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S.2

Probability and Statistics_2

The Poisson Distribution Notes

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Contents:

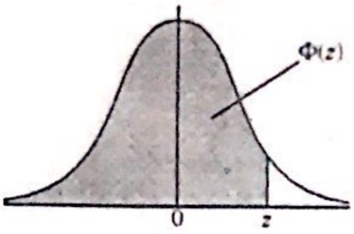
1. The Poisson distribution: $P(X=x)=\frac{e^{-\lambda}\cdot\lambda^{x}}{x!}$
2. Poisson distribution for different intervals.
3. Poisson distribution as an approximation to Binomial distribution.
4. Normal distribution as an approximation to Poisson distribution.

THE NORMAL DISTRIBUTION FUNCTION

If Z has a normal distribution with mean 0 and variance 1 then, for each value of z, the table gives the value of $\Phi(z)$, where

$$\Phi(z) = P(Z \leq z).$$

For negative values of z use $\Phi(-z) = 1 - \Phi(z)$.



z	0	1	2	3	4	5	6	7	8	9	ADD 1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	4	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0

Critical values for the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that

$$P(Z \leq z) = p.$$

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

§ The Poisson Distribution:

It is a discrete probability distribution, in which the events occur, singly, at random and independently, in a given-interval of space or time.

The mean and variance of a Poisson distribution are equal. Hence a Poisson distribution has only one parameter 'λ' [$\mu = \sigma^2 = \lambda$ (let)]

We write $P(X = r) = \dfrac{e^{-\lambda} \cdot \lambda^r}{r!}$; $\lambda > 0$; $r = 0, 1, 2, ---$

We write $X \sim Po(\lambda)$ Here $E(X) = \lambda$, $Var(X) = \lambda$

Example 1: A clinic deals only with flu vaccinations. The number of patients arriving every 15 minutes is modelled by random variable X with distribution $Po(4 \cdot 2)$

(i) State two assumptions required for the Poisson model to be valid. --- [2]

(ii) Find the probability that at least 1 patent will arrive in a 15-minute period. -- [2]

[S-10 | 73 | Q 10]

Solution (i) Patients arrive at constant mean rate.
Patients arrive at random.
Patients arrive independently.
Patients arrive singly.

(ii) $P(X \geq 1) = 1 - P(0)$

$= 1 - \dfrac{e^{-\lambda} \cdot \lambda^0}{0!}$

$= 1 - \dfrac{e^{-4 \cdot 2} \times 1}{1}$

$= 1 - e^{-4 \cdot 2}$

$= 1 - 0 \cdot 0145$

$= 0 \cdot 985$ ✓

$\left\{ \begin{array}{l} P(X = r) = \dfrac{e^{-\lambda} \cdot \lambda^r}{r!} \\ Po(4 \cdot 2) \Rightarrow \lambda = 4 \cdot 2 \end{array} \right.$

Example 2: The number of goals scored per match by Everly Rovers is represented by the random variable X, which has mean 1.8.

(i) State two conditions for X to be modelled by a Poisson distribution. ---[2]

Assume now that $X \sim Po(1.8)$

(ii) Find $P(2 < X < 6)$ ---[2]

(iii) The manager promises the team a bonus if they score at least 1 goal in each of the next 10 matches. Find the probability that they win the bonus. $\boxed{S\text{-}11}\boxed{72}\boxed{Q3}$ ---[3]

Solution (i) Constant average rate of goals scored,
Goals at random
Goals independent.

(ii) $X \sim Po(1.8) \Rightarrow \lambda = 1.8$ $\quad [P(X=r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!}]$

$$P[2 < X < 6] = P(3) + P(4) + P(5)$$

$$= e^{-1.8} \times \frac{(1.8)^3}{3!} + e^{-1.8} \cdot \frac{(1.8)^4}{4!} + e^{-1.8} \cdot \frac{(1.8)^5}{5!}$$

$$= e^{-1.8}\left[\frac{(1.8)^3}{3!} + \frac{(1.8)^4}{4!} + \frac{(1.8)^5}{5}\right]$$

$$= 0.259 \checkmark$$

(iii) In each match. $P(\text{at least 1 goal}) = P(X \geq 1)$

$$= 1 - P(X=0)$$

$$= 1 - \frac{e^{-\lambda} \cdot \lambda^0}{r!}$$

$$= 1 - e^{-1.18}$$

$\therefore$ Prob. that in next 10 matches they score at least one goal:

$$P(\text{win bonus}) = (1 - e^{-1.8})^{10} = (0.8347)^{10}$$

$$= 0.164 \checkmark$$

3. The numbers, M and F, of male and female students who leave a particular school each year to study engineering have means 3·1 and 0·8 respectively.

(i) State, in context, one condition required for M to have a Poisson distribution. [1]

Assume that M and F can be modelled by independent Poisson distribution.

(ii) Find the probability that the total number of students who leave to study engineering in a particular year is more than 3.

(iii) Given that the total number of students who leave to study engineering in a particular year is more that 3, find the probability that no female students leave to study engineering in that year. ...[3]

S-18/73/Q4

Solution (i) Males leave at random. (to do engineering)
or Males leave indepent of other males leaving (to do eng)
or Number of males leaving each year at a constant rate (to do eng.)

(ii) For both males and females combined $\lambda = 3\cdot1 + 0\cdot8 = 3\cdot9$ (each year)

$P_0(3.9)$.

$$P(M+F>3) = 1 - P((M+F)=0,1,2,3)$$

$$= e^{-3.9}\left(1+3\cdot9+\frac{3.9^2}{2!}+\frac{3\cdot9^3}{3!}\right) = 0.54675$$

$$= \underline{0.547} \checkmark (3 \text{ sf})$$

(iii) $P(F=0 \text{ and } M>3)$ for $F \to \lambda_1 = 0.8$
for $M \to \lambda_2 = 3.1$

$$= P(F=0) \times P(M>3)$$

$$= e^{-0.8} \times \left[1 - P(M=0,1,2,3)\right]$$

$$= e^{-0.8} \cdot \left[1 - e^{-3.1}\left(1+3.1+\frac{3.1^2}{2!}+\frac{3.1^3}{3!}\right)\right]$$

$$= 0.16857$$

Now $P(F=0 \text{ and } M=3 / M+F>3) = \dfrac{P(F=0 \text{ and } M=3)}{P(M+F>3)}$
(Conditional Prob.)

$$= \frac{0.16857}{0.54675}$$

$$= \underline{0.308} \checkmark (3 \text{ sf})$$

§ Poisson distribution for different intervals:

In Poisson distribution, events occur at a constant rate; The mean average number of events in a given interval is proportional to that interval.

Example 4: Customers arrive at an enquiry desk at a constant rate of 1 every 5 minutes.

(i) State one condition for the number of customers arriving in a given period to be modelled by a Poisson distribution. --[1]

(ii) Find the probability that exactly 5 customers will arrive during a randomly chosen 30-minute period. --[2]

(iii) Find the probability that fewer than 3 customers will arrive during a randomly chosen 12-minute period. --[3]

[W-11/71/Q6]

Solution: (i) Customers arrive independently/singly and arrive at random (times)

(ii) for a 5 minute period $\lambda = 1$

$\therefore$ for a 30-minute period $\lambda = \frac{1}{5} \times 30 = 6$ ✓

$$\therefore P(X=5) = e^{-6} \cdot \frac{6^5}{5!} = \underline{0.161} ✓ \qquad \left\{ P(X=r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!} \right.$$

(iii) Now for a 12-minute period, $\lambda = \frac{1}{5} \times 12 = 2.4$

$$P(X<3) = P(X=0) + P(X=1) + P(X=2)$$

$$= e^{-2.4}\left(\frac{2.4^0}{0!} + \frac{2.4^1}{1!} + \frac{2.4^2}{2!}\right)$$

$$= \underline{0.570} ✓$$

Example 5. The number of planes arriving at an airport every hour day time is modelled by the random variable X with distribution $Po(5.2)$.

(i) State two assumptions required for the Poisson model to be valid in this context. ---[2]

(ii) (a) Find the probability that the number of planes arriving in a 15-minute period is greater than 1 and less than 4. ---[3]

(b) Find the probability that more than 3 planes will arrive in a 40-minute period. [M-17/72/Q7] --[2]

Solution (i) Planes arrive at constant mean rate.

Planes arrive at random.

(ii)(a) $P(5.2)$ and planes arrive every hour (60 minutes) $\Rightarrow \lambda = 5.2$

for 15 minutes period $\rightarrow \lambda = \frac{15}{60} \times 5.2 = 1.3$ ✓

$$P(1 < X < 4) = P(2) + P(3) \qquad \left\{ P(X=r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!} \right.$$

$$= e^{-1.3}\left[\frac{1.3^2}{2!} + \frac{1.3^3}{3!}\right]$$

$$= \underline{0.330} \checkmark$$

(b) In 40 minute period. $\lambda = \frac{40}{60} \times 5.2 = 3.467$

$$P(X>3) = 1 - P(0,1,2,3)$$

$$= e^{-3.467}\left[1 + 3.467 + \frac{3.467^2}{2!} + \frac{3.467^3}{3!}\right]$$

$$= 0.456 \ (3 \text{ s.f})$$

.6 The numbers of eagles seen per hour in a certain location has the distribution $Po(1.8)$. The number of vultures seen per hour in the same location has the independent distribution $Po(2.6)$

(i) Find the probability that, in a randomly chosen hour, at least 2 eagles are seen. ---[2]

(ii) Find the prob. that, in a randomly chosen half-hour period, the total number of eagles and vultures seen is less than 5, ---[3]

Alex wants to be at least 99% certain of seeing at least 1 eagle.

(iii) Find the minimum time for which she should watch for eagles. ---[3]

[M-19 | 72 | Q5]

Solution (i) For eagles $Po(1.8) \to \lambda = 1.8$

$$P(X \geq 2) = 1 - P(X = 0, 1)$$
$$= 1 - e^{-1.8}(1 + 1.8) = 1 - 0.463 = 0.537 \text{ (3 sf)}$$

(ii) Per hour $Po(1.8)$ and $Po(2.6)$
for half-hour period combine for both eagles and vultures:

$$\lambda = \frac{1}{2}(1.8 + 2.6) = 2.2$$

$Po(2.2)$

$$P(X+Y < 5) = P(X+Y = 0, 1, 2, 3, 4)$$
$$= e^{-2.2}\left[1 + 2.2 + \frac{2.2^2}{2!} + \frac{2.2^3}{3!} + \frac{2.2^4}{4!}\right] = 0.928 \text{ (3 sf)}$$

(iii) To watch eagles: let t hour period $\to \lambda = 1.8t$ $\quad Po(1.8t)$

$$P(X > 0) = 1 - e^{-1.8t} \geq 0.99 \quad (99\%)$$
$$\Rightarrow e^{-1.8t} \leq 0.01$$
$$\Rightarrow -1.8t \leq \ln 0.01$$
$$\Rightarrow -1.8t \leq -4.60517$$
$$t \geq \frac{-4.60517}{-1.8} = 2.5584$$
$$t \geq 2.56 \text{ (3sf)}$$

She must watch for at least 2.56 (hours)

§ Poisson distribution as an approximation to binomial distribution:

$X \sim B(n, p)$, when the value of 'n' is large and 'p' is small, such that np is moderate ($n > 50$ and $np \leq 5$ (or $p < 0.1$), the Poisson distribution with mean np can be used as an approximation for the binomial distribution.

$X \sim B(n, p) \longrightarrow P_o(\lambda)$ where $\lambda = np$.

Example 7: On average, 1 in 400 microchips made at a certain factory are faulty. The number of faulty microchips in a random sample of 1000 is denoted by X.

(a) State the distribution of X, giving the values of any parameters.--[1]

(b) State an approximating distribution for X, giving the values of parameters.-[2]

(c) Use this approximating distribution to find each of the following.
(i) $P(X = 4)$ (ii) $P(2 \leq X \leq 4)$ ---[2]+[2]

(d) Use a suitable approximating distribution to find the probability that, in a random sample of 700 microchips, there will be at least 1 faulty one.

[M-21/62/Q4] ---[3]

Solution (a) $p = \frac{1}{400}$, $n = 1000 \Rightarrow$ Binomial distribution: $X \sim B(1000, \frac{1}{400})$.

(b) n is very large and p is small hence Binomial distribution approximates to Poisson: $\lambda = np = 1000 \times \frac{1}{400} = 2.5 < 5$
or $P_o(2.5)$

(c) (i) $P(X = 4) = e^{-2.5} \times \frac{2.5^4}{4!} = \underline{0.134}$ (3 s.f) $\{P(X=r) = e^{-\lambda} \cdot \frac{\lambda^r}{r!}$

(ii) $P(2 \leq X \leq 4) = P(X=2) + P(X=3) + P(X=4)$

$= e^{-2.5}\left[\frac{2.5^2}{2!} + \frac{2.5^3}{3!} + \frac{2.5^4}{4!}\right] = \underline{0.604}$ (3 s.f.)

(d) Now $\lambda = \frac{1}{400} \times 700 = 1.75$ $\{\lambda = np\}$

$P(X \geq 1) = 1 - P(X = 0)$ $P(0) = e^{-\lambda}$

$= 1 - e^{-1.75} = \underline{0.826}$

Example 8. The booklets produced by a certain publisher contain, on average, 1 incorrect letter per 30000 letters and these errors occur randomly. A randomly chosen booklet from this publisher contains 12500 letters.
Use a suitable approximating distribution to find the probability that this booklet contains atleast 2 errors. --- [3]
[M-20/ 62/ Q1]

Solution: $\lambda = np = 12500 \times \frac{1}{30,000} = \frac{5}{12} = 0.417 < 5$ $\left\{ \begin{array}{l} p = 1/30000 \\ n = 12,500 \end{array} \right.$

$\boxed{B(n,p) \sim Po(\lambda)}$

$P(X \geq 2) = 1 - P(X=0 \text{ or } X=1)$

$= 1 - e^{-0.417}(1 + 0.417) = 0.0661$ ✓ $\left\{ P(X=2) = \frac{e^{-\lambda} \cdot \lambda^2}{2!} \right.$

Example 9: In the data-entry department of a certain firm, it is known that 0.12% of data items are entered incorrectly, and that these errors occur randomly and indepently.

(a) A random sample of 3600 data items is chosen. The number of these data items that are incorrectly entered is denoted by X.

(i) State the distribution X, including the values of any parameters. --[1]

(ii) State an appropriate approximating distribution for X, including the values of parameters. Justify your choice of approx. distribution. --[3]

(iii) Use your approximating distribution to find $P(X > 2)$ ---[2]

(b) Another large random sample of 'n' data items is chosen. The prob. that the sample contains no data items that are entered incorrectly is more than 0.1. Find the largest value of n. [S-20/ 62/ Q3] --[3]

Solution (a) (i) $B(3600, 0.0012)$ $\{ p = 0.12\%$

(ii) $Po(4.32)$; $\left. \begin{array}{l} n = 3600 \\ p = .0012 \end{array} \right\} \Rightarrow \lambda = np = 4.32 < 5$

$\boxed{B(n,p) \rightarrow Po(\lambda)}$

(iii) $P(X > 2) = 1 - P(0, 1, 2)$

$= 1 - e^{-4.32}\left[1 + 4.32 + \frac{4.32^2}{2}\right]$

$= \underline{0.805}$ ✓ (3 s.f)

(b) $P(X = 0) = e^{-\lambda} > 0.1$ (Given)

$\Rightarrow -\lambda > \ln 0.1$

$\Rightarrow \lambda < -\ln 0.1$

$\Rightarrow \lambda < \ln 10$

$\Rightarrow 0.0012n < \ln 10$

$\Rightarrow n < \frac{\ln 10}{0.0012}$

$n < 1918.8$

$\therefore$ Largest value of $n = \underline{1918}$ ✓

Example 10: On average, 1 in 75 000 adults has a certain genetic disorder.

(a) Use a suitable approximating distribution to find the probability that, in a random sample of 10 000 people, at least 1 has a genetic disorder. ---[3]

(b) In a random sample of n people, where n is large, the probability that no-one has the genetic disorder is more than 0·9.
Find the largest possible value of n. ---[4]

S-21/61/Q5

Solution (a) $n = 10\,000,\ p = \frac{1}{75000}$ $\quad$ n is large and p is small

$np = 10000 \times \frac{1}{75000} = \frac{2}{15} < 5$

hence it is Poisson distribution. $\lambda = \frac{2}{15}$

$Po\left(\frac{2}{15}\right)$ $\qquad$ $B(n,p) \longrightarrow Po(\lambda)$

$P(X \geq 1) = 1 - P(X=0) = 1 - e^{-2/15}$ $\qquad \{ P(0) = e^{-\lambda}$

$= \underline{0{\cdot}125}$ (3 sf)

(b) $\lambda = \frac{n \times 1}{75\,000}$ $\qquad \left\{ \lambda = np = n \times \frac{1}{75{,}000} \right.$

$P(X=0) = e^{-\frac{n}{75000}} > 0{\cdot}9$ (Given)

$\Rightarrow -\frac{n}{75000} > \ln 0{\cdot}9$

$\Rightarrow -n > -0{\cdot}10536 \times 75000$

$\Rightarrow n < 7902{\cdot}04$

$\therefore$ Largest value of $\underline{n = 7902}$

11. Most plants of a certain type have three leaves, However, it is known that, on average, 1 in 10000 of these plants have four leaves and plants with four leaves are called "lucky". The number of lucky plants in a random sample of 25000 plants is denoted by X.

(a) State, with a justification, an approximating distribution for X, giving the values of any parameters. ---[2]

Use your approximating distribution to answer parts (b) and (c)

(b) Find $P(X \leq 3)$ ---[2]

(c) Given that $P(X=k) = 2P(X=k+1)$, find k. ---[2]

The number of lucky plants in a random sample of n plants, where n is large, is denoted by Y.

(d) Given that $P(Y \geq 1) = 0.963$, correct to 3 significant figures, use a suitable approximating distribution to find the value of n. ---[3]

S-21/63/Q5

Solution: $n = 25000,\ p = \frac{1}{10{,}000}$

(a) $\lambda = np = 25000 \times \frac{1}{10{,}000} = 2.5 < 5$

$n > 50$ and $p = 0.0001 < 0.1$

$\therefore$ Poission distribution $Po(\lambda)$
(as an approx. to Binomial dis) $= \underline{Po(2.5)}$ ✓

(b) $P(X=r) = e^{-\lambda} \cdot \frac{\lambda^r}{r!}$

$P(X \leq 3) = P(X=0,1,2,3)$
$= e^{-2.5} \times \left(1 + 2.5 + \frac{2.5^2}{2!} + \frac{2.5^3}{3!}\right)$
$= \underline{0.758}$ (3sf)

(c) $P(X=k) = 2P(X=k+1)$
$\Rightarrow e^{-2.5} \times \frac{2.5^k}{k!} = 2 \times e^{-2.5} \times \frac{2.5^{k+1}}{(k+1)!}$
$\Rightarrow k+1 = 5$
$\Rightarrow \underline{k = 4}$ ✓

(d) $P(Y \geq 1) = 0.963$
$\Rightarrow 1 - P(Y=0) = 0.963$
$\Rightarrow 1 - e^{-\lambda} = 0.963$
$\Rightarrow e^{-\lambda} = 0.037$
$\Rightarrow -\lambda = \ln 0.037$
$\Rightarrow \lambda = -(-3.296837\ldots)$
$np = 3.296837$
$n \times 0.0001 = 3.296837$
$n = 32968.37$
$\underline{n = 33000}$ ✓ (3sf)

12 It is known that, on average, 1 in 300 flowers of a certain kind are white. A random sample of 200 flowers of this kind is selected.

(a) Use an approximating distribution to find the probability that more than 1 flower in the sample is white. ---[3]

(b) Justify the approximating distribution used in part (a) ---[1]

The probability that a randomly chosen flower of another kind is white is 0.02. A random sample of 150 of these flowers is selected.

(c) Use an appropriate approximating distribution to find the probability that the total number of white flowers in two samples is less than 4. ---[3]

W-20/61/Q1

Solution:

(a) $P(\text{white}) = p = \frac{1}{300}$, $n = 200 \Rightarrow \lambda = np = \frac{1}{300} \times 200 = \frac{2}{3} < 5$

$Po(2/3)$

$$P(X>1) = 1 - P(X=0,1) = 1 - e^{-2/3}\left(1+\frac{2}{3}\right) \quad \left\{P(X=r) = e^{-\lambda} \cdot \frac{\lambda^r}{r!}\right.$$

$$= 1 - 0.856\ldots$$

Using Poisson distribution as an approximation to Binomial dis. $= 0.144$ (3 s.f.)

(b) $n > 50$ and $np = \frac{2}{3} < 5$ (or $p = \frac{1}{300} < 0.1$ and $n > 50$)

(c) for the first kind $\lambda_1 = 2/3$

Now for another kind: $\lambda_2 = np = 150 \times 0.02 = 3 < 5$

$\therefore$ for combined $\lambda = \lambda_1 + \lambda_2 = \frac{2}{3} + 3 = \frac{11}{3} < 5$

$$\therefore P(X<4) = P(X=0,1,2,3)$$

$$= e^{-\frac{11}{3}}\left(1 + \frac{11}{3} + \frac{\left(\frac{11}{3}\right)^2}{2!} + \frac{\left(\frac{11}{3}\right)^3}{3!}\right) \quad \left(\because \lambda = \frac{11}{3}\right)$$

$$= \underline{0.501} \text{ (3 s.f.)}$$

13. (i) The random variable X has a distribution $B(300, 0.01)$. Use a Poisson approximation to find $P(2<X<6)$ ---[3]

(ii) The random variable Y has the distribution $Po(\lambda)$, and $P(Y=0) = P(Y=2)$. Find λ. ---[2]

(iii) The random variable Z has the distribution $Po(5.2)$ and it is given that $P(Z=n) < P(Z=n+1)$.

(a) Write down an inequality in n. ---[1]

(b) Hence or otherwise find the largest possible value of n. ---[2]

W-19/73/Q5

Solution (i) $B(300, 0.01) \sim Po(\lambda)$; $\lambda = np = 300\times0.01 = 3 < 5$

$P(2<X<6) = P(X=3,4,5)$

$= e^{-3}\left(\frac{3^3}{3!}+\frac{3^4}{4!}+\frac{3^5}{5!}\right) = \underline{0.493}$ (3sf) $\left\{P(X=r) = e^{-\lambda}\frac{\lambda^r}{r!}\right.$

$\boxed{B(n,p) \rightarrow Po(\lambda)}$

(ii) $P(0) = P(2) \Rightarrow e^{-\lambda} = e^{-\lambda}\cdot\frac{\lambda^2}{2} \Rightarrow \lambda^2 = 2 \Rightarrow \lambda = \sqrt{2}$ or $\underline{1.41}$ (3sf)

(iii)(a) $Po(5.2)$, $\lambda = 5.2$

$P(Z=n) < P(Z=n+1)$

$\Rightarrow e^{-5.2}\cdot\frac{5.2^n}{n!} < e^{-5.2}\cdot\frac{5.2^{n+1}}{(n+1)!}$

(b) $\Rightarrow 1 < \frac{5.2}{n+1} \Rightarrow n+1 < 5.2 \Rightarrow n < 4.2$

$\therefore$ largest value of $\underline{n=4}$

14. A random variable X has the distribution $B(75, 0.03)$

(i) Use Poisson approximation to the binomial distribution to calculate $P(X<3)$ ---[3]

(ii) Justify the use of the Poisson approximation. ---[1]

S-18/73/Q1

Solution (i) $B(75, 0.03) \rightarrow n=75,\ p=0.03 \Rightarrow \lambda = np = 75\times0.03 = 2.25 \le 5$

$\sim Po(2.25) \rightarrow P(X<3) = e^{-2.25}\left(1+2.25+\frac{2.25^2}{2!}\right) = \underline{0.609}$ (3sf)

(ii) $n = 75 > 50$ and $\lambda = 2.25 < 5$

n is large.

$\boxed{B(n,p) \rightarrow Po(\lambda)}$

§ Normal distribution as an approximation to Poisson distribution:

For $\lambda > 15$, the Poisson distribution with λ can be approximated by the normal distribution with mean $\mu = \lambda$ and variance $\sigma^2 = \lambda$, with a continuity correction applied.

The accuracy of this approximation improves as λ increases

$$X \sim Po(\lambda) \to N(\mu, \sigma^2) \to N(\lambda, \lambda) \text{ for } \lambda > 15.$$

Example 15. The number of enquiries received per day at a customer service desk has a Poisson distribution with mean 45.2. If more than 60 enquires are received in a day, the customer service desk cannot deal with them all.

Use a suitable approximating distribution to find the probability that, on a randomly chosen day, the customer service desk cannot deal with all the enquires that are received. ---[4]

[W-21/61/Q2]

Solution: $Po(\lambda) = P_o(45.2) \Rightarrow \lambda = 45.2$ ($\lambda > 15$, leads to Normal distribution)

Poisson distribution approaches $\rightarrow$ Normal distribution $N(\mu, \sigma^2)$

$$\rightarrow N(\lambda, \lambda) = N(45.2, 45.2) \{\mu = \sigma^2 = \lambda\}$$

$$P(X > 60) = P\left(z > \frac{X-\mu}{\sigma}\right) = P\left(Z > \frac{X-\lambda}{\sqrt{\lambda}}\right) = P\left(z > \frac{60.5-45.2}{\sqrt{45.2}}\right)$$

$$= P(z > 2.276)$$

$$= 1 - P(z < 2.276)$$

$$= 1 - \phi(2.276)$$

$$= 1 - 0.9886$$

$$= 0.0114$$

Continuity correction.
$x > 60 \rightarrow x \geq 61$
$x \rightarrow 60.5$

Example 16. In a certain document, typing errors occur at random and at a constant mean rate of 0.2 per page.

(a) Find the probability that there are fewer than 3 typing errors in 10 randomly chosen pages. ---[2]

(b) Use an approximating distribution to find the probability that there are more than 50 typing errors in 200 randomly chosen pages. ---[4]

In same document, formatting errors occurs at random and at a constant mean rate of 0.3 per page.

(c) Find the probability that the total number of tying and formatting errors in 20 randomly chosen pages is between 8 and 11 inclusive. ---[3]

W-21/62/Q5

Solution: (a) $n = 10,\ p = 0.2 \Rightarrow \lambda = np = 10 \times 0.2 = 2 < 5$, Use $Po(2)$

$$P(X<3) = P(X=0, X=1, X=2) = e^{-\lambda}\left[1+\lambda+\frac{\lambda^2}{2!}\right) \qquad \left\{ P(X=r) = \frac{e^{-\lambda}\cdot\lambda^r}{r!} \right.$$

$$= e^{-2}\left[1+2+\frac{2^2}{2!}\right) = \underline{0.677} \checkmark (3\text{s.f})$$

(b) $n = 200,\ p = 0.2 \Rightarrow \lambda = np = 200 \times 0.2 = 40 > 15$

$\therefore$ Poisson distribution approximating to Normal distribution $N(\mu, \sigma^2)$

$\mu = \sigma^2 = \lambda = 40 \quad \therefore X \sim N(40, 40) \checkmark$

$$P(X > 50) = P\left(z > \frac{50.5 - 40}{\sqrt{40}}\right) \qquad \left\{ \begin{array}{l} \text{Continuity correction} \\ P(X>50) = P(X \geq 51) \sim N \to X > 50.5 \end{array} \right.$$

$$= P(z > 1.660) = 1 - \phi(1.660)$$
$$= 1 - 0.9515$$
$$= \underline{0.0485} \checkmark$$

(c) Now $n = 20$, p = typing and formatting error per page $= 0.2 + 0.3$, $p = 0.5 \checkmark$

$\therefore \lambda = np = 20 \times 0.5 = 10 \checkmark$

$$P(8 \leq X \leq 11) = P(X = 8, 9, 10, 11) = e^{-\lambda}\left[\frac{\lambda^8}{8!} + \frac{\lambda^9}{9!} + \frac{\lambda^{10}}{10!} + \frac{\lambda^{11}}{11!}\right)$$

$$= e^{-10}\left(\frac{10^8}{8!} + \frac{10^9}{9!} + \frac{10^{10}}{10!} + \frac{10^{11}}{11!}\right)$$

$$= \underline{0.477} \checkmark (3\text{sf})$$

17. The number of phone calls arriving in a 10-minute period at a Switchboard is modelled by the random variable X which has the distribution $Po(4.1)$. Use an approximating distribution to find the probability that more then 90 calls arrive in a 4-hour period. --[5]

[M-18/72/Q2]

Solution: For 10-minute period $\lambda = 4.1 \Rightarrow$ in 4-hour period, $\lambda = \frac{4.1}{10} \times 4 \times 60 = 98.4 > 15$

now $Po(98.4) \sim N(98.4, 98.4) \rightarrow \mu = \sigma^2 = 98.4$

Continuity Correction
$x > 90 \rightarrow x \geq 91$
$x \rightarrow 90.5$

$$P(X > 90) = P\left(Z > \frac{90.5 - 98.4}{\sqrt{98.4}}\right) = P(Z > -0.796) = P(Z < 0.796)$$

$$= \phi(0.796) = 0.787 \text{ (3 sf)}$$

18. The number of e-readers sold in 10-day period in a Shop is modelled by the distribution $Po(5.1)$. Use an approximating distribution to find the probability that fewer than 140 e-readers are sold in a 300-day period. --[4]

[S-18/71/Q3]

Solution: In 10 day-period $\lambda = 5.1 \Rightarrow$ in 300 day-period, $\lambda = \frac{5.1}{10} \times 300 = 153 > 15$

$\therefore Po(153) \sim N(153, 153) \rightarrow \mu = 153 = \sigma^2$

$$P(X < 140) = P\left(Z < \frac{139.5 - 153}{\sqrt{153}}\right)$$

continuity Correction
$X < 140 \Rightarrow X \leq 139$
for Normal ~ 139.5

$$= P(Z < -1.091)$$

$$= 1 - \phi(1.091)$$

$$= 1 - 0.862$$

$$= 0.138 \text{ (3 sf)}$$