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FMe

# Further Mechanics

## Equilibrium of a Rigid body.

## Revision.

| SP-20 | S-20 | W-20 | S-21 |
|---|---|---|---|

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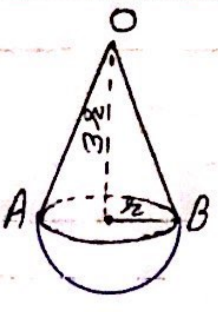
# Equilibrium of a rigid body - Revision.



1. A child's toy consists of a uniform solid circular cone of vertical height '$3r$' and radius '$r$', and a uniform solid hemisphere of radius '$r$'. The circular bases of the cone and the hemisphere are joined together so that they coincide. The cone and the hemisphere are made of the same material. Show that the centre of mass of the toy is at a distance of $\frac{27}{10}r$ from the vertex of the cone. --[4]

[SP-20/03/Q1]




**Solution:**

| | Volume | COM from Vertex |
|---|---|---|
| Cone: | $\frac{1}{3}\pi r^2 \times 3r$ | $\frac{3}{4}\times 3r$ |
| Hemisphere: | $\frac{2}{3}\pi r^3$ | $3r + \frac{3}{8}r$ |
| Combined Solid | $\frac{5}{3}\pi r^3$ | $\bar{x}$ |

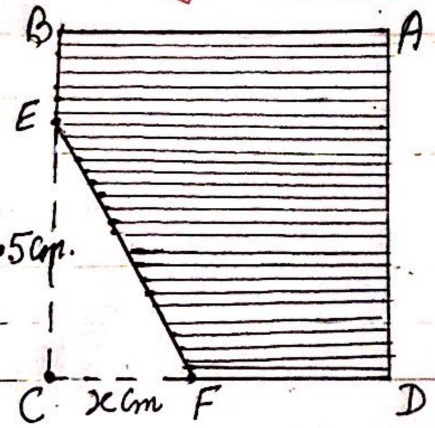
Taking moments about the vertex O.

$$\frac{5}{3}\pi r^3 \cdot \bar{x} = \pi r^3 \cdot \frac{9}{4}r + \frac{2}{3}\pi r^3 \times \frac{27}{8}r$$

$$\Rightarrow \bar{x} = \frac{27}{10}r$$

The distance of Centre of mass (COM) from the vertex $= \frac{27}{10}r$. ✓

2. A uniform square lamina ABCD has sides of length 10 cm. The point E is on BC with EC = 7.5 cm, and the point, F, is on DC with CF = $x$ cm. The triangle EFC is removed from ABCD. The centre of mass of the resulting shape ABEFD is at a distance $\bar{x}$ cm, from CB and a distance $\bar{y}$ cm from CD.




(a) Show that $\bar{x} = \dfrac{400 - x^2}{80 - 3x}$ and find a corresponding expression for $\bar{y}$. ---[4]

The shape ABEFD is in equilibrium in a vertical plane with the edge DF resting on a smooth horizontal surface.

(b) Find the greatest possible value of $x$, giving your answer in the form $a + b\sqrt{2}$, where $a$ and $b$ are constants to be determined --[3]

S-20/31/Q4

Solution
(a)

| | Area | Centre of mass from BC | Centre of mass from DC |
|---|---|---|---|
| Square | 100 | 5 | 5 |
| Triangle | $\frac{1}{2}x \times \frac{15}{2}$ | $\frac{1}{3}x$ | $5/2$ |
| Shape ABEFD | $100 - \frac{15}{4}x$ | $\bar{x}$ | $\bar{y}$ |

Taking moment about BC: $\left(100 - \frac{15}{4}x\right)\cdot\bar{x} = 100\times5 - \frac{15}{4}x \times \frac{x}{3}$

$\Rightarrow \bar{x} = \dfrac{400 - x^2}{80 - 3x}$ ✓

Taking moment about DC;

$\left(100 - \frac{15}{4}x\right)\cdot\bar{y} = 100\times5 - \frac{15}{4}x \times \frac{5}{2} \Rightarrow \bar{y} = \dfrac{800 - 15x}{160 - 6x}$ ✓

(b) For the greatest value of $x$, for remaining in equilibrium:

$\bar{x} \geq x$

$\Rightarrow \dfrac{400 - x^2}{80 - 3x} \geq x$

$\Rightarrow 400 - x^2 \geq 80x - 3x^2$

$\Rightarrow 2x^2 - 80x + 400 \geq 0$

$\Rightarrow x^2 - 40x + 200 \geq 0$

$\Rightarrow x \leq (20 - 10\sqrt{2})$ or $x \geq (20 + 10\sqrt{2})$ ✗

critical values

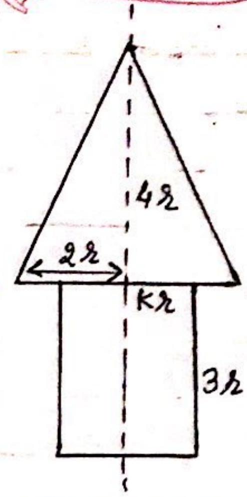
$x = \dfrac{40 \pm \sqrt{800}}{2}$ $\left\{ b^2 - 4ac = 1600 - 4\times1\times200 = 800 \right.$

$= 20 \pm 10\sqrt{2}$

$= (20 - 10\sqrt{2})$ ; $(20 + 10\sqrt{2})$ ✗

$\therefore$ Greatest value $x = (20 - 10\sqrt{2})$

3. A uniform solid circular cone, of vertical height '4r' and radius '2r', is attached to a unifor solid cylinder, of height '3r' and radius 'kr', where k is a constant less than '2'. The base of the cone is joined to one of the circular faces of the cylinder so that the axes of symmetry of the two solids coincide. The cone and the cylinder are made of the same meterial.




(a) Show that the distance of the centre of mass of the combined solid from the vertex of the cone is $\frac{(99k^2+96)r}{18k^2+32}$. ---[4]

The point C is on the circumference of the base of the cone. When the combined solid is freely suspended from C and hanging in equilibrium, the diameter through C makes an angle $\alpha$ with the downward vertical, where $\tan\alpha = \frac{1}{8}$.

(b) Given that the centre of mass of the combined solid is within the cylinder, find the value of k. [S-20/33/Q4] ---[4]

**Solution:**

(a)

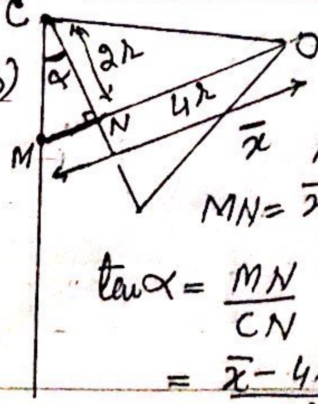
| | Mass | Centre of Mass from Vertex of cone |
|---|---|---|
| Cylinder | $\pi(kr)^2\cdot 3r\cdot\rho$ | $(4r+\frac{3}{2}r)$ |
| Cone | $\frac{1}{3}\pi(2r)^2\cdot 4r\rho$ | $\frac{3}{4}\times 4r$ |
| Combined | $(\pi(kr)^2\cdot 3r+\frac{1}{3}\pi(2r)^2\,4r)\rho$ | $\bar{x}$ |

Taking moment about Vertex;

$$\bar{x}\left(\pi(kr)^2\cdot 3r+\frac{1}{3}\pi(2r)^2\cdot 4r\right) = \pi(kr)^2\cdot 3r\times\frac{11}{2}r+\frac{1}{3}\pi(2r)^2\times 4r\times 3r$$

$$\Rightarrow \bar{x} = \frac{(99k^2+96)r}{18k^2+32} \quad \checkmark \ \text{---} (1)$$

(b)




$MN = OM - ON$

$MN = \bar{x} - 4r$

$$\tan\alpha = \frac{MN}{CN} = \frac{\bar{x}-4r}{2r} = \frac{1}{8}$$

$$\Rightarrow \bar{x} = \frac{17}{4}r \ \text{---} (2)$$

from (1) and (2)

$$\frac{99k^2+96}{(18k^2+32)} = \frac{17}{4}r$$

$$\Rightarrow k = \frac{4}{3} \checkmark$$

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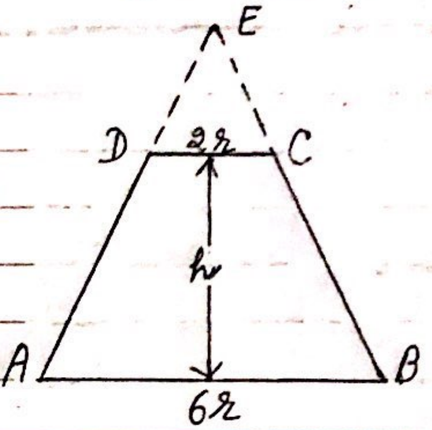
4 The diagram shows the cross-section ABCD of a uniform solid object which is formed by removing a cone with cross-section DCE from the top of a larger cone with cross-section ABE. The perpendicular distance between AB and DC is 'h', the diameter AB is '6r' and the diameter DC is '2r'.




(a) Find an expression, in terms of h, for the distance of the centre of mass of the solid object from AB. ---[4]

The object is freely suspended from the point B and hangs in equilibribrium. The angle between AB and the downward vertical through B is $\theta$.

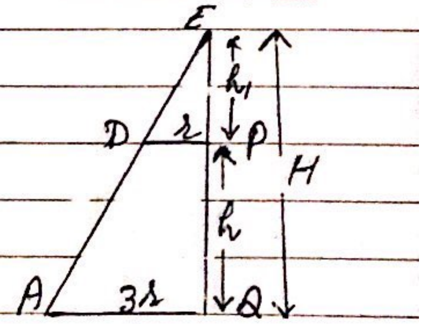
(b) Given that $h = \frac{13}{4}r$, find the value of $\tan\theta$. ---[2]

[W-20/31/Q4]

Solution:

(a)

| | Volume | Centre of Mass from AB |
|---|---|---|
| Small Cone | $\frac{1}{3}\pi r^2 \cdot h/2$ | $h+\frac{1}{4}\cdot\frac{h}{2}=\frac{9h}{8}$ |
| Large Cone | $\frac{1}{3}\pi(3r)^2\cdot\frac{3h}{2}$ | $\frac{1}{4}\cdot\frac{3h}{2}=\frac{3}{8}h$ |
| Object | $\frac{26}{6}\cdot\pi r^2 h$ | $\bar{x}$ |




$\frac{r}{3r}=\frac{h_1}{H} \Rightarrow H=3h_1$

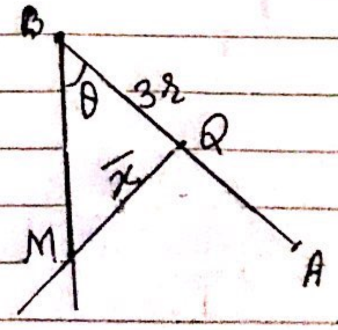
$h_1+h=3h_1$

$h_1=h/2$

Taking moments about AB;

$\frac{13}{3}\pi r^2 h\cdot\bar{x}=\frac{27}{6}\pi r^2 h\cdot\frac{3h}{8}-\frac{1}{6}\pi r^2 h\cdot\frac{9h}{8}$

$\Rightarrow \bar{x}=\frac{9h}{26}$




(b) $\tan\theta=\frac{\bar{x}}{3r}=\frac{\frac{9h}{26}}{\frac{3r}{1}}=\frac{3}{26r}h$

$\Rightarrow \tan\theta=\frac{3}{26r}\times\frac{13}{4}r$ [Given $h=\frac{13}{4}r$]

$\tan\theta=\frac{3}{8}$

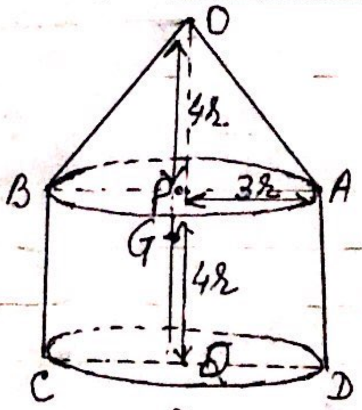
5. An object consists of a uniform solid circular cone, of the vertical height '4r' and radius '3r', and a uniform solid cylinder, of height '4r' and height '3r'. The circular base of the cone and one of the circular faces of the cylinder are joined together so that they coincide. The cone and the cylinder are made of the same material.

(a) Find the distance of the centre of mass of the object from the end of the cylinder that is not attached to the cone. --[4]

(b) Show that the object can rest in equilibrium with the curved surface of the cone in contact with a horizontal surface. --[3] [W-20/32/Q3]

Solution

| | Volume | Centre of mass from base |
|---|---|---|
| Cone | $\frac{1}{3}\pi(3r^2)\cdot 4r$ | $4r+r$ |
| Cylinder | $\pi(3r)^2\cdot 4r$ | $2r$ |
| Combined | $\frac{4}{3}\pi(3r)^2\cdot 4r$ | $\bar{x}$ |




Taking moments about base of Cylinder

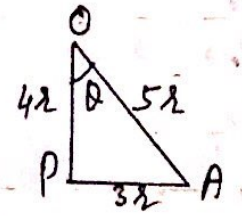
$\frac{4}{3}\pi(3r)^2\cdot 4r\cdot\bar{x} = \frac{1}{3}\pi(3r)^2\cdot 4r\times 5r + \pi(3r)^2\cdot 4r\times 2r$

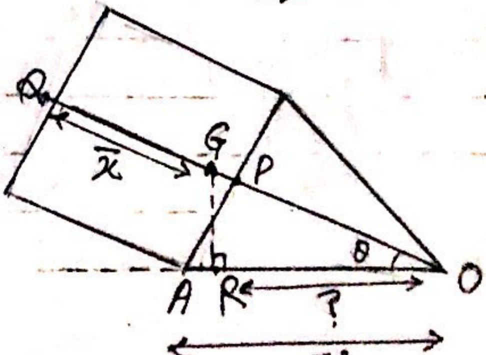
$\Rightarrow \bar{x} = \frac{11}{4}r \checkmark = GQ$

$\Rightarrow PG = 4r - \frac{11}{4}r$

$PG = \frac{5}{4}r$

$\cos\theta = \frac{4}{5}$




(b) Object will rest with curved surface of Cone OA if OR < OA | $\triangle$ OGR

or if $\frac{21}{4}r\cos\theta < 5r$

$\frac{21}{4}r\times\frac{4}{5} < 5r$

$\Rightarrow 21 < 25$

True

$\left[\frac{OR}{OG} = \cos\theta\right.$

$OR = OG\cos\theta$

$= \frac{21}{4}r\cdot\cos\theta$

$GP = \frac{5}{4}r$

$OG = 4r + \frac{5}{4}r = \frac{21}{4}r$

Hence the result.

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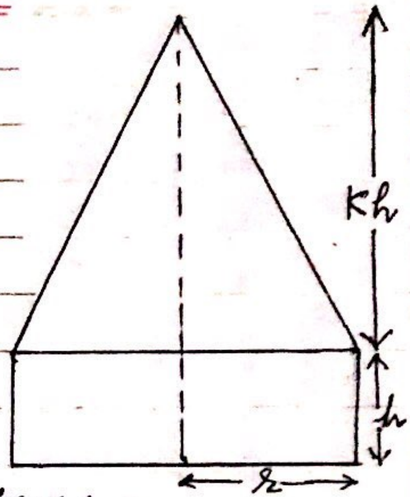
6. A uniform solid circular cone has vertical height '$kh$' and radius '$r$'. A uniform solid cylinder has height '$h$' and radius $r$.
The base of the cone is joined to one of the circular faces of the cylinder so that the axes of symmetry of the two solides coincide. The cone and the cylinder are made of the same material.




(a) Show that the distance of the centre of mass of the comined solid from the base of the cylinder is: $\dfrac{h(k^2+4k+6)}{4(3+k)}$ --[4]

The solid is placed on a plane that is inclined to the horizontal at an angle $\theta$. The base of the cylinder is in contact with the plane. The plane is sufficiently rough to prevent sliding. It is given that $3h = 2r$ and that the solid is on the point of toppling when $\tan\theta = \frac{4}{3}$.

(b) Find the value of $k$. [S-21/31/Q4] --[3]

**Solution:** (a)

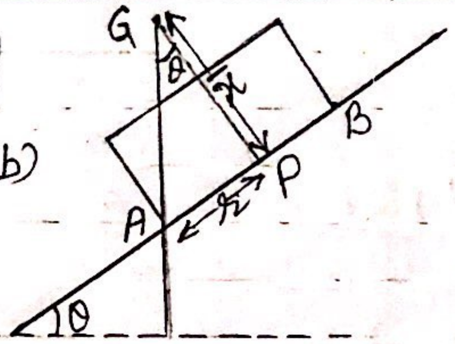
| | Volume | Centre of mass from the base of Cylinder |
|---|---|---|
| Cone | $\frac{1}{3}\pi r^2 \cdot kh$ | $h + \frac{1}{4}kh$ |
| Cylinder | $\pi r^2 h$ | $h/2$ |
| Combined | $\pi r^2 h\left(\frac{1}{3}k+1\right)$ | $\bar{x}$ |

Taking moment about base:

$$\pi r^2 h\left(\tfrac{1}{3}k+1\right)\bar{x} = \tfrac{1}{3}\pi r^2 \cdot kh\left(h+\tfrac{kh}{4}\right) + \pi r^2 h \cdot h/2$$

$$\Rightarrow \bar{x} = \frac{h(k^2+4k+6)}{4(3+k)}$$

(b)




In rt $\Delta GAP$

Given $2r = 3h$, $r = \frac{3}{2}h$

$$\tan\theta = \frac{AP}{GP} = \frac{r}{\bar{x}}$$

$$\Rightarrow \frac{4}{3} = \frac{\frac{3}{2}h}{\frac{h(k^2+4k+6)}{4(3+k)}}$$

$$\Rightarrow \frac{4}{3} = \frac{6(k+3)}{k^2+4k+6}$$

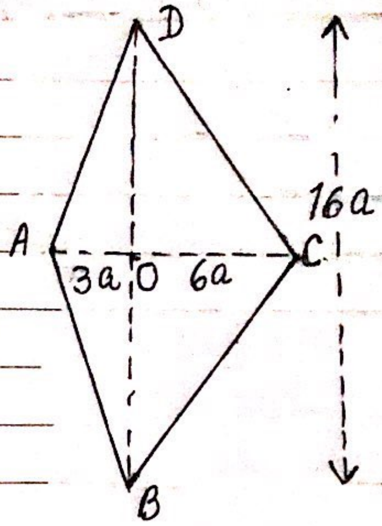
$$\Rightarrow 2k^2 - k - 15 = 0$$

$$(k-3)(2k+5) = 0$$

$\underline{k = 3}$ ✓ ; $k = -\frac{5}{2}$ ✗

7. A uniform lamina ABCD consists of two isosceles triangles ABD and BCD. The diagonals of ABCD meet at the point O. The length AO is $3a$, the length OC is $6a$ and the length of BD is $16a$. Find the distance of the centre of mass of the lamina from BD. ...[3]

S-21/33/Q1




Solution:

| | Area | Centre of mass from DB. |
|---|---|---|
| $\Delta ABD$ | $\frac{1}{2} \times 3a \times 16a = 24a^2$ | $-a$ |
| $\Delta BCD$ | $\frac{1}{2} \cdot 6a \times 16a = 48a^2$ | $2a$ |
| Combined | $72a^2$ | $\bar{x}$ |

Taking moments about BD:

$$72a^2 \cdot \bar{x} = 24a^2 \times -a + 48a^2 \times 2a$$
$$72a^2 \cdot \bar{x} = 72a^3$$
$$\Rightarrow \bar{x} = a \checkmark$$