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F.Me

Further Mechanics

Circular Motion
Notes and Revision

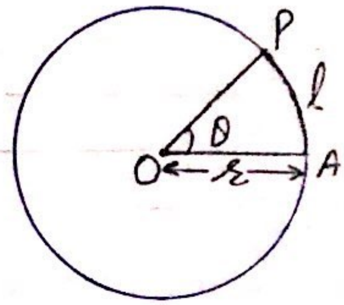
SP-20	S-20	W-20	S-21

Suresh Goel
(Former Director)
Alliance World School
Noida, Delhi NCR
INDIA

(+91. 9810444804)

Circular Motion - Notes and Revision.

- **Length of arc:**

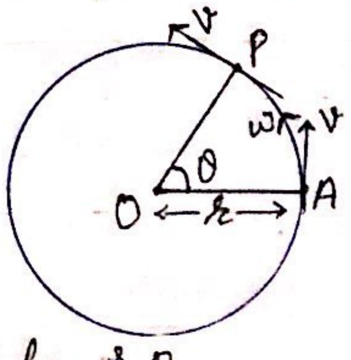


Given a circle with centre at O and radius r and the arc $\widehat{AP}$ of the circle of length l, subtends an angle θ radians at centre O.

Then, Length of arc $l = r\theta$

(Note: 'θ' is measured in radians; π rad $= 180°$)

- **Speed of a particle moving in a circular path:**



If a particle P is moving in a circular path of radius 'r' with constant speed v.

$$v = \frac{dl}{dt}$$

$$v = \frac{d}{dt}(r\theta) \quad [\because l = r\theta$$

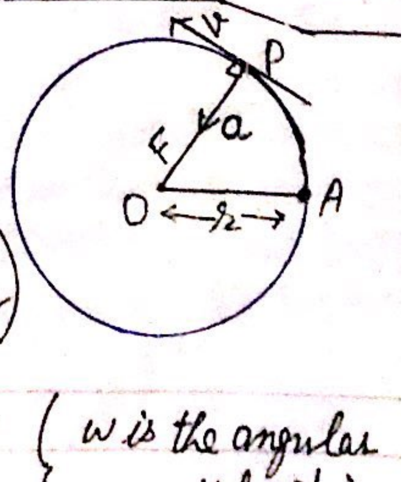
$$v = r.\frac{d\theta}{dt} = r\omega \quad \left[\omega = \frac{d\theta}{dt}\right.$$

$\therefore \underline{V = r\omega}$; here ω is the angular speed.
(ω is measured in radians per seconds.

or $\omega = \frac{V}{r}$

- Note: One complete circle is 2π radians.
  ∴ Time to complete one revolution $T = \frac{2\pi}{\omega}$ ✓

- **Acceleration 'a' of a particle, moving in a circle of radius r, and with constant speed v.**



Acceleration $a = \frac{v^2}{r}$ ✓ (or $\frac{(r\omega)^2}{r} = r\omega^2$ ✓)

Force toward the centre F.

$F = ma = \frac{mv^2}{r}$ or $\underline{mr\omega^2}$ (ω is the angular Velocity)

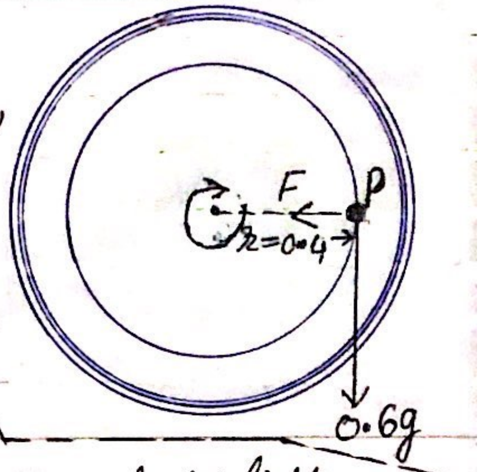
Example 1. A particle P of mass 0.6 Kg is on the rough surface of a horizontal disc with centre O. The distance OP is 0.4 m. The disc and the particle P rotate with angular speed 3 rad s^{-1}, about a vertical axis passing through O. Find the magnitude of the frictional force which the disc exerts on the particle, and state the direction of this force. [M2 / S-15 / 52 / Q1] ---[3]

Solution: Force of friction $F = ma$

$$= m \cdot w^2 \cdot r$$
$$= 0.6 \times 3^2 \times 0.4$$
$$= 2.16 \text{ N.}$$

$\left\{ \begin{array}{l} m = 0.6 \text{ kg} \\ w = 3 \text{ rad } s^{-1} \\ r = 0.4 m \end{array} \right.$

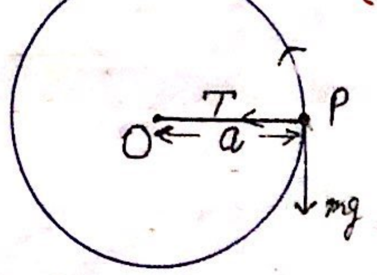
direction of force is $\vec{PO}$.



Example 2: A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O on a smooth horizontal plane. The particle P moves in horizontal circle about O. The tension in the string is 4mg. Find in terms of a and g, the time that P take to make one revolution. [S-20 / 33 / Q1 / ---[2]

Solution:

$$T = maw^2 = 4mg \text{ (given)} \left[\begin{array}{l} T = mrw^2 \\ \text{radius } r = a \end{array} \right.$$



$$\Rightarrow \quad w^2 = \frac{g}{a} \Rightarrow w = \sqrt{\frac{g}{a}} \quad ---(i)$$

Time per revolution $= \frac{2\pi}{w}$

$$= \frac{2\pi}{\sqrt{\frac{g}{a}}} \qquad \left(\text{from (i) } w = \sqrt{\frac{g}{a}}\right.$$

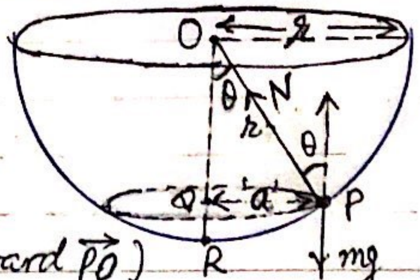
$$\text{Time} = 2\pi \cdot \sqrt{\frac{a}{g}} \checkmark$$

3. A partical 'P' of mass 'm' is moving in a horizontal circle with angular speed 'w' on the smooth inner surface of a hemispherical shell of radius 'r'. The angle between the vertical and the normal section of the surface on 'P' is 'θ'.

(a) Show that $\cos\theta = \frac{g}{w^2 r}$ ---[3]

The plane of the circular motion is at a height 'x' above the lowest point of the shell. When the angular speed is doubled, the plane of the motion is at a height '4x' above the lowest point of the shell.

(b) Find x in terms of r. [W-20/32/Q4] ---[4]



Solution: Let the partical is moving in a horizontal circle with centre at 'Q' and radius 'a'.

angle $POQ = \theta$; $a = r\sin\theta$

Let the normal section at $P = N$ (Toward $\vec{PO}$)

Vertical component of N at P; $N\cos\theta = mg$ --- ①

Horizontal component of N at P,

$N\sin\theta = m \cdot r\sin\theta \cdot w^2$ [maw^2; $a = r\sin\theta$

$\Rightarrow N = mrw^2$ --- ②

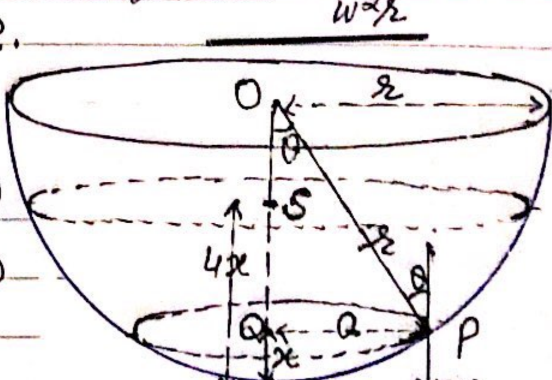
from ① and ② $\cancel{m}rw^2 \cdot \cos\theta = \cancel{m}g \Rightarrow \cos\theta = \frac{g}{w^2 r}$ --- ③

(b) Let the lowest point of the shell is at R.

$QR = x$, In ΔOQP, $\cos\theta = \frac{OQ}{OP}$

$\Rightarrow \cos\theta = \frac{r-x}{r} = \frac{g}{w^2 r}$ (from ③)

$\Rightarrow r - x = r \cdot \frac{g}{w^2 r}$ --- ④



In new situation:

OS : $r - 4x = r \cdot \frac{g}{(2w)^2 r}$ [New angular speed = 2w]

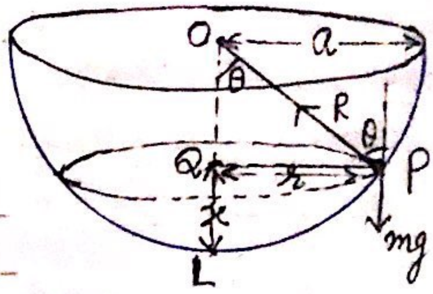
$r - 4x = r \cdot \frac{g}{4w^2 r}$ --- ⑤

from ④ and ⑤ $r - x = 4(r - 4x)$ [from ④ and ⑤]

$\Rightarrow x = \frac{1}{5}r$ ✓

4. A hollow hemispherical bowl of radius 'a' has a smooth inner surface and is fixed with its axis vertical. A partical 'P' of mass 'm' moves in horizontal circles on the inner surface of the bowl, at a height 'x' above the lowest point of the bowl. The speed of P is $\sqrt{\frac{8}{3}ga}$. Find x in terms of a. --[6]

[S-21/31/Q2]



Solution: Let the lowest point of the bowl is at L.
Given $QL = x$, Let radius of circle $= r$
Normal reaction R at P is towards $\overrightarrow{PO}$
Vertical component of R: $R\cos\theta = mg$ ---(1)
Horizontal component of R, $R\sin\theta = \frac{mv^2}{r}$

$$R\sin\theta = \frac{m\cdot\frac{8}{3}g\cancel{a}}{\cancel{a}\sin\theta} \qquad \left[\begin{array}{l} r = a\sin\theta \\ v = \sqrt{\frac{8}{3}ga} \end{array}\right.$$

$$\Rightarrow R\sin^2\theta = \frac{8}{3}mg$$

$$\Rightarrow \cancel{R}(1-\cos^2\theta) = \frac{8}{3}\times\cancel{R}\cos\theta \quad \left[\begin{array}{l}\text{from (1)} \\ mg = R\cos\theta\end{array}\right]$$

$$\Rightarrow 3(1-\cos^2\theta) = 8\cos\theta$$

$$\Rightarrow 3\cos^2\theta + 9\cos\theta - 3 = 0$$

$$\Rightarrow (3\cos\theta - 1)(\cos\theta + 3) = 0$$

$$\cos\theta = \frac{1}{3} \text{ ---(2)} \quad \text{or} \quad \cos\theta = -3 \times$$

In ΔOQP.

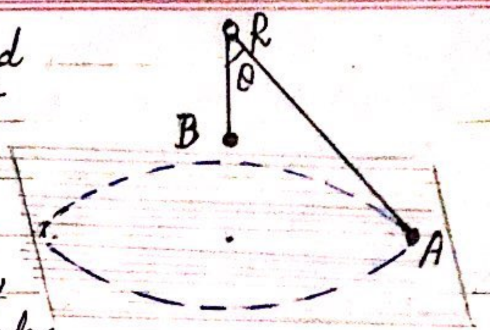
$$\frac{OQ}{OP} = \cos\theta$$

$$\Rightarrow \frac{a-x}{a} = \frac{1}{3} \qquad \left\{\begin{array}{l} OP = OL = a \\ OQ = OL - QL = a - x \\ \cos\theta = \frac{1}{3} \text{ from (2)} \end{array}\right.$$

$$\Rightarrow 3(a-x) = a$$

$$\Rightarrow x = \frac{2}{3}a \checkmark$$

5. Particals 'A' and 'B', of masses $3m$ and m respectively, are connected by a light inextencible string of length 'a' and that passes through fixed smooth ring R. Particle 'B' hangs in equilibrium vertically below the ring. A moves in horizonal cicles on a smooth horizontal surface with speed $\frac{2}{5}\sqrt{ga}$. The angle between AR and BR is θ. The normal rection between A and the surface is $\frac{12}{5}mg$.



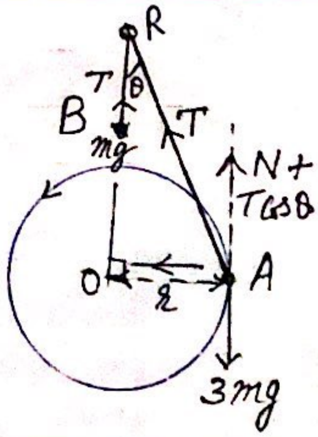
(a) Find $\cos\theta$. --[3]

(b) Find in terms of a, the distance of B below the ring. --[3]

$\boxed{S\text{-}21/33/Q3}$

Solution: Given Normal reaction at A; $N = \frac{12}{5}mg$ --(1)

(a) For B; $T = mg$ ---(2)



Vertical Component. For A, $N + T\cos\theta = 3mg$

$\Rightarrow \frac{12}{5}mg + T\cos\theta = 3mg$ $\left[\because N = \frac{12}{5}mg\right]$

$\Longrightarrow T\cos\theta = \frac{3}{5}mg$ ---(3)

from (2) and (3) $mg.\cos\theta = \frac{3}{5}mg$

$\Rightarrow \cos\theta = 3/5$ ✓ ———(4)

(b) Horizontal Component of T at A: towards $\overrightarrow{AO}$

$T\sin\theta = \frac{Mv^2}{r}$ $\left[\because M = 3m,\ v = \frac{2}{5}\sqrt{ga}\right.$

(rt Δ ROA: $r = AR\sin\theta$

$\Rightarrow \cancel{mg}.\sin\theta = \frac{3\cancel{m}.\frac{4}{5}ga}{AR\sin\theta}$ ($T = mg$

$\Rightarrow AR.\sin^2\theta = \frac{12}{25}a$

$AR.\left(\frac{4}{5}\right)^2 = \frac{12}{25}a$

from (4) $\cos\theta = 3/5$

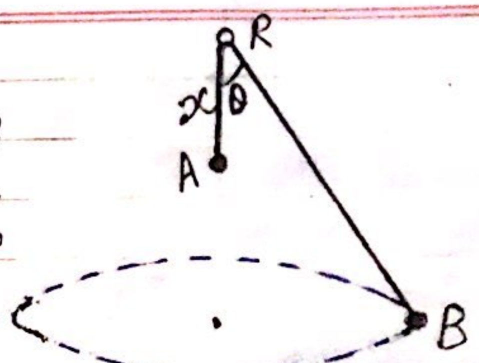
$\Rightarrow \sin\theta = \frac{4}{5}$

$\Rightarrow AR = \frac{12}{25}a \times \frac{25}{16} = \frac{3}{4}a$

$\therefore BR = a - AR = a - \frac{3}{4}a$ $[AR + BR = a]$

$BR = \frac{1}{4}a$

6. A light inextensible string of length 'a' is threaded through a fixed smooth ring 'R'. One end of the string is attached to a particle 'A' of mass $3m$. The other end of the string is attached to a particle 'B' of mass m. The particle 'A' hangs in equilibrium at a distance x vertically below the ring. The angle between AR and BR is 'θ'. The particle B moves in a horizonel circle with constant angular speed $2\sqrt{\frac{g}{a}}$.



Show that $\cos\theta = \frac{1}{3}$ and find x interms of a. ---[5]

S-20/31/Q2

Solution: Length of string $AR + RB = a \Rightarrow x + RB = a$

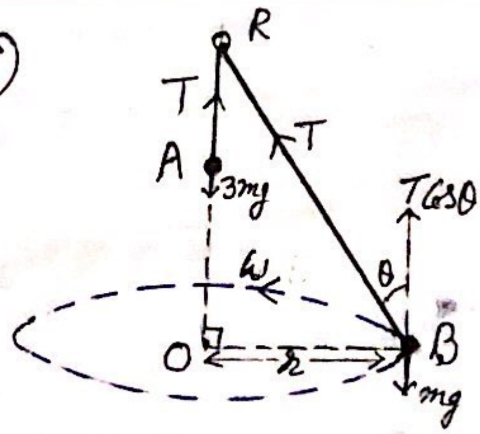
$\Rightarrow RB = (a-x)$ --- (i)

For A; Vertically; $T = 3mg$ --- (ii)

For B; Vertical Component; $T\cos\theta = mg$ --- (iii)

from (ii) and (iii) $3mg\cos\theta = mg$

$\Rightarrow \cos\theta = \frac{1}{3}$ ✓ --- (iv)



For B; horizontal Component.

$T\sin\theta = mr\omega^2$

$\Rightarrow T\,\cancel{\sin\theta} = m\cdot(a-x)\cdot\cancel{\sin\theta}\cdot\frac{4g}{a}$

$\Rightarrow 3\cancel{m}g = \cancel{m}(a-x)\cdot\frac{4g}{a}$

$\Rightarrow 3a = 4(a-x)$

$\Rightarrow 4x = a$

$\Rightarrow x = \frac{a}{4}$ ✓

In rt ΔROB.

$\frac{OB}{RB} = \sin\theta$

$OB = RB\sin\theta$

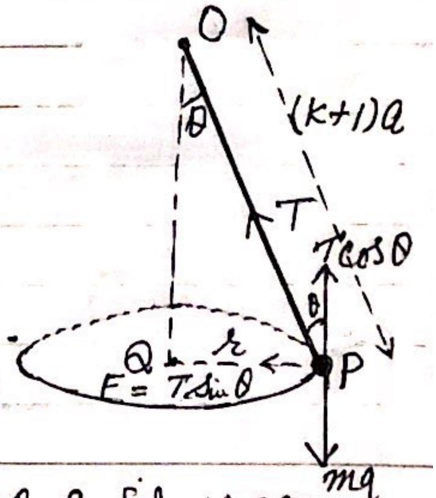
$r = (a-x)\sin\theta$

$\omega = 2\sqrt{\frac{g}{a}} \Rightarrow \omega^2 = \frac{4g}{a}$

7. One end of a light elastic string, of natural length 'a' and modulus of elasticity '4mg', is attached to a fixed point O. The other end of the string is attached to a particle of mass 'm'. The particle moves in a horizontal circle with a constant angular speed $\sqrt{\frac{g}{a}}$ with the string inclined at an angle θ to the downward vertical through O. The length of the string during this motion is $(k+1)a$.

(a) Find the value of k. ---[4]

(b) Find the value of $\cos\theta$. $\boxed{W\text{-}20 \,|\, 31 \,|\, Q3}$ ---[2]



Solution: Natural length $L = a$.

(a) Length of extened string $OP = (k+1)a$

$$\text{Extension } e = (k+1)a - a = ka ✓$$

Angular speed $w = \sqrt{g/a}$, $\lambda = 4mg$

Using Hook's Law:

$$T = \frac{\lambda e}{L} = \frac{4mg \cdot ka}{a} = 4kmg \text{ --- (1)}$$

For circular motion:

Force towards centre; $F = mrw^2 = m \cdot a(k+1)\sin\theta \cdot \frac{g}{a}$

$\Rightarrow T \cdot \cancel{\sin\theta} = m(k+1)\cancel{a} \cdot \frac{g}{\cancel{a}} \cancel{\sin\theta}$

In rt $\triangle OPQ$
$\frac{PQ}{OP} = \sin\theta$
$r = PQ = OP\sin\theta$
$r = (k+1)a\sin\theta ✓$
$w = \sqrt{\frac{g}{a}} \Rightarrow w^2 = \frac{g}{a}$

$$T = m(k+1)g \text{ ---- (ii)}$$

from (i) and (ii) $m(k+1)g = 4kmg$

$$\Rightarrow k+1 = 4k \Rightarrow 3k = 1 \Rightarrow k = \frac{1}{3} ✓$$

(b) Vertical component at P:

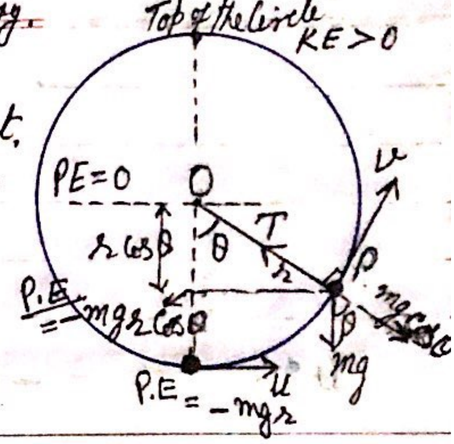
$T = 4kmg$ from (1)
$\therefore T = 4 \times \frac{1}{3} mg$ for $k = \frac{1}{3}$ ✓

$$T\cos\theta = mg$$

$$\frac{4}{3} mg \cdot \cos\theta = mg$$

$$\Rightarrow \cos\theta = 3/4 ✓$$

Circular motion in Vertical circles - Notes



We can use the principle of conservation of energy. The partical is at rest in equilibrium before being projected with speed 'u' from the lowest point. Consider zero P.E. when the particle is at a height level with the centre of the circle O.

Initially, $PE = -mgr$; $K.E = \frac{1}{2}mu^2$.

In general at 'P' $\rightarrow \begin{cases} P.E = -mg.r\cos\theta \\ KE = \frac{1}{2}mv^2 \end{cases}$

Consider $F = ma$ towards the centre at P.

$$\Rightarrow T - mg\cos\theta = \frac{mv^2}{r} \quad \text{---- (i)}$$

and from the conservation of energy:

$$\frac{1}{2}mu^2 - mgr = \frac{1}{2}mv^2 - mg.r\cos\theta$$

$$\Rightarrow \frac{mv^2}{r} = \frac{mu^2}{r} - 2mg + 2mg\cos\theta \quad \text{---- (ii)}$$

from (i) and (ii)

$$T - mg\cos\theta = \frac{mu^2}{r} - 2mg + 2mg\cos\theta \quad \text{--- (iii)}$$

If the tension are such that $T \geq 0$

Then from (iii)

$$\frac{mu^2}{r} \geq 2mg - 3mg\cos\theta \quad \text{--- (iv)}$$

Now at the top point $\theta = 180° \Rightarrow \cos\theta = -1$

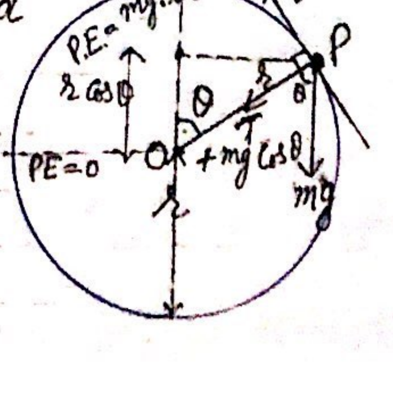
from (iv) $\frac{u^2}{r} \geq 5g$

$$\Rightarrow u \geq \sqrt{5gr} \checkmark$$

This the condition for complete vertical circles for a circle of radius r. $u \geq \sqrt{5gr}$

Note: If $\overrightarrow{OP}$ makes an angle 'θ' with the upward vertical through O. Then

$$T + mg\cos\theta = \frac{mv^2}{r}$$

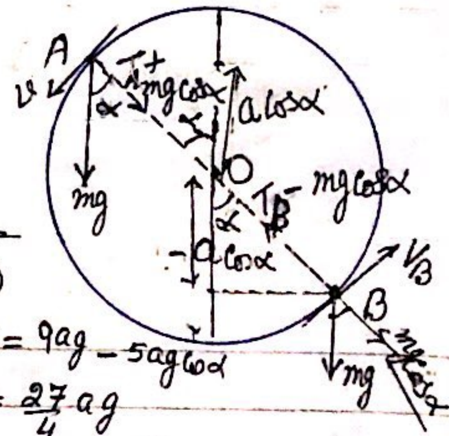


8. A particle 'P' of mass 'm' is attached to one end of a light inextensible string of length 'a'. The other end of the string is attached to a fixed point O. The particle 'P' is moving in a complete vertical circle about O. The points A and B are on the circle, at the opposite ends of a diameter, and such that OA makes an acute angle 'α' with the upward vertical through O. The speed of P, as it passes through A is $\frac{3}{2}\sqrt{ag}$. The tension in the string when P is at B is four times the tension in the string when P is at A.

(i) Show that $\cos\alpha = 3/4$ ---[6]

(ii) Find the tension in the string when P is at B. ---[2]

$\boxed{S\text{-}19/23/Q2}$



Solution: For A, radial component: $F = ma$

(i) $T_A + mg\cos\alpha = \frac{mV_A^2}{a}$ $\left[\begin{array}{l} V_A = \frac{3}{2}\sqrt{ag} \\ V_A^2 = \frac{9}{4}ag \text{ --- (1)}\end{array}\right.$

$\Rightarrow T_A = \frac{m(\frac{9}{4}ag)}{a} - mg\cos\alpha$

$\Rightarrow T_A = \frac{9}{4}mg - mg\cos\alpha$ --- (2)

at B, tension T_B radially from $F = ma$,

$T_B - mg\cos\alpha = \frac{mV_B^2}{a}$

$\Rightarrow T_B = \frac{mV_B^2}{a} + mg\cos\alpha$ --- (3)

Using conservation of energy at B,

$\frac{1}{2}mV_B^2 = \frac{1}{2}mV_A^2 + 2mga\cos\alpha$

$\Rightarrow V_B^2 = \frac{9}{4}ag + 4ga\cos\alpha$ $\left\{\text{from (1) } V_A^2\right\}$

$V_B^2 = ag\left(\frac{9}{4} + 4\cos\alpha\right)$ --- (4)

Now given $T_B = 4T_A$

from (2) & (3) $\Rightarrow \frac{mV_B^2}{a} + mg\cos\alpha = 9mg - 4mg\cos\alpha$

$\Rightarrow V_B^2 = 9ag - 5ag\cos\alpha$ --- (5)

from (4) & (5)

$\frac{9ag}{4} + 4ag\cos\alpha = 9ag - 5ag\cos\alpha$

$\Rightarrow 9ag\cos\alpha = \frac{27}{4}ag$

$\Rightarrow \cos\alpha = \frac{3}{4}$ ✓ --- (6)

(ii) from (2) $T_A = \frac{9}{4}mg - mg\cos\alpha$

$= \frac{9}{4}mg - mg \cdot \frac{3}{4}$ (∵ (6))

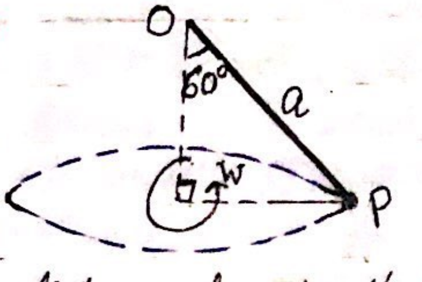
$T_A = \frac{6}{4}mg$ --- (7)

Given $T_B = 4T_A$

$= 4 \times \frac{6}{4}mg$ (from (7))

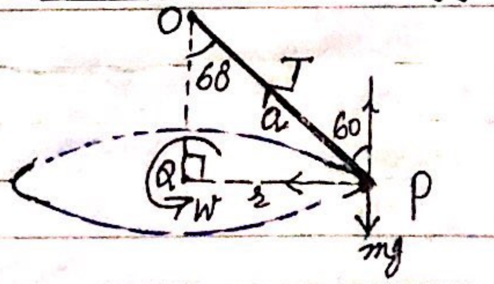
$T_B = 6mg$ ✓

9. A particle P of mass m is attached to one end of a light inextensible string of length 'a'. The other end of the string is attached to a fixed point O.



(a) The particle P moves in a circle with constant angular speed 'w' with the string inclined at '60°' to the downward vertical through O. Show that $w^2 = 2g/a$. ---[4]

(b) The particle now hangs at rest a distance 'a' vertically below O. It is then projected horizontally so that it begins to move in a vertical circle with centre O. When the string makes an angle of '60°' with the downward vertical through 'O', the angular speed of P is $\sqrt{2g/a}$. The string first goes slack when 'OP' makes an angle 'θ' with the upward vertical through O.
Find the value of $\cos\theta$. [SP-20/03/Q5] ---[6]



Solution: (a) Vertically: $T\cos 60° = mg \Rightarrow T = 2mg$ ---①

Horizontally: $T\sin 60 = mrw^2$ $\quad (r = a\sin 60°)$ (ΔOPQ)

$\Rightarrow T\sin 60° = m \cdot a\sin 60°\, w^2$

$2\cancel{m}g = \cancel{m}aw^2$ [from ① $T = 2mg$]

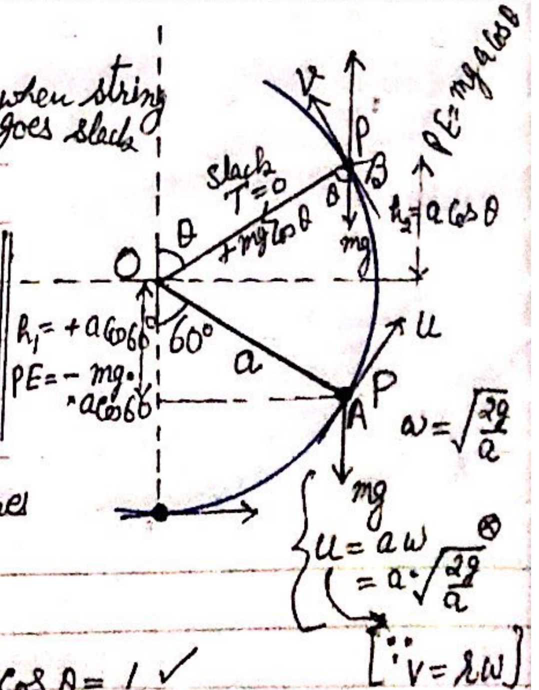
$\Rightarrow w^2 = 2g/a$ ✓

(b) Energy equation: from P at A, to P at B when string goes slack
(60° with downward vertical)

$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mg.a\cos\theta - (-mg.a\cos 60°)$

$\Rightarrow$ ⊛ $u^2 - v^2 = 2ag\cos\theta + ga$

$a^2 \times \frac{2g}{a} - v^2 = 2ag\cos\theta + ga \Rightarrow v^2 = ga - 2ga\cos\theta$ ---②



Now using $F = ma$ at B

$T + mg\cos\theta = \frac{mv^2}{a}$

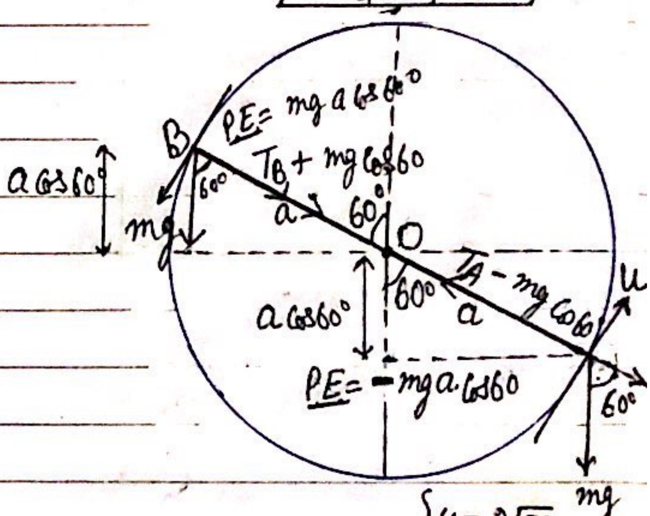
$\Rightarrow 0 + \cancel{m}g\cos\theta = \frac{\cancel{m}v^2}{a}$ [$T = 0$, for string goes slack at B]

$g\cos\theta = \frac{1}{a}(ga - 2ga\cos\theta)$ [from ②, v^2]

$\cos\theta = 1 - 2\cos\theta \Rightarrow 3\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{3}$ ✓

10. A partical 'Q' of mass 'm' is attached to a fixed point 'O' by a light inextensible string of length 'a'. The partical moves in complete vertical circle about O. The point 'A' and 'B' are on the path of Q with AB a diameter of circle. 'OA' makes an angle of '60°' with the downward vertical through O and 'OB' makes an angle of '60°' with the upward vertical through O. The speed of Q when it is at A is '$2\sqrt{ag}$'.
Given that T_A and T_B are the tensions in the string at A and B respectively, find the ratio $T_A : T_B$. ---[6]

S-20/33/Q3



$\left\{\begin{array}{l} u = 2\sqrt{ag} \\ u^2 = 4ag \end{array}\right.$

Solution: Radially at A: $F = ma$

$\Rightarrow T_A - mg\cos 60 = \dfrac{mu^2}{a}$

$\Rightarrow T_A - \frac{1}{2}mg = \frac{m}{a} \times 4ag$

$\Rightarrow T_A = \frac{9}{2}mg$ ---- ①

Now Radially at B, $F = ma$

$T_B + mg\cos 60 = \dfrac{mv^2}{a}$

$T_B + \frac{1}{2}mg = \frac{m}{a}v^2$ ---- ②

Using Energy equation from A to B.

$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mg\,a\cos 60 - (-mg\,a\cos 60)$

$\frac{1}{2}m \cdot 4ag - \frac{1}{2}mv^2 = \frac{1}{2} \times 2mga$ $\quad$ Given ($\because u^2 = 4ag$)

$\Rightarrow v^2 = 2ag$ --- ③

from ② and ③ $T_B + \frac{1}{2}mg = \frac{m}{a} \times 2ag$

$\Rightarrow T_B = \frac{3}{2}mg$ ------ ④

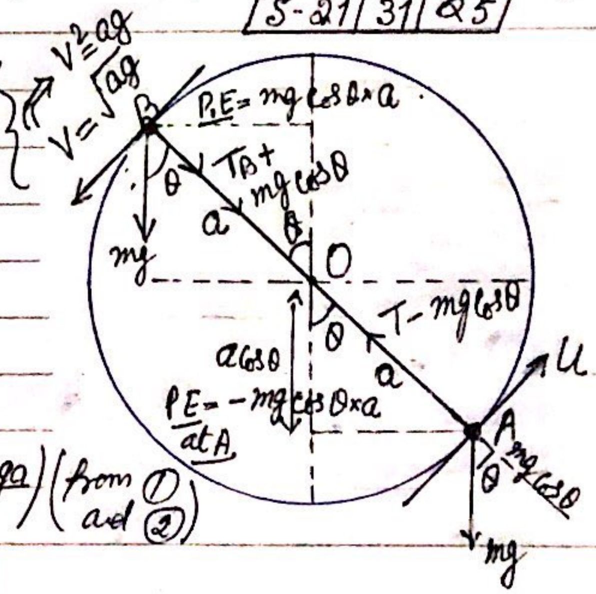
from ① and ④ $\dfrac{T_A}{T_B} = \frac{9}{2}mg \Big/ \frac{3}{2}mg = \dfrac{3}{1}$

$\therefore T_A : T_B = \underline{3:1}$ ✓

P-12

11. A particle P of mass 'm' is attached to one end of a light inextensible string of length 'a'. The other end of the string is attached to a fixed point O. The particle completes vertical circle through O. 'OA' makes an angle 'θ' with the downward vertical through O and OB makes an angle 'θ' with the upward vertical through O.
The speed of P when it is at 'A' is 'u' and the speed of 'P' when it is at 'B' is $\sqrt{ag}$. The tensions in the string at A and B are T_A and T_B respectively. It is given that $T_A = 7T_B$. --- [8]
Find the value of 'θ' and an expression for u in terms of a and g.

S-21/31/Q5



Solution: Radial components at A: $F = ma$

$\Rightarrow T_A - mg\cos\theta = \frac{mu^2}{a}$ --- (1)

and Radial component at B: $F = ma$

$\Rightarrow T_B + mg\cos\theta = \frac{mV^2}{a}$

$\Rightarrow T_B + mg\cos\theta = \frac{m}{a} \cdot ag$ --- (2)

Given $T_A = 7T_B$

$\Rightarrow mg\cos\theta + \frac{mu^2}{a} = 7\left(-mg\cos\theta + \frac{mga}{a}\right)$ (from (1) and (2))

$\Rightarrow u^2 = ag(7 - 8\cos\theta)$ --- (3)

Now using Energy equation from A to B.

$\frac{1}{2}mu^2 - \frac{1}{2}mV^2 = mga\cos\theta - (-mga\cos\theta)$

$\Rightarrow \frac{1}{2}mu^2 - \frac{1}{2}mga = 2mga\cos\theta$

$\Rightarrow u^2 = ag(4\cos\theta + 1)$ --- (4)

from (3) and (4)

$ag(7 - 8\cos\theta) = ag(4\cos\theta + 1)$

$12\cos\theta = 6 \Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = 60°$ ✓

from (4) $u^2 = ag\left(4 \times \frac{1}{2} + 1\right)$ $\left[\because \cos\theta = \frac{1}{2}\right]$

$u^2 = 3ag$ $\Rightarrow u = \sqrt{3ag}$ ✓

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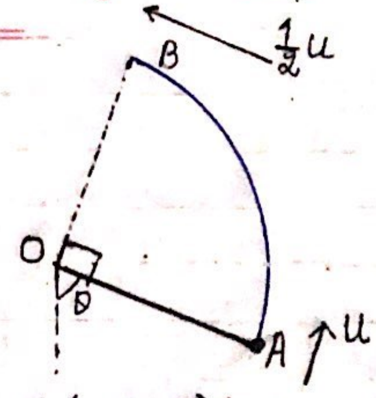
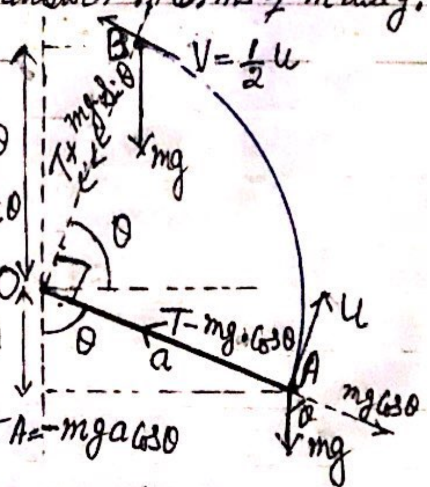


12. A particle of mass 'm' is attached to one end of a light inextensible string of length 'a'. The other end of the string is attached to a fixed point 'O'. The particle is initially held with the string taut at the point 'A', where OA makes an angle 'θ' with the downward vertical through O. The particle is then projected with speed 'u' perpendicular to $\vec{OA}$ and begins to move upward in part of a vertical circle. The string goes slack when the particle is at the point 'B' where angle AOB is a right angle. The speed of the particle when it is at 'B' is '$\frac{1}{2}u$'. Find the tension in the string at 'A', giving your answer in terms of m and g.



Solution: Energy equation from A to B.

$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mga\sin\theta - (-mga\cos\theta)$

$\Rightarrow \frac{1}{2}mu^2 - \frac{1}{2}m(\frac{1}{2}u)^2 = mga(\cos\theta + \sin\theta)$

$\Rightarrow \frac{3}{4}u^2 = 2ag(\cos\theta+\sin\theta)$ --- ①

$\left\{ \begin{array}{l} a\sin\theta \\ \text{P.E. at B} \\ = mga\sin\theta \end{array} \right.$

$\left\{ \begin{array}{l} a\cos\theta \\ \text{P.E. at A} = -mga\cos\theta \end{array} \right.$

Now radial component at B: $F=ma$

$\Rightarrow T + mg\cdot\sin\theta = \frac{mv^2}{a}$

$0 + mg\cdot\sin\theta = \frac{m}{a}\times(\frac{u}{2})^2$

[$\because$ At B the string goes slack $\Rightarrow T=0$]

$\Rightarrow u^2 = 4ag\sin\theta$ --- ②

from ① and ② eliminating u^2

$\frac{3}{4}\cdot 4ag\sin\theta = 2ag(\cos\theta+\sin\theta)$

$\Rightarrow 3\sin\theta = 2\cos\theta + 2\sin\theta$

$\Rightarrow \tan\theta = 2$ --- ③

Now radial component at A: $F=ma$

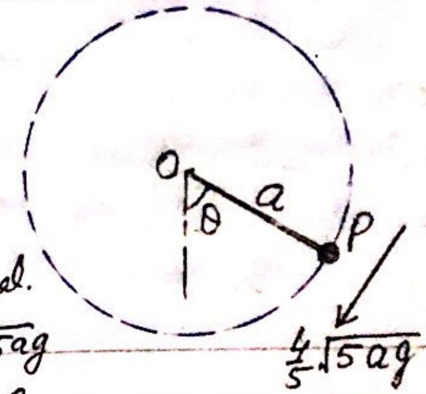
$\Rightarrow T - mg\cos\theta = \frac{mu^2}{a}$

$T - mg\cdot\frac{1}{\sqrt{5}} = \frac{m}{a}\cdot\frac{8ag}{\sqrt{5}}$

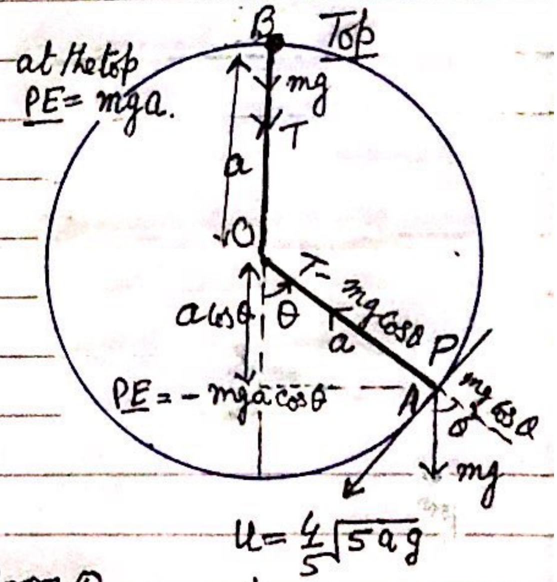
$\Rightarrow T = \frac{9}{\sqrt{5}}mg = \frac{9\sqrt{5}}{5}mg = 4.02mg$

[$\tan\theta = 2$; $\sin\theta = \frac{2}{\sqrt{5}}$; $\cos\theta = \frac{1}{\sqrt{5}}$
from ② $u^2 = 4ag\sin\theta = 4ag\cdot\frac{2}{\sqrt{5}} = \frac{8ag}{\sqrt{5}}$]

13. A particle P is attached to one end of a light inextensible string of length 'a'. The other end of the string is attached to a fixed point O. The particle P is held with the string taut and making an angle θ with the downward vertical. The particle P is then projected with speed $\frac{4}{5}\sqrt{5ag}$ perpendicular to the string and just complete a vertical circle.
Find the value of $\cos\theta$.



W-20/31/Q2 --[5]



Solution: Radial Eqn in general.

$$T - mg\cos\theta = \frac{mv^2}{a}$$

At the top $T = 0$, $\theta = 180^\circ \Rightarrow \cos\theta = -1$

$$\therefore 0 + mg = \frac{mv^2}{a}$$

$$\Rightarrow v^2 = ag \quad \text{---(i)}$$

Energy equation from A to B:

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mga - (-mga\cos\theta)$$

$$\Rightarrow u^2 - v^2 = 2ga(1+\cos\theta)$$

$$\Rightarrow \frac{16}{5}ag - ag = 2ag(1+\cos\theta)$$

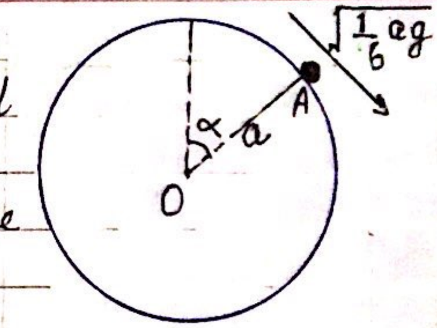
$$\frac{16}{5} - 1 = 2(1+\cos\theta)$$

$$\Rightarrow 2\cos\theta = \frac{1}{5}$$

$$\cos\theta = \frac{1}{10} \checkmark$$

from (i) $v^2 = ag$ and $u^2 = \frac{16}{25} \times 5ag = \frac{16ag}{5}$

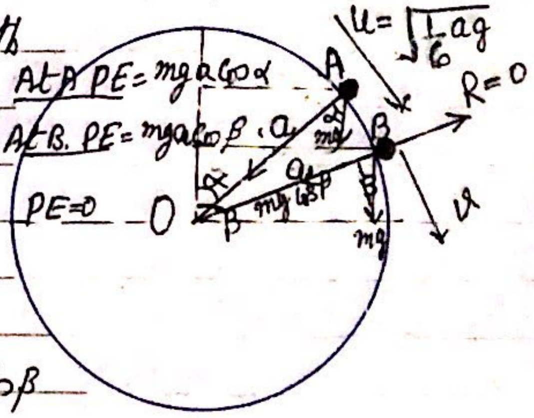
14. A fixed smooth sphere has centre O and radius 'a'. A particle of mass 'm' is projected downwards with speed $\sqrt{\frac{1}{6}ag}$ from the point A on the surface of the sphere, where OA makes an angle 'α' with the upward vertical through O. The particle moves in a part of a verticle circle on the surface of the sphere. It loses contact with the sphere at the point B, where OB makes an angle 'β' with the upward vertical through O



Given that $\cos\alpha = 2/3$; find the value of $\cos\beta$. ...[5]

[W-20/32/Q1]

Solution: At B the particle losses the contact with the surface. $R = 0$, ∴ Radially at B →



∴ $mg\cos\beta = \frac{mV^2}{a}$

$\Rightarrow V^2 = ag\cos\beta$ —— (1)

Energy equation from A to B.

$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mga\cos\alpha - mga\cos\beta$

$\Rightarrow v^2 - u^2 = 2ag(\cos\alpha - \cos\beta)$

$ag\cos\beta - \frac{1}{6}ag = 2ag\left(\frac{2}{3} - \cos\beta\right)$

$\left\{ \begin{array}{l} v^2 = ag\cos\beta \text{ from (1)} \\ u^2 = \left(\sqrt{\frac{1}{16}ga}\right)^2 = \frac{1}{16}ag \\ \cos\alpha = 2/3 \end{array} \right.$

$\cos\beta - \frac{1}{6} = \frac{4}{3} - 2\cos\beta$

$\Rightarrow 3\cos\beta = \frac{9}{6} = \frac{3}{2}$

∴ $\cos\beta = \frac{1}{2}$ ✓

15. A hollow cylinder of radius 'a' is fixed with its axis horizontal. A particle P, of mass 'm', moves in part of a vertical circle of radius 'a' and centre O on the smooth inner surface of the cylinder. The speed of 'P' when it is at the lowest point A of its motion is $\sqrt{\frac{7}{2}ga}$.

The particle P loses contact with the surface of the cylinder when OP makes an angle 'θ' with the upward vertical through O.

(a) Show that $\theta = 60°$ ---[5]

(b) Show that in its subsequent motion P strikes the cylinder at the point A. --[5]

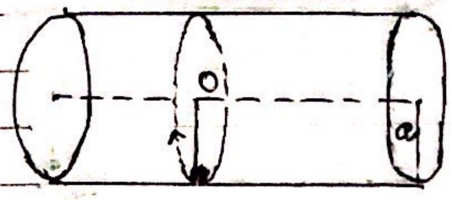
S-20/31/Q7

Solution: Radially at B; $R + mg\cos\theta = \frac{mv^2}{a}$

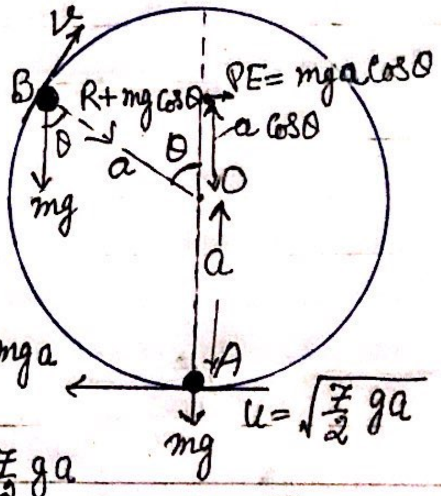
as the Partical P loses contact at B → R = 0

$\therefore 0 + mg\cos\theta = \frac{mv^2}{a}$

$\Rightarrow v^2 = ag\cos\theta$ ---- ①



Now Energy equation from A to B:



$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mga\cos\theta - (-mga)$

$u^2 - v^2 = 2g(a + a\cos\theta)$

$\frac{7}{2}ga - v^2 = 2ga(1+\cos\theta)$

$\Rightarrow v^2 = ga\left[\frac{7}{2} - 2\right] - 2ga\cos\theta$

$ga\cos\theta = ga \cdot \frac{3}{2} - 2ga\cos\theta$

$\Rightarrow 3\cos\theta = \frac{3}{2} \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = 60°$ ✓

PE = −mga
$\therefore$
$u^2 = \frac{7}{2}ga$
$v^2 = ga\cos\theta$ fm ①

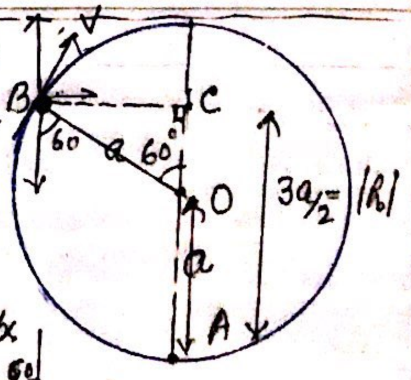
(b) Time from BC = $\frac{a\sin 60}{V\cos 60} = \frac{a\sqrt{3}}{\sqrt{\frac{1}{2}ag}}$ from ① $V^2 = ag\cos 60 = \frac{1}{2}ag$

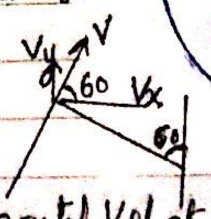
$\Rightarrow T = \sqrt{\frac{6a}{g}}$ To cover horizonal dis 'BC' $V = \sqrt{\frac{1}{2}ag}$ ✓

Vertical distance $h = V\sin 60 \cdot T - \frac{1}{2}gT^2$ $BC = a\sin 60$

$= \sqrt{\frac{1}{2}ag} \cdot \frac{\sqrt{3}}{2} \cdot \sqrt{\frac{6a}{g}} - \frac{1}{2}g \times \frac{6a}{g}$

$h = -\frac{3a}{2} \rightarrow |CA| = \frac{3a}{2}$ |h|



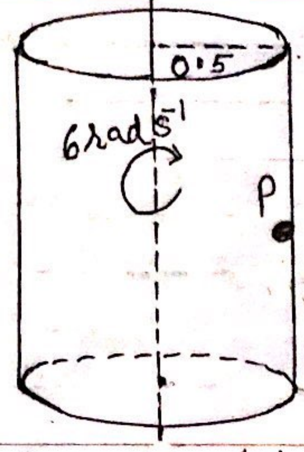


$\therefore$ Particle strikes at A (BC, CA)

Horizontal Vel at B = $V\cos 60 = V_x$

Vertical Vel. at B = $V\sin 60 = V_y$

16. A hollow cylinder with a rough inner surface has radius "0·5 m". A particle P of mass "0·4 kg" is in contact with the inner-surface of the cylinder. The particle and the cylinder rotate together with angular speed "6 rad s⁻¹" about the vertical axis of the cylinder, so that the particle moves in a horizontal circle. Given that P is about to slip downwards, find the coefficient of friction between 'P' and the surface of the cylinder. [M₂ / W-17 / 51 / Q 1] ---[4]



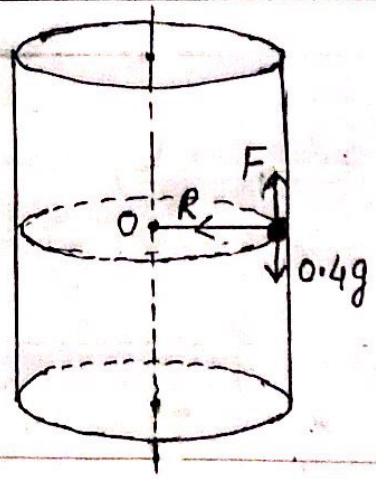
Solution: Radially along PO; $R = ma$
Horizontally →
$\Rightarrow R = m\omega^2 r$
$= 0{\cdot}4 \times 6^2 \times 0{\cdot}5$
$R = 7{\cdot}2\ N$ ---①

Force of friction $F = mg = 0{\cdot}4 \times 10 = 4\ N$ --②
(As P is about to slip.

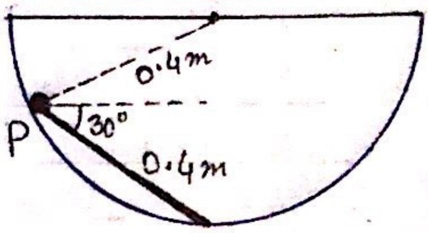
Now $F = \mu R \Rightarrow \mu = \dfrac{F}{R}$

$= \dfrac{4N}{7{\cdot}2}$ (from ① and ②)

$\therefore \mu = \underline{0{\cdot}566}$ ✓



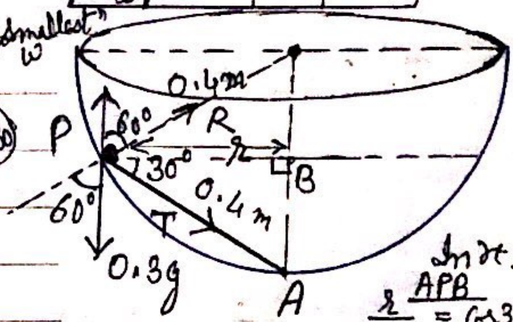
17. One end of a light inextensible string of length "0·4 m" is attached to lowest point of a hemisphere of radius "0·4 m" fixed with its axis vertical. A partical of mass 0·3 kg is attached to the other end of the string.



The string is straight and makes an angle of '30°' with the horizontal. 'P' moves on the smooth inner surface of the hemisphere in a horizontal circle.

(i) Calculate the smallest possible angular speed of P. ---[4]

(ii) Given that the greatest possible tension in the string is 5N, calculate the greatest possible speed of P. | M_2 | M-18 | 52 | Q8 | --[4]



Solution (i) $r = BP = 0.4 \cos 30°$ $\qquad T = 0$ for smallest ω

Vertically at P: $R \cos 60° = 0.3g + T \sin 30$

$\Rightarrow R = 6 \text{ N} \text{ --- } (1)$

Horizontally (radially) at P.

In $\triangle APB$: $\frac{r}{0.4} = \cos 30°$

$F = ma$

$R \cos 30° = m r \omega^2$ $\qquad$ horizontal circle radius $r = 0.4 \cos 30°$

$\Rightarrow 6 \cancel{\cos 30°} = 0.3 \times 0.4 \cancel{\cos 30°} \times \omega^2$ ($R = 6N$ from (1))

$\Rightarrow \omega^2 = \frac{6}{0.12} = 50 \Rightarrow \underline{\omega = 7.07 \text{ rad s}^{-1}}$ ✓

(ii) Now when $T = 5$; Vertically at P:

$R \cos 60° = 0.3g + T \sin 30°$ $\quad (T = 5)$

$R \cdot \frac{1}{2} = 3 + 5 \times \frac{1}{2} \Rightarrow R = 11N \text{ -- } (2)$

Radially (Horizontally) at P:

$$R \cos 30 + T \cos 30 = \frac{mv^2}{r}$$

$$\Rightarrow 11 \cos 30° + 5 \cos 30° = \frac{0.3 v^2}{0.4 \cos 30°}$$

$$\Rightarrow v^2 = \frac{4}{3}[11 \cos^2 30 + 5 \cos^2 30] = \frac{4}{3} \times \frac{3}{4} [16] = 16$$

$[\cos^2 30° = \frac{3}{4}]$

$\therefore v = \sqrt{16}$

or $\quad v = \underline{4 \text{ m s}^{-1}}$ ✓

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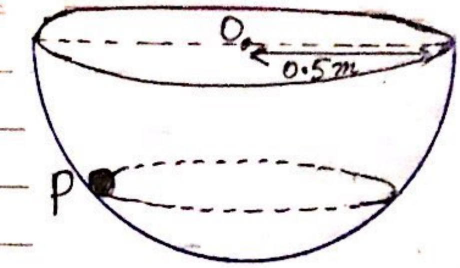
18 A partical P of mass '0.4 kg' moves with constant speed in a horizontal circle on the smooth inner surface of a fixed hollow hemisphere with centre O, and radius '0.5m'.

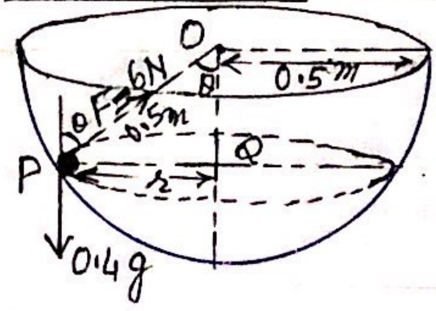


(i) Given that the speed of the particle is $4 \, m s^{-1}$ and its angular speed is $10 \, rad s^{-1}$, Calculate the angle between OP and the vertical. ---[2]

(ii) Given instead the magnitude of the force exerted on P by the hemisphere is 6N. Calculate:

(a) the angle between OP and the vertical. ---[2]

(b) the angular speed of P. $\boxed{M_2 / W\text{-}15 / 53 / Q4}$ --[3]



Solution: Angular speed $\omega = 10 \, rad s^{-1}$, $v = 4 \, m s^{-1}$

(i) $V = r\omega \Rightarrow r = \frac{V}{\omega} = \frac{4}{10} = 0.4 m$

In rt Δ OQP, $\sin\theta = \frac{PQ}{OP} = \frac{0.4}{0.5} = 0.8$

$\sin\theta = 0.8 \Rightarrow \theta = \sin^{-1} 0.8 = \underline{53.1^\circ}$ ✓

(ii) (a) Vertically at P:

$F\cos\theta = mg \Rightarrow 6\cos\theta = 0.4 \times 10$ (Given $F = 6N$)

$\Rightarrow \cos\theta = \frac{4}{6} \Rightarrow \theta = \cos^{-1} 4/6$

$\theta = 48.2^\circ$ ✓

(b) Horizontally (radially):

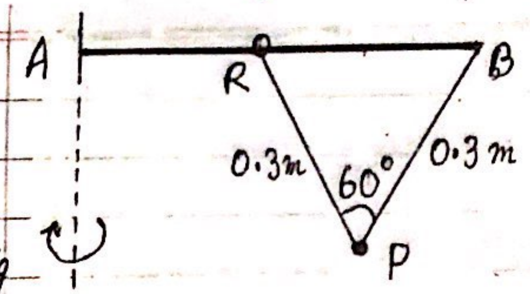
$F\sin\theta = m\omega^2 r$

$6 \cdot \cancel{\sin\theta} = 0.4 \times \omega^2 \times 0.5 \cancel{\sin\theta}$

$\omega^2 = 30$

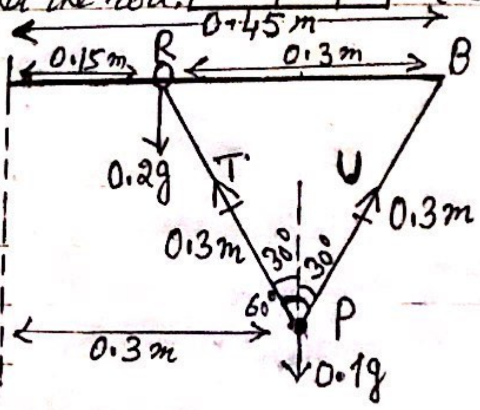
$\underline{\omega = 5.48 \, rad s^{-1}}$ ✓

19. A rough horizontel rod AB of length 0.45m rotates with constant velocity '6 rad s⁻¹' about a vertical axis through 'A'. A small ring 'R' of mass 0.2 kg can slide on the ring. A particle 'P' of mass 0.1 kg is attached to the midpoint of a light inextensible string of length 0.6 m. One end of the string is attached to B, with angle RPB = 60°. R and P move in horizontel circles as the system rotates, R is in equilibrium. --- [5]



(i) Show that the tension in the portion PR of the string is 1.66 N, correct to 3 s.f.

(ii) Find the coefficient of friction between the ring and the rod. W-18/52/Q7 --[5]

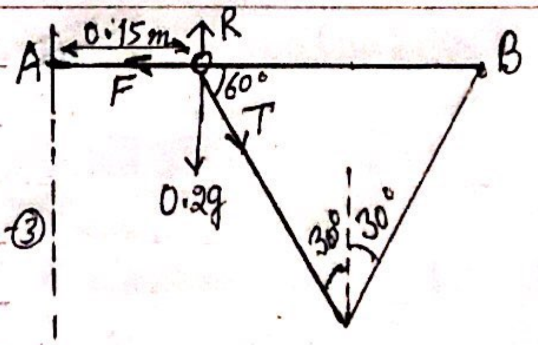


Solution: Vertically at P:

(i) $T\cos 30° + U\cos 30° = 0.1g = 1$ --- ①

Horizontally: (Radially) $T\cos 60 - U\cos 60 = m\omega^2 r = 0.1\times 6^2 \times 0.3$
$= 1.08$ --- ②

Solving ① & ② $T = 1.6573 = \underline{1.66\,N}$ (3 s.f.)



(ii) Radially (Horizontally) at R:

$F - T\cos 60° = m r \omega^2 = 0.2\times 6^2 \times 1.5$

$F - 1.66 \times \frac{1}{2} = 0.2 \times 36 \times 1.5$

$\Rightarrow F = 1.9086 = \underline{1.91\,N}$ -- ③

Vertically at R:

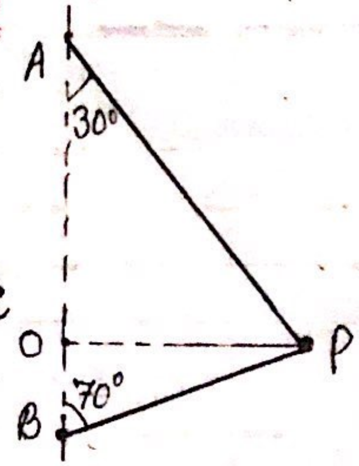
Normal reaction $R = 0.2g + T\cos 30° = 3.4353$.

$\Rightarrow R = 3.44\,N$ -- ④

$F = \mu R \Rightarrow \mu = \frac{F}{R} = \frac{1.9086}{3.4353} = 0.556$ [from ③ & ④]

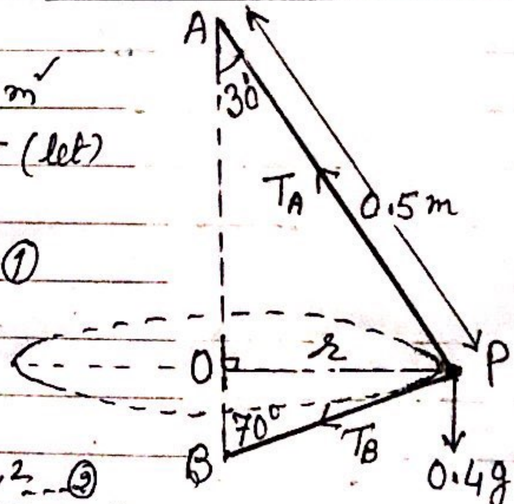
$\underline{\mu = 0.556}$

20 A and B are two fixed points on a vertical axis with A above B. A particle P of mass 0·4 kg is attached to A by a light inextensible string of length 0·5 m. The particle P is attached to B by another light inextensible string. P moves with constant speed in horizontal circle with centre O, between A and B. Angle BAP = 30° and angle ABP = 70°.



(i) Given that the tensions in the two strings are equal, find the speed of P. ---[5]

(ii) Given instead that the angular speed of P is 12 rad s⁻¹, find the tensions in the strings. [M2 | W-19 | 52 | Q5] ---[5]



Solution: In △ APO, $r = 0.5 \sin 30° = 0.25\,m$ ✓

(i) Vertically at P: Given: $T_A = T_B = T$ (let)

$$T\cos 30 - T\cos 30° = 0.4g$$

$$\Rightarrow T = 7.6335 \quad \text{----} ①$$

Horizontally (Radially) at P:

$$F = ma$$

$$T\sin 30 + T\sin 70° = \frac{mV^2}{r}$$

$$= \frac{0.4 \times V^2}{0.25} \quad \text{--} ②$$

[r = 0.25]

from ① and ② $V = 2.62\,ms^{-1}$ ✓

(ii) Vertically at 'P': Given $w^2 = 12\,rad\,s^{-1}$ and $T_A \neq T_B$

$$T_A \cos 30° - T_B \cos 70° = 0.4g \quad \text{----} ③$$

Radially (Horizontally) at P: $F = ma = mrw^2$

$$T_A \sin 30° + T\sin 70° = 0.4 \times 0.25 \times 12^2 \quad \text{--} ④$$

from ③ & ④

$$T_A = 8.82\,N \checkmark$$

$$T_B = 10.6\,N \checkmark$$