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FMe

Further Mechanics

Linear Motion
Under a Variable Force
Notes and Revision.

SP-20 / S-20 / W-20 / S-21 /

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Liner Motion under a variable force.

Notes

Newtons' Second Law of Motion :

Force $F = m.a$

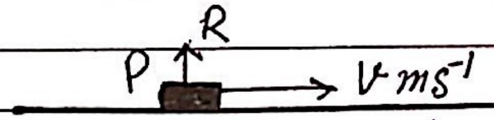
$$\left\{\begin{array}{l} \text{Acceleration } a = \dfrac{dv}{dt} \\ \text{or} \end{array}\right.$$

$$\left\{\begin{array}{l} a = \dfrac{dv}{dx} \times \dfrac{dx}{dt} \\ a = v \cdot \dfrac{dv}{dx} \end{array}\right.$$

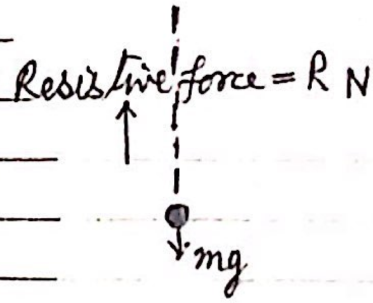
$$F = m.\frac{dv}{dt}$$

or

$$F = m v \frac{dv}{dx}$$



Note 1: If a particle P is moving with a velocity $v\ ms^{-1}$ on a horizontal surface and it loses its contact with the surface then $R = 0$ (R is the normal contact force)



Note 2: If a particle of mass m kg falling under gravity and a resistive force of R N acting on it. Then

$$F = (mg - R)\ N$$

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Linear Motion under a variable force
Revision

1 A particle P is moving along a straight line with acceleration $3ku - kv$ where v is its velocity at time t. u is its initial velocity and k is a constant. The velocity and acceleration of P are both in the direction of increasing displacement from the initial position.

(a) Find the time taken for P to achieve a velocity of $2u$. ---[3]

(b) Find an expression for the displacement of P from its initial position when its velocity is $2u$. ---[5]

[S-20/31/Q5]

Solution: Given acceleration $a = 3ku - kv$

$$\text{or } \frac{dv}{dt} = k(3u - v)$$

$$\Rightarrow \int \frac{dv}{(3u-v)} = \int k\,dt$$

$$\Rightarrow -\ln(3u - v) = kt + c \text{ ---(i)}$$

Now at $t=0$, $v=u \Rightarrow -\ln(3u-u) = 0 + c \Rightarrow c = -\ln 2u$

from (i) $-\ln(3u-v) = kt - \ln 2u$

$$\Rightarrow kt = \ln\left(\frac{2u}{3u-v}\right) \text{ ---(ii)}$$

Now when $v = 2u$

$$\Rightarrow kt = \ln\left(\frac{2u}{3u-2u}\right) \Rightarrow kt = \ln 2 \Rightarrow t = \frac{1}{k}\ln 2$$

(b) $a = k(3u - v)$

$$v\cdot\frac{dv}{dx} = k(3u-v) \quad \left[a = v\cdot\frac{dv}{dx}\right]$$

$$\Rightarrow \int \frac{v}{(3u-v)}\,dv = \int k\,dx$$

$$\Rightarrow \int\left[-1 + \frac{3u}{(3u-v)}\right]dv = \int k\,dx$$

$$\Rightarrow -v - 3u\ln(3u-v) = kx + c \text{ --(iii)}$$

Now $x=0$, $v=u$

$$\Rightarrow c = -u - 3u\ln 2u$$

from (iii)

$$-v - 3u\ln(3u-v) = kx - u - 3u\ln 2u$$

$$kx = u - v + 3u\ln\left(\frac{2u}{3u-v}\right)$$

for $v = 2u$

$$kx = u - 2u + 3u\ln\left(\frac{2u}{3u-2u}\right)$$

$$\Rightarrow kx = -u + 3u\ln 2$$

$$\Rightarrow x = \frac{u}{k}(3\ln 2 - 1)$$

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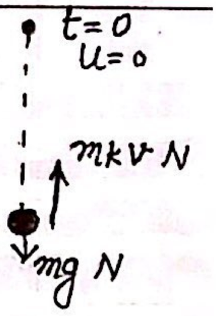
2. A particle Q of mass m kg falls from rest under gravity. The motion of Q is resisted by a force of magnitude mkv N, where $v\ ms^{-1}$ is the speed of Q at time t s and k is a positive constant. Find an expression for v in terms of g, k and t. --[6]

[S-20/33/Q2]

Solution: By Newtons second law of motion:



$$F = ma$$

$$(mg - mkv) = ma$$

$$\Rightarrow a = (g - kv)$$

$$\Rightarrow \frac{dv}{dt} = (g - kv)$$

$$\Rightarrow \int \frac{1}{(g-kv)} dv = \int 1 \cdot dt$$

$$\Rightarrow -\frac{1}{k} \ln(g - kv) = t + c \quad \text{---(i)}$$

for $t=0,\ v=0 \Rightarrow -\frac{1}{k}\ln g = c$

from (1)

$$-\frac{1}{k}\ln(g-kv) = t - \frac{1}{k}\ln g$$

$$\Rightarrow t = -\frac{1}{k}\left[-\ln g + \ln(g-kv)\right] = -\frac{1}{k}\ln\left(\frac{g-kv}{g}\right)$$

$$\Rightarrow \ln\left(\frac{g-kv}{g}\right) = -kt$$

$$\Rightarrow \frac{g-kv}{g} = e^{-kt}$$

$$\Rightarrow g - kv = g e^{-kt}$$

$$\Rightarrow -kv = g\left[e^{-kt} - 1\right]$$

$$v = \frac{g}{k}\left[1 - e^{-kt}\right] \checkmark$$

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3. A particle P moving in a straight line has displacement x m from a fixed point O on the line at time t s. The acceleration of P, in ms^{-2} is given by $\frac{200}{x^2} - \frac{100}{x^3}$ for $x>0$. When $t=0$, $x=1$ and P has velocity $10\ ms^{-1}$, directed towards O.

(a) Show that the velocity $v\ ms^{-1}$ of P is given by $v = \frac{10(1-2x)}{x}$ --[5]

(b) Show that x and t are related by the equation $e^{-40t} = (2x-1)e^{2x-2}$ and deduce what happens to x as t becomes large. ---[5]

[W-20/31/Q7]

Solution (a) $a = v\frac{dv}{dx} = \frac{200}{x^2} - \frac{100}{x^3}$ for $x>0$

$\Rightarrow \int v\,dv = \int\left(\frac{200}{x^2} - \frac{100}{x^3}\right)dx$

$\frac{v^2}{2} = -\frac{200}{x} + \frac{50}{x^2} + C$ ---(i)

Now for $x=1$, $v=-10$,

from (i) $50 = -200+50+C$

$\Rightarrow C = 200$

from (i) $\frac{v^2}{2} = -\frac{200}{x} + \frac{50}{x^2} + 200$

$= \frac{200x^2 - 200x + 50}{x^2}$

$\frac{v^2}{2} = \frac{50[4x^2-4x+1]}{x^2}$

$v^2 = \frac{100(2x-1)^2}{x^2}$

$\Rightarrow v = \pm\frac{10(2x-1)}{x}$

$v = \frac{10(1-2x)}{x}$ ✓; $v = \frac{10(2x-1)}{x}$ ✗ (as $x=1$, $v=-10$)

$\therefore v = \frac{10(1-2x)}{x}$ ✓ ---(ii)

(b) $v = \frac{dx}{dt} = \frac{10(1-2x)}{x}$ [from (ii) Part (a)

$\Rightarrow \int\frac{x}{(1-2x)}dx = \int 10\,dt$

$\frac{1}{2}\int\left(\frac{1}{1-2x} - 1\right)dx = 10t + d$

$\Rightarrow \frac{1}{2}\left[\frac{\ln|1-2x|}{-2} - x\right] = 10t + d$

$-\frac{1}{4}\ln|1-2x| - \frac{x}{2} = 10t + d$ ---(iii)

Now $t=0$, $x=1 \Rightarrow d = -\frac{1}{2}$

from (iii)

$-\frac{1}{4}\ln|1-2x| - \frac{x}{2} = 10t - \frac{1}{2}$

$-\ln|1-2x| - 2x = 40t - 2$

$\Rightarrow 2x-2 = -40t - \ln|1-2x|$

$\Rightarrow \ln|1-2x| = (-40t - (2x-2))$

$\Rightarrow |1-2x| = e^{-40t-(2x-2)}$

$|1-2x| = \frac{e^{-40t}}{e^{2x-2}}$

$\Rightarrow e^{-40t} = (2x-1)e^{2x-2}$

$(|1-2x| = 2x-1)$

Now $t\to\infty$, $e^{-40t} \to 0 \Rightarrow (2x-1) \to 0$ for $x>0$

$\Rightarrow x \to \frac{1}{2}$ ✓

4. A particle of mass m kg moves in a horizontal straight line against a resistive force of magnitude mkv^2 N, where v ms^{-1} is the speed of P after it has moved a distance x m and k is a positive constant. The initial speed of P is u ms^{-1}.

(a) Show that $x = \frac{1}{k}\ln 2$, when $v = \frac{1}{2}u$ ---[4].

Beginning at the instant when the speed of P is $\frac{1}{2}u$, an additional force acts of P. This force has magnitude $\frac{5m}{v}$ N and acts in the direction of increasing x.

(b) Show that when the speed of P has increased again to u ms^{-1}, the total distance travelled by P is given by an expression of the form, $\frac{1}{3k}\ln\left(\frac{A-ku^3}{B-ku^3}\right)$, stating the values of A and B. ---[7]

[W-20/32/Q7]

Solution: $F = ma$

(a) $\Rightarrow -mkv^2 = m.v\frac{dv}{dx}$

$\Rightarrow \frac{dv}{dx} = -kv$

$\Rightarrow \int\frac{dv}{v} = -\int k\,dx$

$\Rightarrow \ln v = -kx + c$ --(i)

Now $x = 0, v = u$

$\Rightarrow c = \ln u$

from (i) $\ln v = -kx + \ln u$

$\Rightarrow kx = \ln\left(\frac{u}{v}\right)$

when $v = \frac{u}{2}$

$kx = \ln\left(\frac{u}{u/2}\right)$

$kx = \ln 2$

$\Rightarrow x = \frac{1}{k}\ln 2$ ✓

(b) Now when initial speed is $\frac{1}{2}u$ and an addition force $\frac{5m}{v}$ N is applied.

$F = ma$

$\Rightarrow \frac{5m}{v} - mkv^2 = m.v\frac{dv}{dx}$

$\Rightarrow \frac{(5-kv^3)}{v} = v\frac{dv}{dx}$

$\Rightarrow \int\frac{v^2}{(5-kv^3)}dv = \int dx$

Put $5 - kv^3 = u$
$-3kv^2 dv = du$

$-\frac{1}{3k}\int\frac{1}{u}du = x + d$

$\Rightarrow -\frac{1}{3k}\ln u = x + d$

$\Rightarrow -\frac{1}{3k}\ln(5-kv^3) = x + d$ ---(ii)

Now when initial speed is $\frac{1}{2}u$

$v = \frac{1}{2}u$, $x = \frac{1}{k}\ln 2$,

$\Rightarrow -\frac{1}{3k}\ln\left(5 - \frac{ku^3}{8}\right) = \frac{1}{k}\ln 2 + d$

$\Rightarrow d = -\frac{1}{3k}\ln(40 - ku^3)$

from (ii) $x = \frac{1}{3k}\ln(40-ku^3) - \frac{1}{3k}(5-kv^3)$

$x = \frac{1}{3k}\left[\ln\frac{(40-ku^3)}{(5-kv^3)}\right]$

$\therefore$ when $v = u \Rightarrow$ $x = \frac{1}{3k}\ln\left(\frac{40-ku^3}{5-ku^3}\right)$ ✓

5. A particle of mass m kg falls from rest under gravity. There is a resistive force of magnitude mkv^2 N, where $v\ ms^{-1}$ is the speed of P after it has fallen a distance x km and k is a positive constant.

(a) By solving an appropriate differential equation, show that
$$v^2 = \frac{g}{k}\cdot(1-e^{-2kx}) \quad ---[7]$$

It is given that $k=0.01$, the speed of P when x becomes large approaches $V ms^{-1}$.

(b) (i) Find V correct to two decimal places. --[1]

(ii) Hence find how far P has fallen when its speed is $\frac{1}{2}V\ ms^{-1}$. --[2]

[SP-20/03/Q3]

Solution: $F = ma$ (Downward motion is taken positive)

(a)
$$\Rightarrow mg - mkv^2 = m\cdot v\frac{dv}{dx}$$
$$\Rightarrow v\frac{dv}{dx} = g - kv^2$$
$$\Rightarrow \int \frac{v}{(g-kv^2)}dv = \int dx$$
$$\Rightarrow -\frac{1}{2k}\int \frac{-2kv\,dv}{(g-kv^2)} = x + c$$
$$\Rightarrow -\frac{1}{2k}\ln(g-kv^2) = x + c \quad ---(i)$$
when $x=0$, $v=0 \Rightarrow c = -\frac{1}{2k}\ln g$

from (i) $x = \frac{1}{2k}\ln g - \frac{1}{2k}\ln(g-kv^2)$
$$\Rightarrow x = \frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right) \quad --(ii)$$
$$\Rightarrow 2kx = \ln\left(\frac{g}{g-kv^2}\right)$$
$$\Rightarrow e^{2kx} = \frac{g}{g-kv^2}$$
$$\Rightarrow g - kv^2 = ge^{-2kx}$$
$$v^2 = \frac{g}{k}(1-e^{-2kx}) \quad --(iii)$$

(b) Now when $x\to\infty$, $e^{-2kx}\to 0$

(i) from Part (a) equation (iii)
$$V^2 = \frac{g}{k}(1-0) = \frac{10}{0.01} = 1000$$
$$\Rightarrow V = \sqrt{1000} = 31.62\ ms^{-1}$$

(ii) Velocity $= \frac{1}{2}V = \frac{1}{2}\times 31.62$

from (ii)
$$x = \frac{1}{2k}\ln\left(\frac{g}{g-0.01\left(\frac{31.62}{2}\right)^2}\right)$$
$$x = \frac{1}{2\times 0.01}\ln\left(\frac{10}{10-0.01\times 250}\right)$$
$$= 50\ln\left(\frac{10}{7.5}\right)$$
$$= 50\ln\frac{4}{3} = 14.371$$
$$\therefore x = 14.4\ m$$

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6. A particle P of mass 1 Kg is moving along a straight line against a resistive force of magnitude $\frac{10\sqrt{v}}{(t+1)^2}$ N, where $v\ ms^{-1}$ is the speed of P at time t s.
When $t=0$, $v=25$.
Find an expression for v in terms of t. ---[5]
[S-21/31/Q1]

Solution: $F = ma \Rightarrow -\frac{10\sqrt{v}}{(t+1)^2} = 1 \times \frac{dv}{dt}$

$$\Rightarrow \int \frac{dv}{\sqrt{v}} = -\int \frac{10\,dt}{(t+1)^2}$$

$$\Rightarrow 2\sqrt{v} = \frac{10}{(t+1)} + C \quad \text{---(i)}$$

$t=0$; $v=25$

$$\Rightarrow 2\sqrt{25} = 10 + C \Rightarrow C = 0$$

from (i) $2\sqrt{v} = \frac{10}{t+1}$

$$\Rightarrow v = \frac{25}{(t+1)^2} \checkmark$$

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7. A particle P of mass m kg is projected vertically upwards from a point O, with speed $20 ms^{-1}$, and moves under gravity. There is a resistive force of magnitude $2mv$ N, where v ms^{-1} is the speed of P at time t s after projection.

(a) Find an expression for v in the terms of t, while P is moving upwards. ---[6]

The displacement of P from O is x m at time t s.

(b) Find an expression for x in terms of t, while P is moving upwards. ---[2]

(c) Find, correct to 3 significant figures, the greatest height above O reached by P. [S-21/33/Q5] ---[2]

Solution: $F = ma$

(a) $-mg - 2mV = m \cdot \frac{dV}{dt}$

$\Rightarrow \frac{dV}{dt} = -(g+2V)$

$= -(10+2V)$

$= -2(5+V)$

$\Rightarrow \int \frac{1}{(5+V)} dV = \int -2dt$

$\Rightarrow \ln(5+V) = -2t + C$ ---(i)

$t=0,\ v=20$

$\Rightarrow \ln 25 = 0 + C$

from (i) $\ln(5+V) = -2t + \ln 25$

$\Rightarrow 2t = \ln\left(\frac{25}{5+V}\right)$ ---(ii)

$\Rightarrow \frac{25}{5+V} = e^{2t}$

$\Rightarrow 5+V = 25e^{-2t}$

$\Rightarrow V = 25e^{-2t} - 5$ ---(iii)

(b) from part (a) $V = \frac{dx}{dt} = 25e^{-2t} - 5$

$\Rightarrow \int dx = \int (25e^{-2t} - 5)$

$\Rightarrow x = -\frac{25}{2} e^{-2t} - 5t + d$

$t=0,\ x=0 \Rightarrow d = \frac{25}{2}$

$\therefore x = -\frac{25}{2} e^{-2t} - 5t + \frac{25}{2}$

$x = \frac{25}{2}\left(1 - e^{-2t}\right) - 5t$ ✓ (iv)

(c) Greatest height, $v=0$

from (ii) $t = \frac{1}{2} \ln\left(\frac{25}{5+0}\right)$

$t = 0.8047$

from (iv)

$x = \frac{25}{2}\left(1 - e^{-2\times 0.8047}\right) - 5\times 0.804$

$= \frac{25}{2}[1 - 0.2] - 4.0235$

$= 10 - 4.0235$

$= 5.9765$

$= 5.98$ m ✓

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8. A particle P of mass 0.4 kg is projected horizontally along a smooth horizontal plane from a point 'O'. After projection the velocity of P is $v\ ms^{-1}$ and its displacement from O is 'x m'. A force of magnitude '$8x$ N' directed away from O acts on P and a force of magnitude '$(2e^{-x}+4)$ N' opposes the motion of P. One end of the elastic string of natural length '0.5 m' is attached to O and other end of the string is attached to P.

(i) Show that $v\frac{dv}{dx} = 20x - 10 - 5e^{-x}$ before the elastic string becomes taut. --- [2]

(ii) Given that the initial velocity of P is $6ms^{-1}$, find v when the string first becomes taut. -- [3]

When the string is taut the acceleration of P is proportion to e^{-x}.

(iii) Find the magnitude of elasticity of the string. -- [2]

(M2) → W-18 | 52 | Q3

Solution: $F = ma = m \cdot v\frac{dv}{dx}$

(i) $\Rightarrow 8x - (2e^{-x}+4) = 0.4 \times v\frac{dv}{dx}$

$\Rightarrow v\frac{dv}{dx} = 20x - 10 - 5e^{-x}$ ✓

(ii) $\int v\,dv = \int (20x - 10 - 5e^{-x})\,dx$

$\frac{v^2}{2} = 10x^2 - 10x + 5e^{-x} + C$

$x=0,\ v=6 \Rightarrow C = 13$

$\therefore v^2 = 2[10x^2 - 10x + 5e^{-x} + 13]$ --- (1)

string is taut at $x = 0.5 \Rightarrow v^2 = 27$

$\Rightarrow v = 5.2 ms^{-1}$ ✓

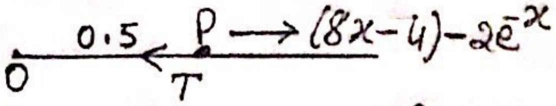
(iii) Using Hooke's Law

$T = \frac{\lambda e}{l} \Rightarrow 8x - 4 = \frac{\lambda(x-0.5)}{0.5}$

$\Rightarrow 4(2x-1) = \lambda(2x-1)$

$\Rightarrow \lambda = 4$ ✓

$[(2x-1) \neq 0]$
$[\lambda \neq \frac{1}{2}]$



$F = [(8x-4) - 2e^{-x}] - T$
$= ke^{-x}$

$\Rightarrow T = 8x - 4$

9. O and A are fixed points on a horizontal surface with OA = '0.5m'. A particle P of mass '0.2kg' is projected horizontally with speed $3ms^{-1}$ from A in the direction OA and moves in a straight line. At t s after projection, the velocity of P is $v\ ms^{-1}$ and its displacement from O is 'x m'. The coefficient of friction between the surface and P is '0.5' and a force of magnitude $0.4/x^2$ N acts on P in the direction PO.

(i) Show that, while the particle is in motion, $v\frac{dv}{dx} = -\left(5+\frac{2}{x^2}\right)$ ---[2]

(ii) Calculate the distance travelled by P, before it comes to rest, and show that P does subsequently move. [M2 | S-11 | 53 | Q6] --[7]

P, $v = 3ms^{-1}$

O ← 0.5 → A ↓0.2g

Solution: $F = ma \Rightarrow mv\frac{dv}{dx} = \text{Friction} - \frac{0.4}{x^2}$

{Force of friction $= \mu R = 0.5 \times 0.2g = 1$ ✓

(i) $\Rightarrow 0.2\,v\frac{dv}{dx} = -1 - 0.4/x^2$

$\Rightarrow v\frac{dv}{dx} = -5 - 2/x^2$

$v\frac{dv}{dx} = -\left(5+\frac{2}{x^2}\right)$ ✓ ---- (1)

(ii) $\int v\,dv = -\int(5+2x^{-2})dx$

$\frac{v^2}{2} = -5x + \frac{2}{x} + C$

$v = 3$ at $x = 0.5 \Rightarrow \frac{3^2}{2} = -5\times0.5 + \frac{2}{0.5} + C \Rightarrow C = 3$

$\therefore \frac{v^2}{2} = -5x + \frac{2}{x} + 3$ ---- (2)

P comes to rest when $v = 0$, from (2) $\Rightarrow 0 = -5x + \frac{2}{x} + 3$

$\Rightarrow 5x^2 - 3x - 2 = 0$

$(x-1)(5x+2) = 0$

$x = 1$ ✓, $x = -\frac{2}{5}$ ✗

$\therefore$ distance travelled by P before it comes to rest $= 1 - 0.5$

$= 0.5\,m$ ✓

Now from (2) $\frac{v^2}{2} = -\frac{1}{x}(5x^2 - 3x - 2) > 0$ [$\because v^2 > 0$]

$\frac{1}{x}(x-1)(5x+2) < 0$

$(x-1)(5x+2) < 0$ [$x = 1$, $x = -2/5$, $x > 0$

$-2/5 < x < 1 \Rightarrow 0 < x < 1$

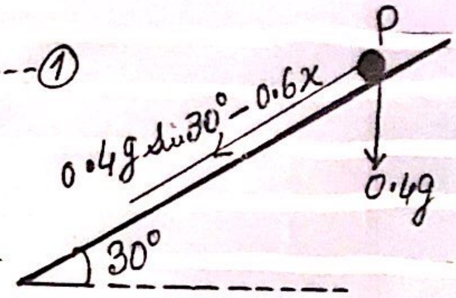
$\therefore$ P does not subsequently move. (after $x = 1$) ✓

10. A particle P of mass 0.4 kg is released from rest at the top of a smooth plane inclined at 30° to the horizontal. The motion of P down the slope is opposed by a force of magnitude '0.6x' N, where 'x' m is the distance P travelled down the slope. P comes to rest before reaching the foot of the slope.
Calculate:
(i) the greatest speed of P during the motion. ---[7]
(ii) the distance travelled by P during its motion. ---[2]

[M2/S-22/51/Q5]

Solution: $F = ma$

(i) $0.4 v\frac{dv}{dx} = 0.4g \sin 30° - 0.6x$ ---①



$\Rightarrow v\frac{dv}{dx} = 5 - 1.5x$

$\Rightarrow \int v\,dv = \int (5 - 1.5x)\,dx$

$\frac{v^2}{2} = 5x - \frac{1.5}{2}x^2 + c$

$x = 0, v = 0 \Rightarrow c = 0$

$\therefore \frac{v^2}{2} = 5x - \frac{1.5}{2}x^2$ ---②

for the greatest speed acc $a = 0$ or from ① $F = 0$

from ① $\Rightarrow 0.4g \sin 30° - 0.6x = 0$

$5 - 1.5x = 0 \Rightarrow x = \frac{10}{3}$ ✓

from ② at $x = \frac{10}{3}$, $\frac{v^2}{2} = \frac{5 \times 10}{3} - \frac{1.5}{2}\cdot\left(\frac{10}{3}\right)^2$

$= \frac{50}{3} - \frac{25}{3} = \frac{25}{3}$

$v^2 = \frac{50}{3} = 16.66\ldots$

$v = \underline{4.08 \text{ ms}^{-1}}$ ✓

(ii) Distance travelled when $v = 0$

from ② $5x - \frac{1.5}{2}x^2 = 0$

$x\left[5 - \frac{1.5}{2}x\right] = 0 \Rightarrow x = 6.67$ or $x = 0$*

$\underline{x = 6.67 \text{ m}}$ ✓