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F.Me

Further Mechanics

Motion of a Projectile
Notes and Revision

SP-20 / S-20 / W-20 / S-21

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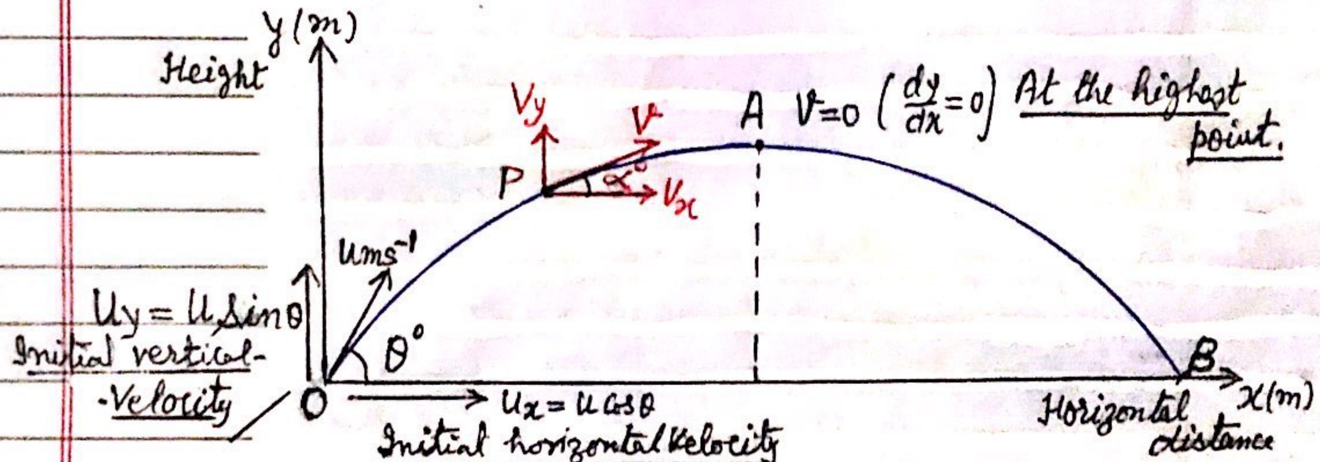
Motion of a Projectile - Notes

If we throw a small object (ball) in air it takes a parabolic path, called its trajectory and the object is called projectile. It happens under certain assumptions:

(a) Object is a small particle, projectile.

(b) The effect of air on its motion is negligible.

(c) It is not powered, it moves under the gravitational force of earth, has a vertically downwards acceleration $g\ ms^{-2}$ and there is no horizontal acceleration.



(i) Let a particle is thrown with a velocity of $U\ ms^{-1}$ inclined at an angle $\theta°$ with the horizontal upwards.

(ii) Let after time t:

Velocity of the particle $= V$

and angle of inclination $= \alpha°$

The horizontal component of the velocity (remains Same) $V_x = U_x \Rightarrow V\cos\alpha = U\cos\theta$

But vertically, $V_y = V\sin\alpha = U\sin\theta - gt$

Initially at $t=0$, $x=0$, $y=0$

Horizontal component of Velocity $U_x = U\cos\theta$

Vertical component of Velocity $U_y = U\sin\theta$

$U^2 = U_x^2 + U_y^2$

and $\tan\theta = \dfrac{U_y}{U_x}$

Horizontal distance $X = U_x \times t = U\cos\theta . t$ ---(1)

Vertical distance $Y = V_y \times t = U\sin\theta . t - \frac{1}{2}gt^2$ ---(2)

$V^2 = \sqrt{V_x^2 + V_y^2}$

$\tan\alpha = \dfrac{V_y}{V_x}$

(iii) To find the maximum height 'H'

For greatest height the vertical component of Velocity $V_y = 0$

$$\Rightarrow u\sin\theta - gt = 0$$

$$\Rightarrow t = \frac{u\sin\theta}{g} \quad \text{---} (3)$$

from (2)

Vertical distance $y = u\sin\theta\, t - \frac{1}{2}gt^2$

$\Rightarrow$ [for Max. height $t = \frac{u\sin\theta}{g}$ from (3)]

$$H = u\sin\theta \cdot \frac{u\sin\theta}{g} - \frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2$$

$$= \frac{u^2}{g}\sin^2\theta - \frac{1}{2}\frac{u^2}{g}\cdot\sin^2\theta$$

$$\therefore H = \frac{u^2\sin^2\theta}{2g} ✓$$

(iv) Time of flight 'T':

The partical hits the ground when $y = 0$

from (2) $y = u\sin\theta t - \frac{1}{2}gt^2 = 0$

$$t(u\sin\theta - \frac{1}{2}gt) = 0$$

$$\Rightarrow t = \frac{2u\sin\theta}{g} \quad \text{or } (t = 0 \text{ at } O)$$

$$\therefore T = \frac{2u\sin\theta}{g} \text{ --- } (4)$$

[Note: (from (3) and (4))
Time of flight = 2× time to reach the heighest point.]

(v) Range 'R'

from (1) $x = u\cos\theta \times t$

$\Rightarrow$ Range $= u\cos\theta \times \frac{2u\sin\theta}{g}$ [for range $t = \frac{2u\sin\theta}{g}$ frm (4)]

$\Rightarrow R = \frac{u^2\sin2\theta}{g}$

(VI) Equation of trajectory of P:

$y = u\sin\theta\cdot t - \frac{1}{2}gt^2$ frm (2) [from (1) $x = u\cos\theta\cdot t \Rightarrow t = \frac{x}{u\cos\theta}$]

$$y = u\sin\theta \times \frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$= x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

$$y = x\tan\theta - \frac{gx^2}{2u^2}\sec^2\theta. ✓$$

Example 1. A particle is projected with speed $24\,ms^{-1}$ at an angle of 30° above the horizontal. Find the speed and direction of motion of the particle at the instant 4 s after projection. -- [5]

(M2) | M-19 | 52 | Q6

Solution:

$u = 24\,ms^{-1};\ \theta = 30^\circ$

Horizontal Component initially $u_x = 24\cos 30 = 12\sqrt{3}$ ---(i)

Vertical " " $u_y = 24\sin 30^\circ = 12$ --(ii)

After 4 s, Let the speed $= V\,ms^{-1}$ at the inclination $= \alpha^\circ$

$V_x = V\cos\alpha = u\cos\theta = 12\sqrt{3}$ -- ③ (from ①)

$V_y = u\sin\theta^\circ - gt = 24\sin 30 - 10\times 4 = 12 - 40 = -28$ -- ④

$\therefore\ V^2 = V_x^2 + V_y^2 = (12\sqrt{3})^2 + (-28)^2 = 1216 \Rightarrow V = \sqrt{1216} = \underline{34.9\,ms^{-1}}$

and $\tan\alpha = \frac{V_y}{V_x} = -\frac{28}{12\sqrt{3}} = -1.347 \Rightarrow \alpha = -\tan^{-1}1.347 = -53.4^\circ$

or $\alpha = 53.4^\circ$ below the horizontal.

Reverse

Example 2: A particle is projected with speed $V\,ms^{-1}$ at an angle θ° above the horizontal. At the instant 4 s after projection the speed of the particle is $16\,ms^{-1}$ and its direction of motion is 30° above the horizontal. Find V and θ.

(M2) | S-19 | 51 | Q2 | ---[5]

Solution: Initially $\begin{cases} \text{Horizontal Comp. of Velocity } V_x = V\cos\theta & \text{--①} \\ \text{Vertical Comp. of Velocity } V_y = V\sin\theta & \text{--②} \end{cases}$

Now at $t = 4\,s$, Velocity (let) $W = 16\,ms^{-1}$

and direction above horizontal $\alpha = 30^\circ$

Horizontal Comp. $W_x = W\cos\alpha = 16\cos 30 = V\cos\theta$

$\Rightarrow V\cos\theta = 8\sqrt{3}$ --③

and the vertical comp. $W_y = V\sin\theta - 4g\ (t=4)$

$\Rightarrow W\sin\alpha = V\sin\theta - 4g$

$\Rightarrow 16\sin 30^\circ = V\sin\theta - 4g$

$\Rightarrow V\sin\theta = 48$ ---④

$\therefore V^2 = V_x^2 + V_y^2 = (8\sqrt{3})^2 + (48)^2 = 2496$

$V = \sqrt{2496} = \underline{49.96}$

and $\tan\theta = \frac{V_y}{V_x} = \frac{48}{8\sqrt{3}} = 2\sqrt{3} \Rightarrow \theta = \tan^{-1}(2\sqrt{3}) = \underline{73.9^\circ}$

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3. A particle P is projected with speed u at an angle α above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t are denoted by x and y respectively.

(a) Derive the equation of the trajectory of P in the form:

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha. \quad ---[3]$$

(b) The greatest height of P above the plane is denoted by H, when P is at a height of $\frac{3}{4}H$, it has travelled a horizontal distance d.
Given that $\tan\alpha = 2$, find, in terms of H, the two possible values of d.

[SP-20/03/Q6] ---[6]

Solution (a) done on page 2, part (vi). (Also done on Page 7)

(b) The greatest height $H = \frac{u^2\sin^2\alpha}{2g}$ --- (i) [See Page 2 - Part (iii)]

$$= \frac{u^2}{2g} \times \frac{4}{5} \qquad \left\{ \begin{array}{l} \tan\alpha = 2 \\ \Rightarrow \sin\alpha = \frac{2}{\sqrt{5}} \end{array} \right\}$$

$$\therefore H = \frac{2u^2}{5g} \quad ---(ii)$$

Now it is given $y = \frac{3}{4}H$, $x = d$

Equation of trajectory: $y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$ $\left[\begin{array}{l} \tan\alpha = 2 \\ \sec^2\alpha = 5 \end{array} \right]$

$$\Rightarrow \frac{3}{4}H = 2d - \frac{gd^2 \times 5}{2u^2} \qquad [\tan\alpha = 2]$$

$$\Rightarrow \frac{3}{4}H = 2d - \frac{gd^2 \times 5}{2 \times \frac{5gH}{2}} \qquad \left[\begin{array}{l} \text{from (ii)} \\ u^2 = \frac{5gH}{2} \end{array} \right.$$

$$\Rightarrow \frac{3}{4}H = 2d - \frac{d^2}{H}$$

$$\Rightarrow 4d^2 - 8dH + 3H^2 = 0$$

$$(2d - 3H)(2d - H) = 0$$

$$\Rightarrow d = \frac{3}{2}H \;;\; d = \frac{1}{2}H \checkmark$$

4 A particle P is projected with speed u at an angle of 30° above the horizontal from a point O on a horizontal plane, and moves freely under gravity. The particle reaches its greatest height at time T after projection.
Find, in terms of u, the speed of P at time $\frac{2}{3}T$ after projection ---[5]

[S-20/31/Q1]

Solution: For Greatest height;

$$T = \frac{u\sin\theta}{g}$$

$$\left\{\begin{array}{l} V_y = u\sin\theta - gt \text{ --(i)} \\ 0 = u\sin\theta - gt \\ \text{(for greatest height)} \\ \Rightarrow t = \frac{u\sin\theta}{g} \end{array}\right.$$

$$\Rightarrow T = \frac{u}{2g} \qquad \left(\begin{array}{l}\text{for } \theta = 30^\circ \\ \sin\theta = \frac{1}{2}\end{array}\right)$$

Now at $t = \frac{2}{3}T = \frac{2}{3}\times\frac{u}{2g} = \frac{u}{3g}$ ✓ --(ii)

Vertical component of $V_y = u\sin\theta - gt$ (from (i))

$$= u\cdot\frac{1}{2} - g\times\frac{u}{3g} \qquad \left(\sin 30^\circ = \frac{1}{2}\right)$$

$$V_y = \frac{u}{2} - \frac{u}{3} = \frac{u}{6} \checkmark \text{ ---(iii)}$$

and Horizontal component;

$$V_x = u\cos\theta = u\cos 30^\circ = \frac{\sqrt{3}}{2}.u \text{ --(iv)}$$

$$\therefore \text{Speed} = \sqrt{V_x^2 + V_y^2}$$

$$= \sqrt{\left(\frac{\sqrt{3}}{2}u\right)^2 + \left(\frac{u}{6}\right)^2}$$

$$= \sqrt{\frac{3u^2}{4} + \frac{u^2}{36}}$$

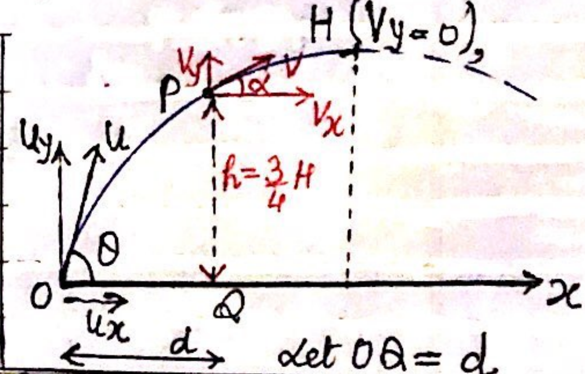
$$= \sqrt{\frac{28u^2}{36}}$$

$$= \frac{\sqrt{7}}{3}u \checkmark$$

5. A particle P is projected with speed u at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The direction of P makes an angle α above the horizontal when P first reaches three-quarters of its greatest height.

(a) Show that $\tan\alpha = \frac{1}{2}\tan\theta$ ---[6]

(b) Given that $\tan\theta = \frac{4}{3}$, find the horizontal distance travelled travelled by P when it first reaches three-quarters of its greatest height. Give your answer in terms of u and g. [S-20/33/Q6] ---[4]



Solution: Greatest height $H = \frac{u^2 \sin^2\theta}{2g}$ ---(i)

(a) At $\frac{3}{4}H \rightarrow$ (at P) $h = \frac{3}{4}\cdot\frac{u^2\sin^2\theta}{2g}$ ---(ii)

Now $u_y = u\sin\theta$ and $V_y = V\sin\alpha$

$V_y^2 = u_y^2 - 2gh$

$\Rightarrow (V\sin\alpha)^2 = (u\sin\theta)^2 - \cancel{2g}\cdot\frac{3}{4}\frac{u^2\sin^2\theta}{\cancel{2g}}$

$\Rightarrow V^2\sin^2\alpha = u^2\sin^2\theta - \frac{3}{4}\sin^2\theta$

$= \frac{1}{4}u\sin^2\theta \Rightarrow V\sin\alpha = \frac{1}{2}u\sin\theta$ ---(iii)

Now the horizontal component of velocity remain same $\Rightarrow V\cos\alpha = u\cos\theta$ ($V_x = u_x$) ---(iv)

from (iii) and (iv) $\Rightarrow \tan\alpha = \frac{1}{2}\tan\theta$ ✓

(b) Now $\tan\theta = \frac{4}{3} \Rightarrow \sin\theta = \frac{4}{5}$ and $\cos\theta = 3/5$

$\therefore \tan\alpha = \frac{1}{2}\tan\theta$ (from Part (a))

$\tan\alpha = \frac{1}{2}\times\frac{4}{3} = \frac{2}{3}$ ✓

Now at $h = \frac{3}{4}H$ from (iii) $V\sin\alpha = \frac{1}{2}\sin\theta$ ---(iii)

at P

$V_y = u_y - gt$

$\Rightarrow V\sin\alpha = u\sin\theta - gt$

$\Rightarrow \frac{1}{2}u\sin\theta = u\sin\theta - gt$ [from iii]

$\Rightarrow gt = \frac{1}{2}u\sin\theta = \frac{1}{2}\times u\times\frac{4}{5}$

$\Rightarrow t = \frac{2u}{5g}$ ---(V)

Now the horizontal distance

$d = u_x \times t$

$= u\cos\theta \times \frac{2u}{5g}$ (from (V))

$= u \times \frac{3}{5}\times\frac{2u}{5g}$

$\therefore d = \frac{6u^2}{25g}$ ✓

6. A particle P is projected with speed u at an angle α above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t are denoted by x and y respectively.

(a) Derive the equation of trajectory of P in the form:

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha. \quad \text{---[3]}$$

- The point Q is the highest point on the trajectory of P in the case where $\alpha = 45°$.

(b) Show that the x-coordinate of Q is $\frac{u^2}{2g}$ ---[3]

Solution (a) $x = u\cos\alpha \cdot t \Rightarrow t = \frac{x}{u\cos\alpha}$ ---(i)

and $y = u\sin\alpha\, t - \frac{1}{2}gt^2$

$\Rightarrow y = u\sin\alpha \cdot \frac{x}{u\cos\alpha} - \frac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2$ [Eliminate t from (i)]

$\Rightarrow y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$ ✓

(b) Greatest height $H = \frac{u^2\sin^2\alpha}{2g} = \frac{u^2\sin^2 45°}{2g} = \frac{u^2}{4g}$ $(\because \alpha = 45°)$

$V_y = u_y - gt \Rightarrow 0 = u\sin\theta - gt$ [At greatest height $V_y = 0$

At Q $\rightarrow$ $\Rightarrow t = \frac{u\sin\theta}{g} = \frac{u\sin 45°}{g}$ ---(i)

$\therefore$ x-coordinate of Q $= u\cos\alpha \cdot t$

$= u\cos 45 \times \frac{u\sin 45}{g}$ (from (i) $t = \frac{u\sin 45}{g}$)

$= \frac{u^2}{2g}$ ✓

7. A particle P is projected with speed $u\,ms^{-1}$ at an angle θ above the horizontal from a point O on a horizonal plane and moves freely under gravity. The horizonal and vertical displacements of P from O at a subsequent time t s are denoted by x m and y m respectively.

(a) Starting from the $y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta)$ equation of trajectory given in the List of formulae (MF 19). Show that; --- [1]

when $\theta = \tan^{-1}2$, P passes through the point with coordinates $(10,16)$.

(b) Show that there is no value of θ for which P can pass through the point with coordinates $(18,30)$. [W-20/32/Q5] --[6]

Solution: From (MF-19), put $\cos\theta = \frac{1}{\sec\theta}$ and $\sec^2\theta = 1+\tan^2\theta$

we get: $y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta)$ ✓ ---(i) --[1]

(b) for $\tan\theta = 2$; and pass through $(10,16)$ in (i)

$$16 = 10\times 2 - \frac{10\times 10^2}{2u^2}(1+4)$$

$$\Rightarrow u^2 = 625 \Rightarrow u = 25 ✓$$

Now again put the point $(18,30)$ in (i)

$$30 = 18\cdot\tan\theta - \frac{10\times 18^2}{2\times 625}(1+\tan^2\theta) \quad (\text{for } u=25)$$

$$\Rightarrow \quad 2.592\tan^2\theta - 18\tan\theta + 32.592 = 0 \quad ---(ii)$$

Now $b^2 - 4ac = 324 - 4\times 2.592\times 32.592$

Discriminant $= -13.90 < 0$

$\therefore$ Equation (ii) has no real solution for 'θ' ✓

8. A particle P is projected from a point O on a horizontal plane and moves freely under gravity. The initial velocity of P is 100 m s⁻¹ at an angle θ above the horizontal, where $\tan\theta = \frac{4}{3}$. The two times at which P's height above the plane is H m differ by 10 s.

(a) Find the value of H. ---[5]

(b) Find the magnitude and direction of the velocity of P one second before it strikes the plane. [S-21/31/Q7] --[4]

Solution: $u = 100\,ms^{-1}$, $\tan\theta = \frac{4}{3}$, $\sin\theta = \frac{4}{5}$, $\cos\theta = 3/5$

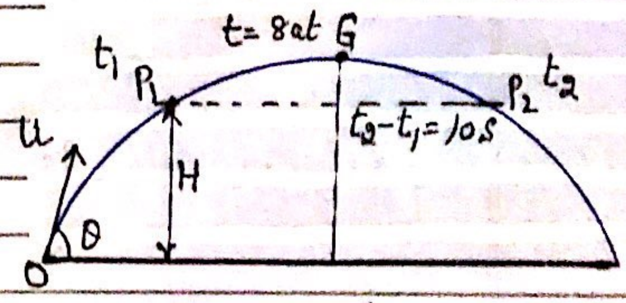
(a) $t_2 - t_1 = 10\,s$ ---(i)

At greatest height $V_y = u\sin\theta - gt$

$\rightarrow 0 = 100\sin\theta - gt$

$\Rightarrow 0 = 100\times\frac{4}{5} - 10t$

$\Rightarrow t = 8\,s$



Now $t - t_1 = t_2 - t = \frac{10}{2} = 5$

$\Rightarrow t_1 = 8 - 5 = 3$ and $t_2 = 8 + 5 = 13$

$H = u_y\cdot t - \frac{1}{2}gt^2 = 100\sin\theta\times 3 - \frac{1}{2}\times 10\times 3^2$ [$t_1 = 3$]

$= 100\times\frac{4}{5}\times 3 - 45$

$\therefore H = \underline{195\,m}$

(b) Total time of flight $= 8\times 2 = 16\,s$

$\therefore$ Time to required point $= 16 - 1 = 15\,s$

$V_y = u_y - gt = 100\sin\theta - 10\times 15 = 100\times\frac{4}{5} - 150 = -70$

$V_y = -70$

and $V_x = 100\cos\theta = 100\times\frac{3}{5} = 60$

$V = \sqrt{V_y^2 + V_x^2} = \sqrt{(-70)^2 + (60)^2} = \underline{92.2\,ms^{-1}}$

Angle below horizontal $= \tan^{-1}\left(\frac{70}{60}\right)$ $\left[\because \tan\theta = \frac{V_y}{V_x} = \frac{-70}{60}\right.$

$= \underline{49.4°}$

9. A particle P is projected with speed u at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t are denoted by x and y respectively.

(a) Use the equation of the trajectory given in the List of formulae (MF 19), together with condition $y=0$, to establish an expression for the range R in terms of u, θ and g. --[2]

(b) Deduce an expression for the maximum height H in terms of u, θ and g. --[2]

It is given that $R = \frac{4H}{\sqrt{3}}$

(c) Show that $\theta = 60°$ --[1]

It is given also that $u = \sqrt{40}\ ms^{-1}$

(d) Find, by differentiating the equation of the trajectory or otherwise, the set of values of x for which the direction of motion makes an angle less than 45° with the horizontal. [S-21/33/Q7] ---[4]

Solution: Equation of trajectory $y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$ ---(i)

(a) Now $y = 0 \Rightarrow x = R$

$\rightarrow 0 = R\tan\theta - \frac{gR^2}{2u^2\cos^2\theta}$

$\Rightarrow R = \frac{2u^2}{g}\cos^2\theta.\tan\theta$

$\Rightarrow R = \frac{2u^2\sin\theta\cos\theta}{g}$ ✓ ---(ii)

(b) Now Maximum height 'H' $\rightarrow x = \frac{R}{2}$

from (ii) $x = \frac{1}{2}R = \frac{u^2\sin\theta\cos\theta}{g}$ and $y = H$ in (i)

$\Rightarrow H = \frac{u^2}{g}\sin\theta\cos\theta.\tan\theta - \frac{g}{2u^2\cos^2\theta}\times\frac{u^4\sin^2\theta\cos^2\theta}{g^2}$

$= \frac{u^2}{g}\sin^2\theta - \frac{1}{2}.\frac{u^2}{g}.\sin^2\theta$

$H = \frac{u^2}{2g}\sin^2\theta$ ✓ ---(iii)

(c) from (ii) $R = \frac{2u^2\sin\theta\cos\theta}{g} = \frac{4H}{\sqrt{3}}$ (Given)

$\Rightarrow \frac{2u^2\sin\theta\cos\theta}{g} = \frac{4}{\sqrt{3}}\times\frac{u^2\sin^2\theta}{2g}$ from (iii)

$\Rightarrow \tan\theta = \sqrt{3}$

$\rightarrow \theta = 60°$ ✓

(d) diff equation (i) w.r.t x;

$\frac{dy}{dx} = \tan\theta - \frac{gx}{u^2\cos^2\theta}$

$\pm 1 = \sqrt{3} - \frac{10x}{40\times\frac{1}{4}}$

$\theta = 60°$, $u = \sqrt{40}$, $u^2 = 40$

$\Rightarrow x = (\sqrt{3}+1), (\sqrt{3}-1)$

$\therefore \sqrt{3}-1 < x < \sqrt{3}+1$ ✓