+Y$ has a Poisson distribution.
   $X \sim Po(\lambda)$ and $Y \sim Po(\mu) \Rightarrow X+Y \sim Po(\lambda+\mu)$

7. Two independent observation of X, X_1 & X_2.
   $E(X_1 + X_2) = E(X_1)+E(X_2)$; $Var(X_1+X_2) = Var X_1 + Var X_2$.

8. $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, p_1 is the profit per Unit of X and p_2 is the profit per Unit of Y

$E(\text{Total Profit}) = p_1 \mu_1 + p_2 \mu_2$
$Var(\text{Total Profit}) = p_1^2 \times \sigma_1^2 + p_2^2 \times \sigma_2^2$

1. The lifetimes, in hours, of Longlive bulbs and Enerlow light bulbs have the independent distributions $N(1020, 45^2)$ and $N(2800, 52^2)$ respectively.

(a) Find the probability that the total of the lifetimes of five randomly chosen Longlive bulbs is less than 5200 hours. --[4]

(b) Find the probability that the lifetime of a randomly chosen Enerlow bulb is at least three times that of a randomly chosen Longlive bulb. --[6]

[SP-20/06/Q4]

Solution: Given Longlive bulb $X \sim N(1020, 45^2)$

(a) for five randomly chosen Longlive bulb, $N(1020\times5,\ 45^2\times5)$

or $X \sim N(5100, 10125)$ $\{N(\mu, \sigma^2)\}$

$$P(X < 5200) = P\left(Z < \frac{5200-5100}{\sqrt{10125}}\right)$$

$$= P(Z < 0.994)$$

$$= \phi(0.994) = \underline{0.840} \checkmark \ (3sf)$$

(b) $E: N(2800, 2704)$ and $L: N(1020, 2025)$

for $E-3L$; $E(E-3L) = E(L) - 3E(L) = 2800 - 3\times1020$

$\therefore E(E-3L) = -260 \checkmark$

and $Var(E-3L) = Var(E) + 3^2\, Var(L) = 2704 + 9\times2025$

$\therefore Var(E-3L) = 20929 \checkmark$

$$P(E-3L > 0) = P\left(Z > \frac{0-(-260)}{\sqrt{20929}}\right)$$

$$= P(Z > 1.797)$$

$$= 1 - \phi(1.797)$$

$$= 1 - 0.9639$$

$$= \underline{0.0361} \checkmark$$

2. The volumes, in millilitres, of large and small cups of tea are modelled by the distributions N(200, 30) and N(110, 20) respectively.

(a) Find the probability that the total volume of a randomly chosen large cup of tea and a randomly chosen small cup of tea is less than 300 ml. ---[4]

(b) Find the probability that the volume of a randomly chosen large cup of tea is more than twice the volume of a randomly chosen small cup of tea. --[6]

[M-20/62/Q6]

Solution (a) Large cup $L \sim N(200, 30)$, Small cup $S \sim N(110, 20)$

$E(L+S) = 200 + 110 = 310$; $Var(L+S) = 30 + 20 = 50$

$$P((L+S) < 300) = P\left(z < \frac{300-310}{\sqrt{50}}\right)$$
$$= P(z < -1.414)$$
$$= \phi(-1.414)$$
$$= 1 - \phi(1.414) = 1 - 0.9214$$
$$= \underline{0.0786}$$

(b) Now for $P(L-2S) > 0$, $E(L-2X) = 200 - 2 \times 110 = -20$

$Var(L-2X) = 30 + 2^2 \times 20 = 110$

Now $N(-20, 110)$

$$P((L-2S) > 0) = P\left(z > \frac{0-(-20)}{\sqrt{110}}\right)$$
$$= P(z > 1.907)$$
$$= 1 - \phi(1.907)$$
$$= 1 - 0.9717$$
$$= \underline{0.0283} \ (3sf)$$

3. The volume, in litres, of juice in large and small bottles have the distributions $N(5.10, 0.0102)$ and $N(2.51, 0.0036)$ respectively.

(a) Find the probability that the total volume of juice in 3 randomly chosen large bottles and 4 randomly chosen small bottles is less than 25.5 litres. --- [5]

(b) Find the probability that the volume of juice in a randomly chosen large bottle is at least twice the volume of juice in a randomly chosen small bottle. --- [5]

[M-21/62/Q5]

Solution: large bottle $L \sim N(5.10, 0.0102)$ and small bottle $S \sim N(2.51, 0.0036)$.

(a) 3 Large and 4 small randomly chosen bottles.

$E(L_1+L_2+L_3+S_1+S_2+S_3+S_4) = 3\times5.1 + 4\times2.51 = 25.34$ ✓

$Var(L_1+L_2+L_3+S_1+S_2+S_3+S_4) = 3\times0.0102 + 4\times0.0036 = 0.045$ ✓

$P(X < 25.5) = P\left(z < \frac{25.5-25.34}{\sqrt{0.045}}\right)$

$= P(z < 0.754) = \phi(0.754) = \underline{0.775}$ ✓ (3sf)

(b) Now $E(L-2S) = 5.10 - 2\times2.51 = 0.08$ ✓

$Var(L-2S) = 0.0102 + 2^2\times0.0036 = 0.0246$ ✓

$P((L-2S) > 0) = P\left(z > \frac{0-0.08}{\sqrt{0.0246}}\right)$

$= P(z > -0.510) = P(z < 0.510)$

$= \phi(0.510)$

$= \underline{0.695}$ ✓ (3 sf)

4. The height of buildings in a large city are normally distributed with mean 18.3m and standard deviation 2.5m.

(a) Find the probability that the total height of 5 randomly chosen building in the city is more than 95m. ---[4]

(b) Find the probability that the difference between the heights of two randomly chosen buildings in the city is less than 1m. ---[5]

[M-22/62/Q5]

Solution: (a) Mean $\mu = 5 \times 18.3 = 91.5$; Variance $= 5 \times 2.5^2 = 31.25$; $N(91.5, 31.5)$

$P(H > 95) = P\left(Z > \frac{95-91.5}{\sqrt{31.5}}\right) = P(Z > 0.626) = 1 - \phi(0.626) = \underline{0.266}$ (3sf)

(b) $E(\text{Difference}) = 18.3 - 18.3 = 0$, $Var(\text{Diff}) = 2 \times 2.5^2 = 12.5$

$P(D < 1) = P\left(Z < \frac{1-0}{\sqrt{12.5}}\right)$ or $P\left(Z < \frac{-1-0}{\sqrt{12.5}}\right)$

$\begin{cases} A - B < 1 \\ \text{or} \\ B - A < 1 \\ \Rightarrow A - B < -1 \end{cases}$

$= P(Z < 0.283)$ or $P(Z < -0.283)$

$= \phi(0.283) - (1 - \phi(0.283)) = 0.6115 - (1 - 0.6115) = \underline{0.223}$

5. The masses, in grams, of large and small packets of Maxwheat cereal have the independent distributions $N(410, 3.6^2)$ and $N(206, 3.7^2)$ respectively.

(a) Find the probability that a randomly chosen large packet has a mass that is more than double the mass of a randomly chosen small packet. ---[5]

The packets are placed in boxes. The boxes are identical in appearance. 60% of the boxe contain exactly 10 randomly chosen large packets. 40% of the boxes contain exactly 20 randomly chosen small packets.

(b) Find the probability that a randomly chosen box contains packets with a total mass of more than 4080 grams. ---[6]

M-23/62/Q5

Solution: $L \sim N(410, 3.6^2)$ and $S \sim (206, 3.7)^2$

(a) $P(L-2S>0) = P\left(z > \frac{0-(-2)}{\sqrt{67.72}}\right)$

$\left\{\begin{array}{l}\text{Let } D = L-2S \\ E(D) = E(L) - 2\cdot E(S) = 410 - 2\times 206 = -2 \checkmark \\ Var\,D = 3.6^2 + 2^2 \times 3.7^2 = 67.72\end{array}\right.$

$= P(z > 0.243)$

$= 1 - \phi(0.243)$

$= \underline{0.404} \checkmark$ (3sf)

(b) $T_L \sim N(410\times 10, 10\times 3.6^2) = N(4100, 129.6)$ (10 large boxes) ↓ Radomly Choosen

$T_S \sim N(20\times 206, 20\times 3.7^2) = N(4120, 273.8)$ (20 small boxes)

$P(T_L > 4080) = P\left(z > \frac{4080-4100}{\sqrt{129.6}}\right) = P(z > -1.757)$

$= P(z < 1.757) = \phi(1.757)$

$= 0.9605 \checkmark$ --- ①

$P(T_S > 4080) = P\left(z > \frac{4080-4120}{\sqrt{273.8}}\right) = P(z > -2.417)$

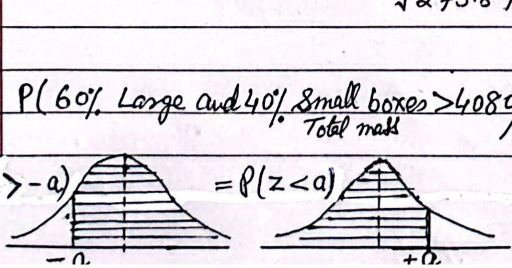
$= P(z < 2.417) = \phi(2.417)$

$= 0.9921 \checkmark$ --- ②

$P(60\%$ Large and 40% Small boxes $> 4080)$ Total mass $= 0.6\times 0.9605 + 0.4\times 0.9921$

$= \underline{0.973} \checkmark$ (3sf)

$P(z>-a) = P(z<a)$



6. The masses, in kilograms, of large sacks of flour and small sack of flour have the independent distribution $N(40, 1.5^2)$ and $N(12, 0.7^2)$ respectively.

(a) Find the probability that the total mass of 6 randomly chosen large sacks of flour is more than 245 kg. ---[4]

(b) Find the probabity that the mass of a randomly chosen large sack of flour is less then 4 times the mass of a randomly chosen small sack of flour. ---[6]

$\boxed{S\text{-}20/61/Q3}$

Solution: 6 randomly chosen large bottles $L \sim N(6\times40, 6\times1.5^2) \sim N(240, 13.5)$

(a) $P(L>245) = P\left(Z > \frac{245-240}{\sqrt{13.5}}\right) = P(Z>1.361)$

$= 1-\phi(1.361)$

$= 1-0.9133 = \underline{0.0867}$ ✓ (3sf)

(b) $E(L-4S) = 40-4\times12 = -8$ ✓

$Var(L-4S) = 1.5^2 + 4^2\times0.7^2 = 10.09$

$L<4S \Rightarrow L-4S<0$

$P(L-4S<0) = P\left(Z< \frac{0-(-8)}{\sqrt{10.09}}\right) = P(Z<2.519)$

$= \phi(2.519) = 0.994$ ✓ (3sf)

7. The masses, in grams, of a certain type of plums have the distribution $N(40.4, 5.2^2)$. The plums are packed in bags, with each bag containing 6 randomly chosen plums. If the total weight of the plums in a bag is less than 220 gr the bag is rejected. Find the percentage of bags that are rejected. ---[4]

$\boxed{S\text{-}20/62/Q1}$

Solution: mass of Plum $\sim N(40.4, 5.2^2)$; $\Rightarrow$ each bag has 6 plums $\Rightarrow E(X) = 6\times40.4 = 242.4g.$

$Var(X) = 6\times5.2^2 = 162.24$

Mass of Plums in bag. $X \sim N(242.4, 162.24)$

$P(X<220) = P\left(Z< \frac{220-242.4}{\sqrt{162.24}}\right) = P(Z<-1.759)$

$= \phi(-1.759)$

$= 1-\phi(1.759) = 0.0393$

$= 3.93\%$

8. Each day at the gym, Sarah completes three runs. The distances, in metres, that she completes in the three runs have independent distributions $W \sim N(1520, 450)$, $X \sim (2250, 720)$ and $Y \sim N(3860, 1050)$.
Find the probability that, on a particular day, Y is less than the total of W and X. ---[5]

S-20/63 | Q2

Solution: $E(Y-(W+X)) = 3860-(1520+2250) = 90$ ✓

$Var(Y-W-X) = 1050+450+720 = 2220$ ✓

$$P(Y-(W+X)<0) = P\left(Z < \frac{0-90}{\sqrt{2220}}\right) = P(Z < -1.910)$$

$$= \phi(-1.910) = 1-\phi(1.910) = \underline{0.0281}$$ ✓

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9. The random variable A has the distribution $Po(1.5)$. A_1 and A_2 are independent values of A.

(a) Find $P(A_1+A_2<2)$ ---[3]

(b) Given that $A_1+A_2<2$, find $P(A_1=1)$ --[4]

(c) Give reason why A_1-A_2 cannot be a Poisson distribution --[1]

S-20/63/Q4

Solution: $A \sim Po(1.5) \Rightarrow A_1+A_2 \sim Po(2\times1.5) = P(3),\ \lambda=3$

(a) $P(A_1+A_2<2) = P(A_1+A_2 = 0 \text{ or } 1) = e^{-3}(1+3) = \underline{0.199}$ ✓

(b) $P(A=1 \text{ and } A_1+A_2<2) = P(A_1=1) \times P(A_2=0)$

$= (e^{-1.5} \times 1.5) \times (e^{-1.5}) = 0.0747$

$$\therefore P(A_1=1 / A_1+A_2<2) = \frac{P(A_1=1 \text{ and } A_1+A_2<2)}{P(A_1+A_2<2)}$$

$$= \frac{(1.5\times e^{-1.5})\times e^{-1.5}}{0.199} = \frac{0.0747}{0.199} =$$

$= \underline{0.375}$ ✓ (3 sf)

(c) Takes negative values.

$E(A_1-A_2) = 1.5-1.5 = 0$ but $\neq Var(A_1-A_2) = 1.5+1.5 = 3)$

10. Accidents at two factories occur randomly and independenly. On average the number of accidents per month are 3.1 at factory A and 1.7 at factory B. Find the probability that the total number of accidents in two factories during a 2-month period is more than 3. --[4]

[S-21/61/Q1]

Solution: $Po(3.1)$ and $Po(1.7) \Rightarrow$ for Total in 2 month $\rightarrow \lambda = 2(3.1+1.7) = 9.6$ ✓

$P(X>3) = 1 - P(X=0,1,2,3) = 1 - e^{-9.6}\left(1 + 9.6 + \frac{9.6^2}{2!} + \frac{9.6^3}{3!}\right) = \underline{0.986}$ ✓ (3sf)

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11. The masses, in kilograms, of large and small sacks of flour have the distributions $N(55, 3^2)$ and $N(27, 2.5^2)$ respectively.

(a) Some sacks are loaded on a boat. The maximum load of flour that the boat can carry safly is 340 kg. Find the probability that the boat can carry 3 randomly chosen large sacks of flour and 6 randomly chosen small sacks of flour. --[5]

(b) Find the probability that the mass of a randomly chosen large sack of flour is greater than the total mass of two randomly chosen small sacks of flour. [S-21/61/Q7] --[5]

Solution: $E(\text{Total mass}) = 3\times55 + 6\times27 = 327$ ✓ and $Var(\text{Total mass}) = 3\times3^2 + 6\times2.5^2 = 64.5$ ✓

$P(\text{Total mass} < 340) = P\left(Z < \frac{340-327}{\sqrt{64.5}}\right) = P(Z < 1.619)$

$= \phi(1.619) = \underline{0.947}$ ✓ (3 sf)

(b) $E(L - S_1 - S_2) = 55 - 2\times27 = 1$ ✓

$Var(L - S_1 - S_2) = 3^2 + 2\times2.5^2 = 21.5$ ✓

$P((L - S_1 - S_2) > 0) = P\left(Z > \frac{0-1}{\sqrt{21.5}}\right)$

$= P(Z > -0.216)$

$= P(Z < 0.216)$

$= \phi(0.216)$

$= \underline{0.586}$ ✓ (3 s.f)

12. The random variable X has the distribution $B(400, 0.01)$

(a) Find $Var(4X+2)$ --[3]

(b) (i) State an appropriate approximating distribution for X, giving the values of parameters. Justify your choice of approximating distribution. --[2]

(ii) Use your approximating distribution to find $P(2 \leq X \leq 5)$ --[2]

[S-21/62/Q2]

Solution (a) $B(400, 0.01) \rightarrow n = 400, p = 0.01, q = 0.99 \Rightarrow Var(X) = npq = 400 \times 0.01 \times 0.99 = 3.96$ ✓

hence $Var(4x+2) = 4^2 \times Var X = 16 \times 3.96 = \underline{63.36}$ ✓

b (i) $np = 400 \times 0.01 = 4 < 5$ and $n = 400 > 50$

$\therefore$ Poisson distribution $Po(4)$: $\lambda = 4$. $\{ P(X=r) = e^{-\lambda} \cdot \frac{\lambda^r}{r!}$

(ii) $P(2 \leq X \leq 5) = e^{-4}\left[\frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}\right] = 0.694$ ✓ (3 s.f)

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13. Wendy's journey to work consists of three parts: walking to train station, riding on the train and then walking to the office. The times, in minutes, for the three parts of her journey are indendent and have distributions $N(15.0, 1.1^2)$, $N(32.0, 3.5^2)$ and $N(8.6, 1.2^2)$ respectively.

(a) Find the mean and variance of the total time for Wendy's journey. ---[2]

If Wendy's journey takes more than 60 minutes, she is late for work.

(b) Find the probability that, on a randomly chosen day, Wendy will be late for work. --[3]

※ (C) Find the probability that the mean of Wendy's journey time over 15 randomly chosen days will be less than 54.5 minute. --[3]

{Unit-4 question: Sample mean}

[S-21/62/Q4]

Solution: (a) Mean. $E(\text{Total time}) = 15 + 32 + 8.6 = \underline{55.6}$ ✓

$Var(\text{Total time}) = 1.1^2 + 3.5^2 + 1.2^2 = \underline{14.9}$ ✓

(b) $P(T > 60) = P\left(z > \frac{60 - 55.6}{\sqrt{14.9}}\right)$

$= P(z > 1.140)$

$= 1 - \phi(1.140)$

$= 1 - 0.873$

$= \underline{0.127}$ ✓ (3sf)

(c)※ for 15 randomly chosen day:

$P(\text{Time} < 54.5) = P\left(z < \frac{54.5 - 55.6}{\sqrt{\frac{14.9}{15}}}\right)$ ※

$= P(z < -1.104)$

$= 1 - \phi(1.104)$

$= 1 - 0.865$

$= \underline{0.135}$ ✓ (3sf)

(※ Expectation and variance of Sample mean $\bar{X}$.)

14. The number of goals scored by a team in a match is independent of other matches, and is denoted by the random variable X, which has a Poisson distribution with mean 1.36. A supporter offers to make a donation of \$5 to the team for each goal that they score in the next 10 matches. Find the expectation and standard deviation of the amount that the supporter will pay. [S-21/63/Q1] --[5]

Solution: $Po(1.36)$ for one match $\Rightarrow$ for 10 matches $\lambda = 10\times1.36 = 13.6$ ✓

$E(\text{amount}) = \$5\times13.6 = \$68$ ✓

$Var(\text{amount}) = 5^2\times13.6 = \$340 \Rightarrow$ Standard deviation $= \sqrt{340} = \$18.44$ ✓ (3sf)

15. The lengths, in centimetres, of two types of insect, A and B, are modelled by the random variables $X \sim N(6.2, 0.36)$ and $Y \sim N(2.4, 0.25)$.

Find the probability that the length of a randomly chosen type A insect is greater than the sum of the lengths of 3 randomly chosen type B insects. --[5]

[S-22/61/Q3]

Solution: $D = X - (Y_1+Y_2+Y_3) \Rightarrow E(D) = 6.2 - 3\times2.4 = -1$ (μ)

$Var(D) = 0.36 + 3\times0.25 = 1.11$ (σ^2)

$$P(D>0) = P\left(Z > \frac{0-(-1)}{\sqrt{1.11}}\right) = P(Z > 0.949)$$

$$= 1 - \phi(0.949)$$

$$= 1 - 0.8287$$

$$= 0.1713$$

$$= 0.171 \ (3s.f)$$

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The independent random variables X and Y have distributions $P_o(2)$ and $B(20, \frac{1}{4})$ respectively.

(a) Find the mean and standard deviation of $X-3Y$. ---[5]

(b) Find $P(Y=15X)$ ---[3]

S-22/61/Q4

Solution: $X \sim P_o(2) \Rightarrow E(X) = \lambda = 2$ & $Var(X) = \lambda = 2$

(a) $Y \sim B(n,p) \rightarrow B(20, \frac{1}{4}) = E(Y) = np = 20 \times \frac{1}{4} = 5$ & $Var(Y) = npq = 20 \times \frac{1}{4} \times \frac{3}{4}$

$Var(Y) = \frac{15}{4}$

Now $E(X-3Y) = E(X) - 3 \cdot E(Y) = 2 - 3 \times 5 = -13$

and $Var(X-3Y) = Var(X) + 3^2(Var\,Y) = 2 + 9 \times \frac{15}{4} = 35.75$

Standard deviation of $(X-3Y) = \sqrt{\frac{143}{4}} = \underline{5.98}$

$X \sim P_o(2)$; $\lambda = 2$

$P(X=r) = e^{-\lambda} \cdot \frac{\lambda^r}{r!}$

$Y \sim B(20, \frac{1}{4})$

$P(Y=r) = {}^nC_r p^r \cdot q^{n-r}$

(b) $Y = 15X$ for $(0,0)$ and $(1,15)$ as for $Y(n=20)$.

$P(Y=15X) = P(X=0 \text{ and } Y=0)$ or $P(X=1 \text{ and } Y=15)$

$= e^{-2} \times (\frac{3}{4})^{20} + e^{-2} \times 2 \times {}^{20}C_{15} \cdot (\frac{1}{4})^{15} \cdot (\frac{3}{4})^5 = 0.000430$ (3 s.f)

17. The masses in kilograms, of large and small sacks of grain the distributions $N(53, 11)$ and $N(14, 3)$ respectively.

(a) Find the probability that the mass of a randomly chosen large sack is greater than four times the mass of a randomly chosen small sack. -[5]

(b) A lift can safly carry a maximum of 1000 kg. Find the prob. that the lift can safly carry 12 randomly chosen large sacks and 25 randomly chosen small sacks. --[5]

S-22/62/Q6

Solution: (a) $E(\text{Difference}) = E(D) = 53 - 4 \times 14 = -3$ $(D = L - 4S)$

$Var(D) = Var(L) + 4^2(Var\,S) = 11 + 4^2 \times 3 = 59$

$P(L - 4S > 0) = P\left(Z > \frac{0-(-3)}{\sqrt{59}}\right) = P(Z > 0.391) = 1 - \phi(0.391) = \underline{0.348}$ (3 s.f)

(b) Total; $T = 12L + 25S \Rightarrow E(12L + 25S) = 12 \times 53 + 25 \times 14 = 986$

and $Var(T) = 12 \times 11 + 25 \times 3 = 207$

$P(T < 1000) = P\left(Z < \frac{1000 - 986}{\sqrt{207}}\right) = P(Z < 0.973) = \phi(0.973)$

$= \underline{0.835}$ (3 s.f)

18. Each box of Seeds & Raisins contains 'S' grams of seeds and 'R' grams of raisins. The weight of box, when empty, is B grams. S, R and B are independent random variables, where $S \sim N(300,45)$, $R \sim N(200,25)$ and $B \sim N(15,4)$. A box full of Seeds and Raisins is chosen at random.

(a) Find the probability that the total weight of the box and its contents is more than 500 grams. ---[5]

(b) Find the probability that the weight of seeds in the box is less than 1.4 times the weight of raisins in the box. --[5]

[S-22/63/Q4]

Solution: $S \sim N(300,45)$; $R \sim N(200,25)$ and $B \sim N(15,4)$.

(a) $\left.\begin{aligned} E(T) &= 300+200+15 = 515 \\ Var(T) &= 45+25+4 = 74 \end{aligned}\right\} T \sim N(515,74) \quad N(\mu,\sigma^2)$

$P(T>500) = P\left(z > \frac{500-515}{\sqrt{74}}\right) = P(Z>-1.744) = \phi(1.744) = 0.959$

(b) $E(S-1.4R) = 300-1.4\times 200 = 20$; $Var(S-1.4R) = 45+1.4^2\times 25 = 94$

$(S-1.4R) \sim N(20,94)$

$P(S-1.4R<0) = P\left(z < \frac{0-20}{\sqrt{94}}\right) = P(z<-2.063)$

$= 1-\phi(2.063) = 0.0196$ (3 s.f)

19. Large packets of rice are packed in cartons, each containing 20 randomly chosen packets. The masses of these packets are normally distributed with mean 1010g and standard deviation 3.4g. The masses of the cartons, when empty, are independently normally distributed with mean 50g and standard deviation 2.0g.

(a) Find the variance of the masses of full cartons. ---[2]

Small packets of rice are packed in boxes. The total masses of full boxes are normally distributed with mean 6730g and standard deviation 15.0g. The masses of the boxes and cartons are distributed independently of each other.

(b) Find the probability that the mass of a randomly chosen full carton is more than three times the mass of a randomly chosen full box. --[5]

[S-23/61/Q5]

Solution: $L_p \sim N(1010, 3.4)$, 20L packets ; $\overset{\text{Empty}}{E_C} \sim (50, 2.0)$

(a) Var(Full Carton) = Var(20L packets + Empty Carton)

$= 20 \times 3.4^2 + 1 \times 2^2 = \underline{235.2}$ ✓ ---(1)

(b) $E(C-3B) = (50 + 20 \times 1010 - 3 \times 6730) = 60g$ from (1)

$Var(C-3B) = 235.2 + 3^2 \times 15 = 2260.2$, [$Var(C) = 235.2$]

$(C-3B) \sim N(60, 2260.2)$

$P((C-3B)>0) = P\left(Z > \frac{0-60}{\sqrt{2260.2}}\right) = P(Z > -1.262)$

$= P(Z < 1.262)$

$= \phi(1.262)$

$= \underline{0.897}$ ✓ (3sf)

20 (a) Two random variables X and Y have the independent distributions $N(7,3)$ and $N(6,2)$ respectively. A random value of each variable is taken. Find the probability that the two values differ by more than 2. ---[5]

(b) Each candidate's overall score in a science test is calculated as follows. The mark for theory is is denoted by T, the mark for practical is denoted by P, and the overall score is given by $T+1.5P$. The variables T and P are assumed to be independent with distributions $N(62,158)$ and $N(42,108)$ respectively. You should assume that no continuity correction are needed when using these distributions.

(i) A pass is awarded to candidates whose overall score is at least 90. Find the proportion of candidates who pass. ---[5]

(ii) Comment on the assumption that the variables T and P are independent [1]

| S-23 | 62 | Q5 |

Solution: $X \sim N(7,3)$ and $Y \sim N(6,2)$ are independent (difference means: ⊗ $X-Y>2$ or $Y-X>2$) Alt.

(a) $E(X-Y) = 7-6 = 1$; $Var(X-Y) = 3+2 = 5$ | and $E(Y-X) = 6-7 = -1$; $Var(Y-X) = 3+2$ $\boxed{=1}$ ✓

$P(X-Y>2) = P\left(z > \frac{2-1}{\sqrt{5}}\right) = P(z>0.447)$ | $P(Y-X) = P\left(z > \frac{2-(-1)}{\sqrt{5}}\right) = P(z>1.342)$

$= 1-\phi(0.447) = 0.327$ ✓ | $= 1-\phi(1.342) = 0.0898$ ✓

$\therefore$ P(difference of X & Y > 2) $= 0.327 + 0.0898 = \underline{0.417}$ ✓ (3sf) (0.4168)

⊗ differ by more than 2 $\Rightarrow |X-Y| > 2$
$\Rightarrow x-y>2$ or $X-Y<-2$
$\Rightarrow P(X-Y<-2)$
$= P\left(z < \frac{-2-1}{\sqrt{5}}\right)$
$P(z<-1.342)$
$= \phi(-1.342)$
$= 1-\phi(1.342)$
$= 0.0898$
"Given in MS" This way.

(b) $T \sim N(62, 158)$ and $P \sim N(42, 108)$

(i) Let X denotes the overall score: $X = T + 1.5P$

$E(X) = E(T) + 1.5 \times E(P) = 62 + 1.5 \times 42 = 125$ ✓

$Var(X) = Var(T) + 1.5^2 \cdot Var(P) = 158 + 1.5^2 \times 108 = 401$ ✓

$\underline{P(X \geq 90)} = P\left(z \geq \frac{90-125}{\sqrt{401}}\right) = P(z \geq -1.748)$

$= \phi(1.748)$

$= \underline{0.960}$ (3sf) or 96%

(ii) Unlikely i.e. A candidate who does well in theory is likely to do well in Practical.

21. The mass, in tonnes, of steel produced per day at a factory is normally distributed with mean 65.2 and standard deviation 3.6. It can be assumed that the mass of steel produced each day is independent of other days. The factory makes \$50 profit on each tonne of steel produced.
Find the probability that the total profit made in a randomly chosen 7-day week is less than \$22000. ---[6]
[S-23/63/Q4]

Solution: Steel produced per day $\sim N(65.2, 3.6^2)$

For 7-days (week): Based on mass

Mean $\mu = 7 \times 65.2 = 456.4$

Var $\sigma^2 = 7 \times 3.6^2 = 90.72$

Steel used for profit \$22000 $= \frac{22000}{50} = 440$ tonne

Now mass $X \sim N(456.4, 90.72)$,

$P(X < 440) = P\left(Z < \frac{440 - 456.4}{\sqrt{90.72}}\right)$

$= P(Z < -1.722)$

$= \phi(-1.722) = 1 - \phi(1.722) = 0.0425$

Alternatively: Based on Profit:

Mean profit in week $= 7 \times 65.2 \times 50 = \$22820$.

Profit: Var $\sigma^2 = 7 \times 3.6^2 \times 50^2 = 226800$.

Profit $X \sim N(22820, 226800)$

$P(X < 22000) = P\left(Z < \frac{22000 - 22820}{\sqrt{226800}}\right)$

$= P(Z < -1.722)$

$= \phi(-1.722) = 1 - \phi(1.722)$

$= 0.425$

22 The masses in kilograms, of female and male animals of a certain species have the distributions $N(102, 27^2)$ and $N(170, 55^2)$ respectively.
Find the probability that a randomly chosen female has a mass that is less than half the mass of a randomly chosen male. --[6]
[W-20/61/Q3]

Solution: $F \sim N(102, 27^2)$ and $M \sim N(170, 55^2)$

Let $X \sim (F - 0.5M)$; $E(X) = 102 - 0.5 \times 170 = 17$

$Var(X) = 27^2 + 0.5^2 \times 55^2 = 1485.25$

$X \sim (17, 1485.25)$

$P(X < 0) = P\left(Z < \frac{0 - 17}{\sqrt{1485.25}}\right) = P(Z < -0.4411) = \phi(-.4411)$

$= 1 - \phi(0.4411) = 0.330$ (3sf)

23. Before a certain type of book is published it is checked for errors, which are then corrected. For costing purposes each error is classified as either minor or major. The minor and major errors in a book are modelled by the independent distributions $N(380, 140)$ and $N(210, 80)$ respectively. You should assume that no continuity corrections are needed.

A book of this type is chosen at random.

(a) Find the probability that the number of minor errors is at least 200 more than the number of major errors. ---[5]

The cost of correcting a minor error and a major error are 20 cents and 50 cents respectively.

(b) Find the probability that the total cost of correcting the errors in the book is less than \$190. ---[5]

[W-20/62/Q7]

Solution: Minor error $M_1 \sim N(380, 140)$ and major error $M_2 \sim N(210, 80)$.

(a) $E(M_1 - M_2) = 380 - 210 = 170$ and $Var(M_1 - M_2) = 140 + 80 = 220$

$$P((M_1 - M_2) \geqslant 200) = P\left(Z \geqslant \frac{200-170}{\sqrt{220}}\right) = P(Z \geqslant 2.023)$$

$$= 1 - \phi(2.023) = 1 - 0.9784 = \underline{0.0216} \text{ (3sf)}$$

(b) $E(\text{total cost}) = 380 \times 20 + 210 \times 50 = 18100$ cent

$Var(\text{total cost}) = 140 \times 20^2 + 80 \times 50^2 = 256000$ cents

{\$190 = 19000 cent}

$$P(\text{Total cost} < 19000) = P\left(Z < \frac{19000 - 18100}{\sqrt{256000}}\right)$$

$$= P(Z < 1.778)$$

$$= \phi(1.778)$$

$$= \underline{0.962} \text{ (3s.f)}$$

24. The mass m, in kilograms, of a block of cheese sold in a supermarket is denoted by the random variable M. The masses of a random sample of 40 blocks are summarised as follows.

$$n = 40, \quad \Sigma m = 20.50, \quad \Sigma m^2 = 10.7280$$

See unit 4 (a) Calculate unbiased estimate of the population mean and variance of M. ... [3]

(b) The price, \$$P$, of a block of cheese of mass M kg is found using the formula $P = 11M + 0.50$

Find estimates of the population mean and variance of P. ---[3]

[W-21/62/Q1]

Solution (a) Unbiased estimate of mean of M, $\mu = \frac{\Sigma m}{n} = \frac{20.50}{40} = \underline{0.5125}$ ✓ $(= \mu)$

$$Var(est\ M) = \frac{n}{(n-1)}\left[\frac{\Sigma m^2}{n} - \mu^2\right]$$

$$s^2 = \frac{40}{39}\left[\frac{10.7280}{40} - (0.5125)^2\right] = 0.0056859$$

$$= \underline{0.00569} ✓ \ (3\ sf)$$

(b) for $P = 11M + 0.50$.

$$est\ Mean\ of\ P = (11\mu + 0.5) = 11 \times 0.5125 + 0.5 = 6.1375$$

$$= \underline{6.14} ✓ (3\ sf)$$

$$est\ Var\ P = 11^2 \times est\sigma^2 = 11^2 \times 0.0056859 = \underline{0.688} ✓ (3\ sf)$$

25. Each month a company sells X kg of brown sugar and Y kg of white sugar, where X and Y have the independent distributions $N(2500, 120^2)$ and $N(3700, 130^2)$ respectively.

(a) Find the mean and standard deviation of the total amount of sugar that the company sells in 3 randomly chosen months. ---[3]

The company makes a profit of \$1.50 per kilograms of brown sugar and makes a loss of \$0.20 per kg of white sugar sold.

(b) Find the probability that, in a randomly chosen month, the total profit is less than \$3000. --[5]

W-22/61/Q4

Solution: (a)

$$\text{Mean} = 3\cdot(2500+3700) = \underline{18600}\ (\text{kg})$$

$$\text{Variance(Total)} = 3(120^2+130^2) = 93900$$

$$\Rightarrow \text{S.D(Total)} = \sqrt{93900} = \underline{306}\ \text{kg}\ (3\text{sf})$$

$$\begin{cases} X \sim N(2500, 120^2) \\ Y \sim N(3700, 130^2) \end{cases}$$

(b)

$$E(1.5\times X - 0.2Y) = 1.5\times 2500 - 0.2\times 3700 = 3010$$

$$Var(1.5X - 0.2Y) = 1.5^2\times 120^2 + 0.2^2\times 130^2 = 33076$$

$$\text{Total Profit: } P = (1.5X - 0.02Y) \sim N(3010, 33076)$$

$$P(P<3000) = P\left(z < \frac{3000-3010}{\sqrt{33076}}\right) = P(z<-0.055)$$

$$= 1-\phi(0.055) = \underline{0.478}\ (3\text{sf})$$

26. The masses, in grams, of small and large bags of flour have the distributions $N(510, 100)$ and $N(1015, 324)$ respectively. André selects 4 small bags of flour and 2 large bags of flour at random.

(a) Find the prob. that the total mass of these 6 bags of flour is less than 4130g. --[5]

(b) Find the prob. that the total mass of 4 small bags is more than the total mass of the 2 large bags. W-22/62/Q6 ---[5]

Solution: (a)

$$E(4S+2L) = 4\times 510 + 2\times 1015 = 4070$$

$$Var(4S+2L) = 4\times 100 + 2\times 324 = 1048$$

$$\begin{cases} S \sim N(510, 100) & : N(\mu, \sigma^2) \\ L \sim N(1015, 324) \end{cases}$$

$$P(\text{Total} < 4130) = P\left(z < \frac{4130-4070}{\sqrt{1048}}\right) = P(z<1.853) = \phi(1.853) = \underline{0.968}\ (3\text{sf})$$

(b)

$$E(D) = 4\times 510 - 2\times 1015 = 10 \qquad [D = 4S - 2L]$$

$$Var(D) = 4\times 100 + 2\times 324 = 1048$$

$$P(D>0) = P\left(z > \frac{0-10}{\sqrt{1048}}\right) = P(z>-0.309) = \phi(0.309) = \underline{0.621}$$