FP.2 Complex Numbers – Notes and Revision.

§ de Moivre's Theorem:

Given $z = r(\cos\theta + i\sin\theta)$ is a complex number.

Then:

$$z^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$$ { modulus of $z = |z| = r$; argument of $z = \theta$

Let $P(n)$: $z^n = r^n(\cos n\theta + i\sin n\theta)$ ---(i): $n \in I$.

To prove (i) for positive integer n (using Principle of mathematical induction)

for $n=1$ in (i) $\Rightarrow z^1 = r^1(\cos\theta + i\sin\theta)$ is True $\Rightarrow P(1)$ is true.

Let $P(K)$ is true: $(r(\cos\theta + i\sin\theta))^K = r^K(\cos K\theta + i\sin K\theta)$ ---(ii)

$$\boxed{\text{Consider for } n=(K+1) \text{ To be proved. } (r(\cos\theta + i\sin\theta))^{K+1} = r^{K+1}[\cos(K+1)\theta + i\sin(K+1)\theta]}$$

multiply both sides of (ii) by $r(\cos\theta + i\sin\theta)$

$$r^K(\cos\theta + i\sin\theta)^K \times r(\cos\theta + i\sin\theta) = r^K(\cos K\theta + i\sin K\theta) \cdot r(\cos\theta + i\sin\theta)$$

$$\Rightarrow r^{K+1}(\cos\theta + i\sin\theta)^{K+1} = r^{K+1}[(\cos K\theta\cos\theta + i^2\sin K\theta\sin\theta) + i(\sin K\theta\cos\theta + \cos K\theta\sin\theta)]$$

$$= r^{K+1}[(\cos K\theta\cos\theta - \sin K\theta\sin\theta) + i(\sin K\theta\cos\theta + \cos K\theta\sin\theta)]$$

$$\Rightarrow r^{K+1}(\cos\theta + i\sin\theta)^{K+1} = r^{K+1}[\cos(K+1)\theta + i\sin(K+1)\theta]$$

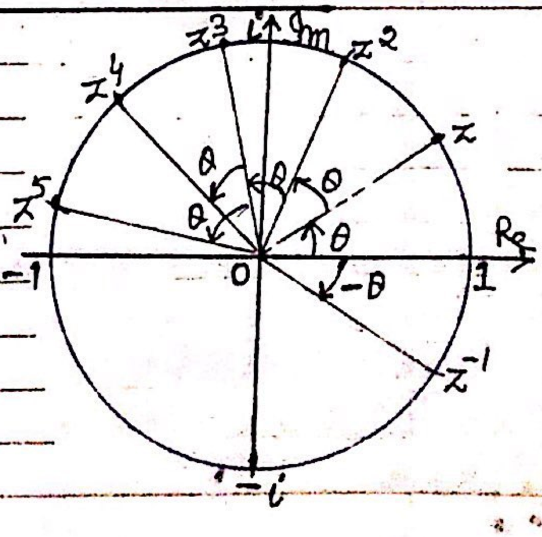
$\Rightarrow$ $P(K+1)$ is true whenever $P(K)$ is true and $P(1)$ is true.

$\therefore$ By P.M.I. $P(n)$ is true for all $n \in N$. (or I^+) ✓

§ Geometrical Representation:

$z = (\cos\theta + i\sin\theta)$; $|z| = r = 1$

Powers of z are equally spaced around the unit circle.



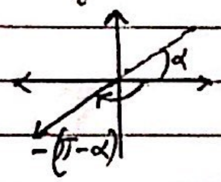
Scanned with CamScanner

Example 1: Find the value of $(-\sqrt{3}-i)^{14}$.

Solution: consider $z = r(\cos\theta + i\sin\theta) = (-\sqrt{3}-i) \Rightarrow \begin{cases} r\cos\theta = -\sqrt{3} \\ r\sin\theta = -1 \end{cases}$

$\therefore z = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$ $\quad \begin{cases} r = \sqrt{(-3)^2+(-1)^2} = 2 \\ \tan\theta = \frac{+1}{+\sqrt{3}} = \tan\frac{\pi}{6} \end{cases}$

$z^{14} = \left[2\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right]^{14}$ $\quad \begin{cases} \theta = -\left(\pi - \frac{\pi}{6}\right) \\ \theta = -\frac{5\pi}{6} \checkmark \end{cases}$



Using de Moivre's theorem:

$= 2^{14}\cos\left(-\frac{5\pi}{6}\times 14\right) + i\sin\left(-\frac{5\pi}{6}\times 14\right)$

$= 16384\left[\cos\left(\frac{70\pi}{6}\right) - i\sin\left(\frac{70\pi}{6}\right)\right]$ $\quad \left\{ \frac{70\pi}{6} = 12\pi - \frac{2\pi}{6} = 12\pi - \frac{\pi}{3} \right.$

$= 16384\left[\cos\left(12\pi - \frac{\pi}{3}\right) - i\sin\left(12\pi - \frac{\pi}{3}\right)\right]$

$= 16384\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = 16384\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \underline{(8192 + i\,8192\sqrt{3})}$ ✓

Example 2: Find the value of $\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)^{8}$ in exact form.

Solution: $\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)^{8} = \cos\left(\frac{8\pi}{12} + i\sin\frac{8\pi}{12}\right)$ $\quad \{$ using de-Moivre's Theorem.

$= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= \cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right)$

$= -\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$= \underline{-\frac{1}{2} + \frac{\sqrt{3}}{2}i}$ ✓

Example 3 (i): Use de Moivre's theorem to prove that:

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta} \quad \text{---[5]}$$

(ii) Hence find the solution of the equation:

$$t^4 - 4t^3 - 6t^2 + 4t + 1 = 0$$

giving your answer in the form $\tan k\pi$, where k is a rational number. [S-17/13/Q7] --[5]

Solution: $(\cos\theta + i\sin\theta)^4 = (\cos 4\theta + i\sin 4\theta)$ ---(i) [Using de Moivre's theorem]

Let $\cos\theta = c$ and $\sin\theta = s$

$(c+is)^4 = c^4 + 4c^3(is) + 6c^2(is)^2 + 4c(is)^3 + (is)^4$ [Using Binomial Theorem]

$\Rightarrow (c+is)^4 = (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3)$ ---(ii)

Comparing real and imaginary parts in (i) & (ii) (R.H.S)

$\left.\begin{aligned}\cos 4\theta &= c^4 - 6c^2s^2 + s^4\\ \sin 4\theta &= 4c^3s - 4cs^3\end{aligned}\right\} \Rightarrow \tan 4\theta = \dfrac{4c^3s - 4cs^3}{c^4 - 6c^2s^2 + s^4}$

(dividing N^r & D^r by c^4)

$$\Rightarrow \tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta} \quad \text{---(iii)}$$

(ii) Now to solve:

$t^4 - 4t^3 - 6t^2 + 4t + 1 = 0$

$\Rightarrow 4t - 4t^3 = -1 + 6t^2 - t^4$

$\Rightarrow 4t - 4t^3 = -1(1 - 6t^2 + t^4)$

$\Rightarrow \dfrac{4t - 4t^3}{1 - 6t^2 + t^4} = -1$ Let $t = \tan\theta$

$\Rightarrow \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = -1$

$\Rightarrow \tan 4\theta = -1$ (from (iii))

$= -\tan\alpha$ $\left[\alpha = \frac{\pi}{4}\right]$

$= \tan(-\alpha)$ $\left[\begin{aligned}&\text{if } \tan\theta = \tan\alpha\\ &\theta = n\pi + \alpha\end{aligned}\right]$

$\therefore 4\theta = n\pi - \alpha$ $n \in I$

$4\theta = n\pi - \frac{\pi}{4}$

$4\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$, for $n = 1, 2, 3, 4$,

$4\theta = \frac{3}{4}\pi, \frac{7}{4}\pi, \frac{11}{4}\pi, \frac{15}{4}\pi$

$\Rightarrow \theta = \frac{3}{16}\pi, \frac{7}{16}\pi, \frac{11}{16}\pi, \frac{15}{16}\pi$

Now $t = \tan\theta$

$\Rightarrow t = \tan\frac{3}{16}\pi, \tan\frac{7\pi}{16}, \tan\frac{11}{16}\pi, \tan\frac{15}{16}\pi$

or $t = \tan k\pi$

for $k = \frac{3}{16}, \frac{7}{16}, \frac{11}{16}, \frac{15}{16}$

Scanned with CamScanner

Example 4 (i) Use de Moivre's theorem to express $\cot 7\theta$ in term of $\cot\theta$ --[4]
(ii) Use the equation $\cot 7\theta = 0$ to show that the roots of the equation $x^6 - 21x^4 + 35x^2 - 7 = 0$ are $\cot\left(\frac{1}{14}k\pi\right)$ for $k = 1,3,5,9,11,13$ and deduce that $\cot^2\left(\frac{1}{14}\pi\right)\cdot\cot^2\left(\frac{3}{14}\pi\right)\cdot\cot^2\left(\frac{5\pi}{14}\right) = 7$. ---[5]

S-16 | 11 | Q6

Solution: Consider $(\cos\theta + i\sin\theta)^7 = (\cos 7\theta + i\sin 7\theta)$ --(i) {using de-Moivre's theorem}

Let $\cos\theta = c$ and $\sin\theta = s$

$$(c+is)^7 = c^7 + 7c^6(is) + 21c^5(is)^2 + 35c^4(is)^3 + 35c^3(is)^4 + 21c^2(is)^5 + 7c(is)^6 + (is)^7$$

$$(c+is)^7 = (c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6) + i(7c^6s - 35c^4s^3 + 21c^2s^5 - s^7) \quad \text{---(ii)}$$

Equating real and imaginary parts in (i) and (ii) →

$$\cos 7\theta = c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6) \quad \text{---(iii)}$$

$$\sin 7\theta = (7c^6s - 35c^4s^3 + 21c^2s^5 - s^7) \quad \text{---(iv)}$$

dividing (iii) by (iv) and N^r and D^r by s^7

$$\cot 7\theta = \frac{\cot^7\theta - 21\cot^5\theta + 35\cot^3\theta - 7\cot\theta}{7\cot^6\theta - 35\cot^4\theta + 21\cot^2\theta - 1} \quad \text{---(V)}$$

(ii) Given $\cot 7\theta = 0 \Rightarrow \cot^7\theta - 21\cot^5\theta + 35\cot^3\theta - 7\cot\theta = 0$

$$\Rightarrow \cot\theta(\cot^6\theta - 21\cot^4\theta + 35\cot^2\theta - 7) = 0$$

$\Rightarrow \cot^6\theta - 21\cot^4\theta + 35\cot^2\theta - 7 = 0$ (for $\cot\theta \neq 0$)

Let $x = \cot\theta \Rightarrow x^6 - 21x^4 + 35x^2 - 7 = 0$ --(VI)

To solve (VI) $\Rightarrow \cot 7\theta = 0 \Rightarrow \cos 7\theta = 0$

$$7\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}$$

$$\theta = \frac{\pi}{14}, \frac{3\pi}{14}, \frac{5\pi}{14}, \frac{9\pi}{14}, \frac{11\pi}{14}, \frac{13\pi}{14}$$

$\left\{ \cot\frac{\pi}{2} = 0 \right.$
$\theta \neq \frac{7\pi}{14}$ or $\frac{\pi}{2}$
(as $\cot\theta \neq 0$

$\therefore$ Req. solutions are $x = \cot\left(\frac{1}{14}k\pi\right)$: $k = 1,3,5,9,11,13$

For equation (VI) the product of roots $= -7$

$$\Rightarrow \cot\frac{\pi}{14}\cdot\cot\frac{3\pi}{14}\cdot\cot\frac{5\pi}{14}\cdot\cot\frac{9\pi}{14}\cdot\cot\frac{11\pi}{14}\cdot\cot\frac{13\pi}{14} = -7$$

$$\Rightarrow -\left[\cot^2\frac{\pi}{14}\cdot\cot^2\frac{3\pi}{14}\cdot\cot^2\frac{5\pi}{14}\right] = -7$$

$$\Rightarrow \cot^2\left(\frac{1}{14}\pi\right)\cdot\cot^2\left(\frac{3}{14}\pi\right)\cdot\cot\left(\frac{5}{14}\pi\right) = 7$$

$\left\{ \cot\frac{13\pi}{14} = \cot\left(\pi - \frac{\pi}{14}\right) = -\cot\frac{\pi}{14} \right.$

$\cot\frac{11\pi}{14} = \cot\left(\pi - \frac{3\pi}{14}\right) = -\cot\frac{3\pi}{14}$

$\cot\frac{\pi}{14} = \cot\left(\pi - \frac{5\pi}{14}\right) = -\cot\frac{5\pi}{14}$

Scanned with CamScanner

Example 5: Using de Moivre's theorem show that:

(i) $$\sec 6\theta = \frac{\sec^6\theta}{32-48\sec^2\theta+18\sec^4\theta-\sec^6\theta}$$ ---[6]

(ii) Hence obtain the roots of the equation:
$$3x^6-36x^4+96x^2-64=0$$
in the form $\sec q\pi$, where q is rational. ---[5]

[W-19/11/Q9]

Solution: Consider $(\cos\theta+i\sin\theta)^6=(\cos 6\theta+i\sin 6\theta)$ ---(i) [Using de-Moivre's Theo]

Now Using Binomial Theo. [Let $\cos\theta=c$ and $\sin\theta=s$]

(i) $$(c+is)^6 = c^6+6c^5 i s+15c^4 i^2 s^2+20c^3 i^3 s^3+15c^2 i^4 s^4+6c\,i^5 s^5+i^6 s^6\ldots$$
$$=(c^6-15c^4s^2+15c^2s^4-s^6)+i(6c^5s-20c^3s^3+6cs^5)\ \text{---(ii)}$$

Comparing real parts in (i) and (ii)
$$\cos 6\theta = c^6-15c^4s^2+15c^2s^4-s^6 = c^6-15c^4(1-c^2)+15c^2(1-c^2)^2-(1-c^2)^3$$
$$= c^6-15c^4(1-c^2)+15c^2(1-2c^2+c^4)-(1-3c^2+3c^4-c^6)$$
$$\Rightarrow \cos 6\theta = 32c^6-48c^4+18c^2-1$$
$$\Rightarrow \sec 6\theta = \frac{1}{\cos 6\theta} = \frac{1}{32c^6-48c^4+18c^2-1} = \frac{\sec^6\theta}{32-48\sec^2\theta+18\sec^4\theta-\sec^6\theta}\ \text{---(iii)}$$
(Dividing N^r and D^r by $\cos^6\theta$)

(ii) Given $3x^6-36x^4+96x^2-64=0$
$$\Rightarrow x^6 = 64-96x^2+36x^4-2x^6 = 2(32-48x^2+18x^4-x^6)$$
$$\Rightarrow \frac{x^6}{32-48x^2+18x^4-x^6}=2$$
for "$x=\sec\theta$"
$$\Rightarrow \sec 6\theta = 2 \quad \text{(from (iii))}$$
$$\Rightarrow \sec 6\theta = \sec\frac{\pi}{3} \quad \left(\cos 6\theta=\frac{1}{2} \Rightarrow \cos 6\theta=\cos\frac{\pi}{3}\right)$$
$$\therefore 6\theta = 2n\pi\pm\frac{\pi}{3} = \frac{6n\pi\pm\pi}{3}$$
$$\Rightarrow \theta = \frac{6n\pi\pm\pi}{18} \quad \text{for } n=0,1,2,3$$
$$\Rightarrow x=\sec\theta; \quad \theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}$$
$$x=\frac{\pi}{18}$$
$$\therefore x = q\pi;\ q=\frac{1}{18}, \frac{5}{18}, \frac{7}{18}, \frac{11}{18}, \frac{13}{18}, \frac{17}{18}$$

Scanned with CamScanner

Example 6(i) Use de Moivre's theorem to show that:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \quad ---[3]$$

(ii) Hence find all the roots of the equation,

$$x^4 - 6x^2 + 1 = 0$$

in the form $\tan q\pi$, where q is a positive rational number ---[5]

[S-18/13/Q3]

Solution: $(\cos\theta + i\sin\theta)^4 = (\cos 4\theta + i\sin 4\theta)$ ---(i) {using de-Moivre's Theorem}

(i) Let $\cos\theta = c$ and $\sin\theta = s$

Now $(c+is)^4 = c^4 + 4c^3 \cdot is + 6c^2 \cdot i^2 s^2 + 4c \cdot i^3 s^3 + i^4 s^4$

$$= (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3) \quad ---(ii)$$

Comparing the real parts in (i) and (ii)

$$\cos 4\theta = c^4 - 6c^2s^2 + s^4$$

or $\underline{\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$ ---(iii)

(ii) from (iii) $\dfrac{\cos 4\theta}{\cos^4\theta} = \tan^4\theta - 6\tan^2\theta + 1$ ---(IV)

now let $x = \tan\theta$, Then

Now for : $x^4 - 6x^2 + 1 = 0$ ---(V)

from (IV) & (V) $\Rightarrow \dfrac{\cos 4\theta}{\cos^4\theta} = x^4 - 6x^2 + 1 = 0$

$$\Rightarrow \cos 4\theta = 0 \quad (\text{and } x = \tan\theta)$$

$$\Rightarrow 4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \quad \left[2n\pi \pm \frac{\pi}{2}\right) \; n \in I$$

$$\text{or } \theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$$

$\therefore$ roots of equation (V) are $x = \tan q\pi$; $\underline{q = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}}$.

Scanned with CamScanner

Example 7: Using de Moivre's theorem, Show that:

(a) $$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$ ---[5]

(b) Hence Show that $x^2 - 10x + 5 = 0$ has roots $\tan^2\left(\frac{1}{5}\pi\right)$ and $\tan^2\left(\frac{2}{5}\pi\right)$ --[5]

[SP-20/02/Q6]

Solution (a) Consider $(\cos\theta + i\sin\theta)^5 = (\cos 5\theta + i\sin 5\theta)$ --(i) [de Moivre's Theorem]

$\therefore$ Also $(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta + 10\cos^3\theta \cdot i^2\sin^2\theta$
$+ 10\cos^2\theta \cdot i^3\sin^3\theta + 5\cos\theta \cdot i^4\sin^4\theta$

$\boxed{(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5}$ (Binomial Theorem)

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + i^5\sin^5\theta$
$+ 5\cos^4\theta\sin\theta + i\sin^5\theta$

$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$ ---(ii)

Comparing real and imaginary parts of (i) and (ii)

$\cos 5\theta = (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta)$ ---(iii)

$\sin 5\theta = (5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$ ---(iv)

Now $\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$

$$= \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta} \quad \text{(from (iii) \& (iv))}$$

$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta} \checkmark \text{---(v)}$$ [dividing N^r and D^r by $\cos^5\theta$]

(b) Now $\tan 5\theta = 0 \Rightarrow 5\theta = n\pi \Rightarrow \theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$ -(vi)

$\Rightarrow 5\tan\theta - 10\tan^3\theta + \tan^5\theta = 0$ from part (a)

or $\tan\theta[\tan^4\theta - 10\tan^2\theta + 5] = 0$

$\Rightarrow \tan^4\theta - 10\tan^2\theta + 5 = 0$ -(vii) [$\tan\theta \neq 0$]

has roots $\frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$ from (vi)

from (vii)

$\left(\tan\theta - \tan\frac{\pi}{5}\right)\left(\tan\theta - \tan\frac{2\pi}{5}\right)\left(\tan\theta - \tan\frac{3\pi}{5}\right)\left(\tan\theta - \tan\frac{4\pi}{5}\right) = 0$

$\Rightarrow \left(\tan\theta - \tan\frac{\pi}{5}\right)\left(\tan\theta - \tan\frac{2\pi}{5}\right)\left(\tan\theta + \tan\frac{2\pi}{5}\right)\left(\tan\theta + \tan\frac{\pi}{5}\right) = 0$ $\left[\begin{array}{l}\tan\frac{4\pi}{5} = -\tan\frac{\pi}{5} \\ \tan\frac{3\pi}{5} = -\tan\frac{2\pi}{5}\end{array}\right.$

$\Rightarrow \left(\tan^2\theta - \tan^2\frac{\pi}{5}\right)\left(\tan^2\theta - \tan^2\frac{2\pi}{5}\right) = 0$

[from (vii) $x = \tan^2\theta$ $\Rightarrow x^2 - 10x + 5 = 0$

$\therefore$ $x^2 - 10x + 5 = 0$ has roots $\tan^2\left(\frac{1}{5}\pi\right)$ and $\tan^2\left(\frac{2}{5}\pi\right)$ $\checkmark$

Scanned with CamScanner

Example 8 (i) Use de-Moivre's theorem to show that,
$$\sin 8\theta = 8\sin\theta\cos\theta\,(1-10\sin^2\theta+24\sin^4\theta-16\sin^6\theta) \quad ---[6]$$
(ii) Use the equation $\frac{\sin 8\theta}{\sin 2\theta}=0$ to find the roots of;
$$16x^6-24x^4+10x^2-1=0$$
in the form $\sin k\pi$, where k is rational. ---[4]

W-18/11/Q7

Solution: consider $(\cos\theta+i\sin\theta)^8=(\cos 8\theta+i\sin 8\theta)$ ---(i) [Using de-Moivre's theorem.

(i) Let $\cos\theta=c$, $\sin\theta=s$

$$(c+is)^8 = c^8+8c^7is+28c^6\cdot i^2s^2+56c^5\cdot i^3s^3+70c^4\cdot i^4s^4+56c^3i^5s^5+28c^2i^6s^6+8ci^7s^7+i^8s^8 \quad ---(ii)$$

Comparing the imaginary part in (i) and (ii)

$$\sin 8\theta = 8c^7s-56c^5\cdot s^3+56c^3s^5-8cs^7$$
$$\Rightarrow = 8cs[c^6-7c^4s^2+7c^2s^4-s^6]$$
$$= 8cs[(1-s^2)^3-7(1-s^2)^2s^2+7(1-s^2)s^4-s^6] \quad [c^2=1-s^2]$$
$$= 8cs[1-10s^2+24s^4-16s^6]$$
$$\sin 8\theta = 8\cos\theta\sin\theta[1-10\sin^2\theta+24\sin^4\theta-16\sin^6\theta] \quad ---(iii)$$

(ii) from (iii) $\sin 8\theta=(\sin 2\theta)\cdot 4[1-10\sin^2\theta+24\sin^4\theta-16\sin^6\theta]$

$$\Rightarrow \frac{\sin 8\theta}{\sin 2\theta} = 4[1-10\sin^2\theta+24\sin^4\theta-16\sin^6\theta]=0 \quad \left[\text{Given } \frac{\sin 8\theta}{\sin 2\theta}=0\right]$$

and let $x=\sin\theta$

$$\Rightarrow 16x^6-24x^4+10x^2-1=0 \quad \text{To solve (IV)}$$

$$\frac{\sin 8\theta}{\sin 2\theta}=0 \Rightarrow \sin 8\theta=0 \quad \left[\sin 2\theta\neq 0,\ \theta\neq 0,\ \theta\neq\frac{\pi}{2}\right]$$
$$\Rightarrow 8\theta=\pm\pi,\pm 2\pi,\pm 3\pi$$
$$\theta=\pm\pi/8,\pm\frac{2\pi}{8},\pm\frac{3\pi}{8}$$
$$\therefore x=\sin\theta$$
$$x=\sin\frac{k\pi}{8};\ k=\pm 1,\pm 2,\pm 3$$

or $[k=1,2,3,9,10,11]$

$$\text{as } \sin\frac{9\pi}{8}=\sin\left(\pi+\frac{\pi}{8}\right) = -\sin\frac{\pi}{8} = \sin\left(-\frac{\pi}{8}\right)$$
$$\sin\frac{10\pi}{8}=\sin\left(-\frac{2\pi}{8}\right)$$
$$\sin 11\pi/8=\sin(-3\pi/8)$$

Scanned with CamScanner

§ Powers of $\sin\theta$ and $\cos\theta$ as $\sin n\theta$ & $\cos n\theta$:

Let $z = \cos\theta + i\sin\theta$ --- (i)

$\frac{1}{z} = \cos\theta - i\sin\theta$ --- (ii)

$$\left\{\begin{aligned} \frac{1}{z} &= \frac{1}{(\cos\theta + i\sin\theta)} \\ &= (\cos\theta + i\sin\theta)^{-1} \\ &= (\cos(-\theta) + i\sin(-\theta)) \\ \frac{1}{z} &= (\cos\theta - i\sin\theta) \end{aligned}\right.$$

add (i) and (ii) $\Rightarrow z + \frac{1}{z} = 2\cos\theta$ --- (iii)

and $z - \frac{1}{z} = 2i\sin\theta$ --- (iv)

Now $z^n = (\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$ --- (v)

$z^{-n} = (\cos\theta + i\sin\theta)^{-n} = (\cos n\theta - i\sin n\theta)$ --- (vi)

add (v) and (vi)

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

and (v) − (vi) $\Rightarrow$

$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

Given $z = (\cos\theta + i\sin\theta)$
$= e^{i\theta}$ or

and $z^{-1} = (\cos\theta - i\sin\theta)$
$= e^{-i\theta}$

$$\left\{\begin{aligned} e^{ni\theta} + e^{-ni\theta} &= 2\cos n\theta \\ e^{ni\theta} - e^{-ni\theta} &= 2i\sin n\theta \end{aligned}\right.$$

$z^n = (\cos n\theta + i\sin n\theta) = e^{in\theta}$

$z^{-n} = (\cos n\theta - i\sin n\theta) = e^{-in\theta}$

Scanned with CamScanner

Example 9 (i) Let $z = (\cos\theta + i\sin\theta)$; show that $z - \frac{1}{z} = 2i\sin\theta$ and hence express $16\sin^5\theta$ in the form $\sin 5\theta + p\sin 3\theta + q\sin\theta$, where p and q are integers to be determined. ---[6]

(ii) Hence find the exact value of $\int_0^{\frac{1}{3}\pi} 16\sin^5\theta\, d\theta$ ---[3]

S-17/11/Q8

Solution: $z = (\cos\theta + i\sin\theta)$ ---(i)

(i) $\frac{1}{z} = (\cos\theta + i\sin\theta)^{-1} = (\cos(-\theta) + i\sin(-\theta))$ [De Moivre's Theorem]

$\Rightarrow \frac{1}{z} = (\cos\theta - i\sin\theta)$ ---(ii)

subtract (ii) from (i) $\Rightarrow z - \frac{1}{z} = 2i\sin\theta$ ---(iii)

Now $(z - \frac{1}{z})^5 = z^5 - 5z^4 \times \frac{1}{z} + 10z^3 \times \frac{1}{z^2} - 10z^2 \cdot \frac{1}{z^3} + 5z \times \frac{1}{z^4} - \frac{1}{z^5}$

or $(z - \frac{1}{5})^5 = (z^5 - \frac{1}{z^5}) - 5(z^3 - \frac{1}{z^3}) + 10(z - \frac{1}{z})$ ---(IV)

from (iii)

$\Rightarrow (2i\sin\theta)^5 = 2i\sin 5\theta - 5 \times 2i\sin 3\theta + 10 \times 2i\sin\theta$

$\Rightarrow 32i\sin^5\theta = 2i(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$ ✓

$\Rightarrow 16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$ ---(IV)

$\left\{ z^n - \frac{1}{z^n} = 2i\sin n\theta \right.$

$z^5 - \frac{1}{z^5} = 2i\sin 5\theta$

$\left. z^3 - \frac{1}{z^3} = 2i\sin 3\theta \right\}$

(ii) $\int_0^{\frac{1}{3}\pi} 16\sin^5\theta\, d\theta = \int_0^{\frac{1}{3}\pi} (\sin 5\theta - 5\sin 3\theta + 10\sin\theta)\, d\theta$ { from (iv)

$= \left[-\frac{\cos 5\theta}{5} + \frac{5\cos 3\theta}{3} - 10\cos\theta \right]_0^{\frac{1}{3}\pi}$

$= \left[\left(-\frac{1}{5}\cos\frac{5\pi}{3} + \frac{5}{3}\cos\frac{3\pi}{3} - 10\cos\frac{\pi}{3}\right) - \left(-\frac{1}{5}\cos 0 + \frac{5}{3}\cos 0 - 10\cos 0\right) \right]$

$= \left[\left(-\frac{1}{5} \times \frac{1}{2} + \frac{5}{3} \times (-1) - 10 \times \frac{1}{2}\right) - \left(-\frac{1}{5} + \frac{5}{3} - 10\right) \right]$

$= \left(-\frac{1}{10} - \frac{5}{3} - 5\right) - \left(-\frac{1}{5} + \frac{5}{3} - 10\right)$

$= \frac{50}{30}$ ✓

Scanned with CamScanner

Scanned with CamScanner

Example 10(a) Use de Moivre's theorem to show that:

$$\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3) \quad \text{---[5]}$$

(b) Find the solution of the differential equation:

$$\frac{dy}{d\theta} + y\cot\theta = \sin^3\theta$$

for which $y = 0$ when $x = \frac{\pi}{2}$; ---[6]

W-20/21/Q 6

Solution: $z - \frac{1}{z} = 2i\sin\theta$

$$\Rightarrow (2i\sin\theta)^4 = \left(z - \frac{1}{z}\right)^4$$

$$= z^4 - 4z^3 \cdot \frac{1}{z} + 6z^2 \cdot \frac{1}{z^2} - 4z \cdot \frac{1}{z^3} + \frac{1}{z^4}$$

$$= \left(z^4 + \frac{1}{z^4}\right) - 4\left(z^2 + \frac{1}{z^2}\right) + 6 \qquad \left[\because z^n + \frac{1}{z^n} = 2\cos n\theta\right.$$

$$\Rightarrow 16\sin^4\theta = (2\cos 4\theta) - 4\cdot(2\cos 2\theta) + 6$$

$$\Rightarrow \sin^4\theta = \frac{1}{16}\times 2(\cos 4\theta - 4\cos 2\theta + 3)$$

$$\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3) \checkmark \quad (i)$$

(b) To solve: $\frac{dy}{d\theta} + y\cot\theta = \sin^3\theta$ ---- (ii)

$$\text{Integrating factor} = e^{\int \cot\theta} = e^{\ln \sin\theta} = \sin\theta$$

$\therefore$ Solution of equation (ii)

$$y\cdot(I.F) = \int \sin^3\theta\cdot(I.F)\,d\theta$$

$$\Rightarrow y\sin\theta = \int \sin^3\theta\cdot\sin\theta\,d\theta$$

$$= \int \sin^4\theta\,d\theta$$

$$= \frac{1}{8}\int(\cos 4\theta - 4\cos 2\theta + 3)\,d\theta \quad (\text{from } (i))$$

$$\Rightarrow y\sin\theta = \frac{1}{8}\left[\frac{\sin 4\theta}{4} - 4\frac{\sin 2\theta}{2} + 3\theta\right] + C \quad \text{---(iii)}$$

Now $y = 0$ for $x = \frac{\pi}{2}$

from (iii) $\quad 0 = \frac{1}{8}\left[0 - 0 + \frac{3\pi}{2}\right] + C \Rightarrow C = -\frac{3}{16}\pi \checkmark$

from (iii) Req. solution: $y\sin\theta = \frac{1}{8}\left[\frac{1}{4}\sin 4\theta - 2\sin 2\theta + 3\theta\right] - \frac{3}{16}\pi$

or $y\sin\theta = \frac{1}{8}\left[\frac{1}{4}\sin 4\theta - 2\sin 2\theta + 3\theta - \frac{3}{2}\pi\right] \checkmark$

Example 11: By considering the binomial expansions of $(z+\frac{1}{z})^5$ and $(z-\frac{1}{z})^5$, where $z=\cos\theta+i\sin\theta$, use de Moivre's theorem to show that: $\tan^5\theta = \dfrac{\sin 5\theta - a\sin 3\theta + b\sin\theta}{\cos 5\theta + a\cos 3\theta + b\cos\theta}$ ---[7]

where a and b are the integers to be determined. [S-21/23/ Q4]

Solution: $z-\frac{1}{z} = 2i\sin\theta \Rightarrow (2i\sin\theta)^5 = (z-\frac{1}{z})^5$

$$\Rightarrow 32i\sin^5\theta = z^5 - 5z^4\cdot\frac{1}{z} + 10z^3\cdot\frac{1}{z^2} - 10z^2\cdot\frac{1}{z^3} + 5z\cdot\frac{1}{z^4} - \frac{1}{z^5}$$

$$= (z^5-\frac{1}{z^5}) - 5(z^3-\frac{1}{z^3}) + 10(z-\frac{1}{z})$$

$$\Rightarrow 32i\sin^5\theta = 2i\sin 5\theta - 5\times 2i\sin 3\theta + 10\cdot 2i\sin\theta$$

$$\Rightarrow 16\sin^5\theta = (\sin 5\theta - 5\sin 3\theta + 10\sin\theta) \text{ ---(i)}$$

$\left[(z^n - \frac{1}{z^n}) = 2i\sin n\theta \right]$

Again $z+\frac{1}{z} = 2\cos\theta \Rightarrow (2\cos\theta)^5 = (z+\frac{1}{z})^5$

$$\Rightarrow (2\cos\theta)^5 = z^5 + 5z^4\cdot\frac{1}{z} + 10z^3\times\frac{1}{z^2} + 10z^2\times\frac{1}{z^3} + 5z\cdot\frac{1}{z^4} + \frac{1}{z^5}$$

$$\Rightarrow 32\cos^5\theta = (z^5+\frac{1}{z^5}) + 5(z^3+\frac{1}{z^3}) + 10(z+\frac{1}{z})$$

$$\Rightarrow 32\cos^5\theta = 2\cos 5\theta + 5\times 2\cos 3\theta + 10\times 2\cos\theta$$

$$\Rightarrow 16\cos^5\theta = (\cos 5\theta + 5\cos 3\theta + 10\cos\theta) \text{ --- (ii)}$$

$\left[\because (z^n + \frac{1}{z^n}) = 2\cos n\theta \right]$

from (i) and (ii)

$$\tan^5\theta = \frac{16\sin^5\theta}{16\cos^5\theta} = \frac{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta}$$

$$\text{or } \tan^5\theta = \frac{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta}$$

---

§ Express $\cos^5\theta$ in terms of $ae^{k\theta}$

From (Example 11)

Solution: (ii) $\cos^5\theta = \frac{1}{16}[\cos 5\theta + 5\cos 3\theta + 10\cos\theta]$

$$= \frac{1}{32}[2\cos 5\theta + 5\times 2\cos 3\theta + 10\times 2\cos\theta]$$

$$= \frac{1}{32}\left[(e^{5i\theta} + e^{-5i\theta}) + 5(e^{3i\theta} + e^{-3i\theta}) + 10(e^{i\theta} + e^{-i\theta})\right]$$

$2\cos n\theta = e^{in\theta} + e^{-in\theta}$

$2\cos 5\theta = e^{5i\theta} + e^{-5i\theta}$

$2\cos 3\theta = e^{3i\theta} + e^{-3i\theta}$

and

$2\cos\theta = e^{i\theta} + e^{-i\theta}$

Scanned with CamScanner

Example 12 (a) Use de Moivre's theorem to show that:

$$\sin^6\theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10) \quad ----[6]$$

It is given that $\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

(b) Find the exact value of $\int_0^{\frac{1}{3}\pi}\left(\cos^6(\tfrac{1}{4}x) + \sin^6(\tfrac{1}{4}x)\right)dx$ ---[4]

(c) Express each root of the equation $16c^6 + 16(1-c^2)^3 - 13 = 0$ in the form $\cos k\pi$, where k is a rational number. $\boxed{S\text{-}20/23/Q8}$ ---[5]

Solution: $\left(z - \frac{1}{z}\right) = 2i\sin\theta \Rightarrow (2i\sin\theta)^6 = \left(z - \frac{1}{z}\right)^6$

(a) $\Rightarrow (2i\sin\theta)^6 = z^6 - 6z^5\times\frac{1}{z} + 15z^4\times\frac{1}{z^2} - 20z^3\times\frac{1}{z^3} + 15z^2\times\frac{1}{z^4} - 6z\times\frac{1}{z^5} + \frac{1}{z^6}$

$\Rightarrow -64\sin^6\theta = \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$

$-64\sin^6\theta = 2\cos 6\theta - 6\times 2\cos 4\theta + 15\times 2\cos 2\theta - 20$

$\Rightarrow \sin^6\theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$ ---(i)

$\left\{\left(z^n + \frac{1}{z^n}\right) = 2\cos n\theta\right.$

Now Given $\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$ ---(ii)

(b) $\cos^6\theta + \sin^6\theta = \frac{1}{32}(12\cos 4\theta + 20)$ ---(iii) $\Rightarrow \cos^6(\tfrac{1}{4}x) + \sin^6(\tfrac{1}{4}x) = \frac{1}{8}[3\cos x + 5]$ ---(iv)

from (iv) $\therefore \int_0^{\frac{1}{3}\pi}\left(\cos^6(\tfrac{1}{4}x) + \sin^6(\tfrac{1}{4}x)\right)dx = \frac{1}{8}\int_0^{\frac{1}{3}\pi}(3\cos x + 5)dx = \frac{1}{8}\left[3\sin x + 5x\right]_0^{\pi/3}$

$= \frac{1}{8}\left[\frac{3\sqrt{3}}{2} + \frac{5\pi}{3}\right]$ ✓

(c) Given $16c^6 + 16(1-c^2)^3 - 13 = 0$ $\left[\text{let } c = \cos\theta \Rightarrow (1-c^2) = 1 - \cos^2\theta = \sin^2\theta\right]$

$\Rightarrow 16\cos^6\theta + 16\sin^6\theta - 13 = 0$

$\Rightarrow 16[\cos^6\theta + \sin^6\theta] - 13 = 0$

$\Rightarrow 16[\frac{1}{8}(3\cos 4\theta + 5)] - 13 = 0$ from (iii)

$\Rightarrow 6\cos 4\theta + 10 - 13 = 0$

$\Rightarrow 6\cos 4\theta = 3$

$\Rightarrow \cos 4\theta = \frac{1}{2} = \cos\frac{\pi}{3}$

$\Rightarrow 4\theta = \frac{\pi}{3}, \frac{5}{3}\pi, \frac{7}{3}\pi, \frac{11}{3}\pi$ ✓

$\theta = \frac{\pi}{12}, \frac{5}{12}\pi, \frac{7}{12}\pi, \frac{11}{12}\pi$

$\left[\because \cos\theta = \cos\alpha,\ \theta = 2n\pi \pm \alpha\right.$

$\left[4\theta = 2n\pi \pm \frac{\pi}{3}\right]$

$\therefore c = \cos\theta = \cos k\pi$: $k = \frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$ ✓

Scanned with CamScanner

Example 13 (i) Let $z = \cos\theta + i\sin\theta$, show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta \; ; \; z^n - \frac{1}{z^n} = 2i\sin n\theta \quad \text{---[2]}$$

(ii) By considering $\left(z - \frac{1}{z}\right)^4 \cdot \left(z + \frac{1}{z}\right)^2$, show that

$$\sin^4\theta \cdot \cos^2\theta = \frac{1}{32}\left[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2\right] \quad \text{--[7]}$$

(iii) Hence find the value of $\int_0^{\frac{1}{4}\pi} \sin^4\theta \cdot \cos^2\theta \, d\theta$ --[3]

| W-16 | 11 | Q 10 |

Solution (i) $z = (\cos\theta + i\sin\theta) \Rightarrow z^n = (\cos n\theta + i\sin n\theta)$ and $z^{-n} = (\cos n\theta - i\sin n\theta)$

$\therefore \; z^n + \frac{1}{z^n} = 2\cos n\theta$ ---(i) and $z^n - \frac{1}{z^n} = 2i\sin n\theta$ ---(ii)

(ii) Consider $\left(z - \frac{1}{z}\right)^4 \cdot \left(z + \frac{1}{z}\right)^2 = \left(z - \frac{1}{z}\right)^2 \cdot \left(z - \frac{1}{z}\right)^2 \left(z + \frac{1}{z}\right)^2$ $\left[\left(z - \frac{1}{z}\right) = 2i\sin\theta \quad z + \frac{1}{z} = 2\cos\theta\right.$

$\Rightarrow (2i\sin\theta)^4 \cdot (2\cos\theta)^2 = \left(z - \frac{1}{z}\right)^2 \cdot \left(z^2 - \frac{1}{z^2}\right)^2$

$\Rightarrow 64\sin^4\theta \cdot \cos^2\theta = \left(z^2 - 2 + \frac{1}{z^2}\right)\left(z^4 - 2 + \frac{1}{z^4}\right)$

$= \left(z^6 + \frac{1}{z^6}\right) - 2\left(z^4 + \frac{1}{z^4}\right) - \left(z^2 + \frac{1}{z^2}\right) + 4$

$64\sin^4\theta \cdot \cos^2\theta = 2\cos 6\theta - 2 \times 2\cos 4\theta - 2\cos 2\theta + 4$

$\Rightarrow \sin^4\theta \cdot \cos^2\theta = \frac{1}{32}\left[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2\right]$ ✓ ....(iii)

(iii) $\int_0^{\frac{1}{4}\pi} \sin^4\theta \cos^2\theta \, d\theta$

$= \frac{1}{32}\int_0^{\frac{1}{4}} (\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2)\, d\theta$

$= \frac{1}{32}\left[\frac{\sin 6\theta}{6} - \frac{2\sin 4\theta}{4} - \frac{\sin 2\theta}{2} + 2\theta\right]_0^{\frac{1}{4}\pi}$

$= \frac{1}{32}\left[-\frac{1}{6} - 0 - \frac{1}{2} + \frac{\pi}{2}\right]$

$= \frac{(3\pi - 4)}{192} = 0.0283$ ✓

Scanned with CamScanner

§ The roots of Unity:

(i) Cube roots of Unity:

Let $z = r(\cos\theta + i\sin\theta)$ is the cube root of Unity (1)

$z^3 = 1 \Rightarrow |z^3| = |z|^3 = 1 \Rightarrow |z| = r = 1.$

$\therefore\ z = (\cos\theta + i\sin\theta)\ ;\ r = 1$

Now $z^3 = 1 = (\cos 0 + i\sin 0)$

$\Rightarrow z^3 = \cos(2k\pi + 0) + i\sin(2k\pi + 0) \qquad ; k \in I$

$z = \cos(2k\pi + i\sin 2k\pi)^{1/3}$

Cube root of Unity $= z = \cos\left(\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right)\ ;\ k = 0, 1, 2$

$\therefore$ Cube root of Unit are $(\cos 0 + i\sin 0), \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right), \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$= 1, \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$ ✓

If z_1, z_2, z_3 are the cube roots of unity:

$z_1 = 1,\ z_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), z_3 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)$ ✓

Or maybe $1, \omega$ and ω^2 ✓

here $1 + \omega + \omega^2 = 0$

and $\omega^3 = 1$

Note: Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

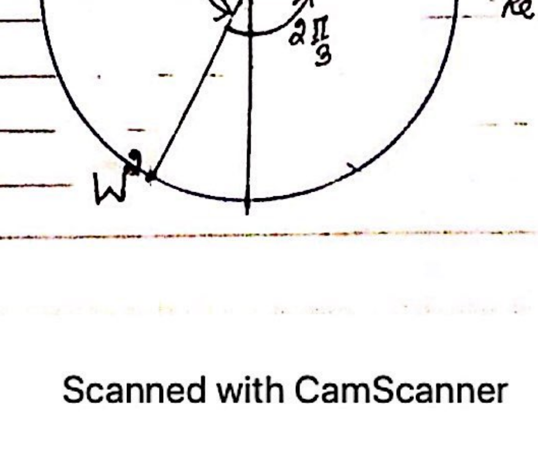
$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \omega^2$

$\left[1 + \omega + \omega^2 = \frac{\omega^3 - 1}{\omega - 1} = 0\right.$

$\therefore$ In exponent form the cube root of unity are $e^{\frac{2k\pi}{3}i}$; $k = 0, 1, 2$

$e^0, e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i}$ ✓

Geometrically:



Scanned with CamScanner

§ n_{th} roots of Unity:

Let z is the n_{th} root of Unity $\quad [1 = (\cos 0 + i\sin 0)]$

$z^n = \cos(0+2k\pi) + i\sin(0+2k\pi)$ for $k = 0, 1, 2, \ldots (n-1)$

$\therefore$ n_{th} root $z = \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)$, $k = 0, 1, 2, \ldots (n-1)$

also denoted by: $1, \omega, \omega^2, \ldots \omega^{n-1}$ [or $e^{\frac{2k\pi}{n}i}$

or $1, e^{\frac{2\pi}{n}i}, e^{\frac{4\pi}{n}i}, \ldots e^{\frac{2(n-1)\pi}{n}i}$.

Also $z^n = 1 \Rightarrow z^n - 1 = 0$

$\Rightarrow (z-1)(1+z+z^2+\ldots z^{n-1}) = 0$

or Sum of 'n' n_{th} roots of Unity: $1+z+z^2+\ldots z^{n-1} = 0$

Example 14 (i) Write down the fifth roots of unity. --- [2]

(ii) Find all the roots of the equation: $z^{10} + z^5 + 1 = 0$

giving each root in the form $e^{i\theta}$ $\boxed{S\text{-}19/13/Q3}$ --- [5]

Solution: (i) 5_{th} root of Unity $= e^{\frac{2k\pi}{5}i}$, $k = 0, 1, 2, 3, 4$ $\quad$ [n_{th} roots of unity $= e^{\frac{2k\pi}{n}i}$, $k = 0, 1, 2, \ldots (n-1)$]

$\therefore$ 5 - 5_{th} root are $1 (e^0), e^{\frac{2\pi}{5}i}, e^{\frac{4\pi}{5}i}, e^{\frac{6\pi i}{5}}, e^{\frac{8\pi i}{5}}$

(ii) Given $z^{10} + z^5 + 1 = 0 \Rightarrow (z^5)^2 + z^5 + 1 = 0$

Let $z^5 = u \Rightarrow u^2 + u + 1 = 0$ $\quad$ $\{b^2 - 4ac = 1 - 4 = -3$

$\therefore z^5 = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ $\quad$ $\left\{u = \frac{-1 \pm \sqrt{-3}}{2} = \left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)\right.$

$z^5 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ or $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$z^5 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ or $\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ $\quad$ $\left\{\cos\theta = -\frac{1}{2},\ \sin\theta = \frac{\sqrt{3}}{2},\ \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}\right.$

$z^5 = e^{\left(2k\pi + \frac{2\pi}{3}\right)i}$ or $e^{\left(2k\pi + \frac{4\pi}{3}\right)i}$

$\therefore z = e^{\frac{\left(2k\pi + \frac{2\pi}{3}\right)}{5}i}$ or $e^{\frac{\left(2k\pi + \frac{4\pi}{3}\right)}{5}i}$

$z = e^{\left(\frac{6k\pi + 2\pi}{15}\right)i}$ or $e^{\left(\frac{6k\pi + 4\pi}{15}\right)i}$; $k = 0, 1, 2, 3, 4$

Not Required [$e^{\frac{2\pi}{15}i}, e^{\frac{8\pi}{15}i}, e^{\frac{14\pi}{15}i}, e^{\frac{20\pi i}{15}}, e^{\frac{26\pi i}{15}}$) or ($e^{\frac{4\pi i}{15}}, e^{\frac{10\pi i}{15}}, e^{\frac{16\pi i}{15}}, e^{\frac{22\pi i}{15}}, e^{\frac{28\pi i}{2}}$)

Example 15 (a) Find the roots of the equation $z^3 = -1-i$, giving your answer in the form $re^{i\theta}$, where $r>0$ and $0 \le \theta \le 2\pi$ ---[5]

Let $W = z_1^{3k} + z_2^{3k} + z_3^{3k}$, where k is a positive integer and z_1, z_2 and z_3 are the roots of $z^3 = -1-i$

(b) Express W in the form $Re^{i\alpha}$, where $R>0$, giving R and α in terms of k. $\boxed{S\text{-}20/21/Q3}$ ---[3]

Solution: $z^3 = -1-i = r(\cos\theta + i\sin\theta)$ $\Rightarrow$ $\left\{ \begin{array}{l} r\cos\theta = -1 \\ r\sin\theta = -1 \\ r = \sqrt{(-1)^2+(-1)^2} = \sqrt{2} \\ \tan\theta = \frac{-1}{-1} = 1 = \tan\frac{\pi}{4} \\ \theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4} \end{array} \right.$

$= \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

or $z^3 = \sqrt{2}\, e^{\frac{5\pi}{4}i}$

$z^3 = \sqrt{2}\, e^{\left(2p\pi + \frac{5\pi}{4}\right)i}$

root of z^3; $z = \left[2^{\frac{1}{2}} e^{\left(2\pi p + \frac{5\pi}{4}\right)i}\right]^{1/3}$

$z = 2^{1/6}\, e^{\frac{\left(8p\pi + \frac{5\pi}{4}\right)}{3}i}$; $p = 0, 1, 2$

$\pi + \alpha$

Cube roots of z^3, $z_1 = 2^{1/6} e^{\frac{5\pi}{12}i}$, $2^{1/6} e^{\frac{13\pi}{12}i}$, $2^{1/6} \cdot e^{\frac{21}{12}\pi i}$ ✓

(b) $(z_1)^{3k} = \left(2^{1/6} e^{\frac{5\pi i}{12}}\right)^{3k} = 2^{\frac{1}{2}k} \cdot e^{\frac{5\pi k i}{4}}$ ---(i)

$(z_2)^{3k} = \left(2^{1/6} e^{\frac{13\pi i}{12}}\right)^{3k} = 2^{\frac{1}{2}k} e^{\frac{13\pi k i}{4}}$

$= 2^{\frac{1}{2}k} e^{\frac{5\pi k}{4}i}$ ✓ ---(ii)

$(z_3)^{3k} = \left(2^{\frac{1}{6}} e^{\frac{21}{12}\pi i}\right)^{3k} = 2^{\frac{1}{2}k} e^{\frac{21\pi k i}{4}}$

$= 2^{\frac{1}{2}k} e^{\frac{5\pi k i}{4}}$ --- (iii)

$e^{\frac{13\pi k i}{4}}$
$= \cos\left(\frac{13\pi k}{4} + i\sin --\right)$
$\cos\left(2\pi k + \frac{5\pi k}{4}\right) + i$ ---
$\cos\left(\frac{5\pi k}{4}\right) + i\sin(--)$
$= e^{\frac{5\pi}{4}ki}$

add (i), (ii) and (iii)

$\Rightarrow W = z_1^{3k} + z_2^{3k} + z_3^{3k} = 3 \cdot 2^{\frac{1}{2}k} \cdot e^{i\frac{5k\pi}{4}}$

Similarly $e^{\frac{21\pi k i}{4}} = e^{\frac{5\pi k i}{4}}$ (as $\frac{21\pi}{4} = \left(4\pi + \frac{5\pi}{4}\right)$)

here $R = |W| = 3 \cdot 2^{\frac{1}{2}k}$ ✓

and $\alpha = \frac{5}{4}k\pi$ ✓

Scanned with CamScanner

Example 16(i) Show that if $z = e^{i\theta}$ and $z \neq -1$, then

$$\frac{z-1}{z+1} = i \tan \frac{1}{2}\theta \quad \text{---[3]}$$

(ii) Hence or otherwise, show that if z is a cube root of unity, then:

$$\frac{z^3-1}{z^3+1} + \frac{z^2-1}{z^2+1} + \frac{z-1}{z+1} = 0 \quad \text{---[5]}$$

(iii) Hence write the three roots of the equation:

$$(z^3-1)(z^2+1)(z+1) + (z^2-1)(z^3+1)(z+1) + (z-1)(z^3+1)(z^2+1) = 0$$

and find the other three roots. Give your answer in exact form. ---[6]

S-18 / 11 / Q 11(i)

Solution (i) $z = e^{i\theta} = (\cos\theta + i\sin\theta)$ $\Rightarrow \begin{cases} z^{\frac{1}{2}} = (\cos\theta/_2 + i\sin\theta/_2) \\ z^{-\frac{1}{2}} = (\cos\theta/_2 - i\sin\theta/_2) \end{cases}$

Consider $\dfrac{z-1}{z+1} = \dfrac{z^{1/2} - z^{-1/2}}{z^{1/2} + z^{-1/2}} = \dfrac{2i\sin\theta/_2}{2\cos\theta/_2}$

$$= i\tan\theta/_2 \quad \text{---(i)}$$

(ii) z is a cube root of unity $\Rightarrow z = 1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}$

Consider $\dfrac{z^3-1}{z^3+1} + \dfrac{z^2-1}{z^2+1} + \dfrac{z-1}{z+1} = 0 + \dfrac{e^{\frac{4\pi i}{3}}-1}{e^{\frac{4\pi i}{3}}+1} + \dfrac{e^{\frac{2\pi i}{3}}-1}{e^{\frac{2\pi i}{3}}+1}$ $\begin{cases} z^3 = 1 \\ z^3 - 1 = 0 \\ z \neq 1 \Rightarrow e^{\frac{2i\pi}{3}} \end{cases}$

$= i\tan\frac{1}{2}\left(\frac{4\pi}{3}\right) + i\tan\frac{1}{2}\left(\frac{2\pi}{3}\right)$ $\begin{cases} \text{from Part (i)} \\ \frac{z-1}{z+1} = i\tan\theta/_2 \end{cases}$

$= i\left[\tan\frac{2\pi}{3} + \tan\frac{\pi}{3}\right] = i\left[\cancel{-\tan\frac{\pi}{3}} + \cancel{\tan\frac{\pi}{3}}\right]$

$= 0$ $\begin{cases} \tan\frac{2\pi}{3} = \tan\left(\pi - \frac{\pi}{3}\right) \\ = -\tan\pi/_3 \end{cases}$

(iii) z is cube root of unity $\Rightarrow z^3 = 1$

hence;

Three roots of the equation are $1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}$

Now $(z^3-1)(z^2+1)(z+1) + (z^2-1)(z^3+1)(z+1) + (z-1)(z^3+1)(z+1) = 0$

$\Rightarrow 3z^6 + z^5 + z^4 - z^2 - z - 3 = 0 \Rightarrow (z^3-1)(3z^3 + z^2 + z + 3) = 0$

$\Rightarrow (z^3-1)(z+1)(3z^2 - 2z + 3) = 0$ $[\because z^3 = 1 \Rightarrow (z^3-1) \text{ is a factor}]$

$\Rightarrow (z+1)(3z^2 - 2z + 3) = 0$ $\begin{cases} \text{Now Roots for} \\ z^3 - 1 = 0 \\ \text{are already there} \end{cases}$

$z = -1$, or $3z^2 - 2z + 3 = 0$

$z = -1$; $z = \dfrac{2 \pm i\sqrt{32}}{6} = \dfrac{1}{3} \pm \dfrac{2\sqrt{2}}{3}$

$\therefore$ other three roots are $-1, \dfrac{1}{3} \pm i\dfrac{2\sqrt{2}}{3}$

Scanned with CamScanner

Example 17: Find the roots of the equation:

$$z^4 = 8+8\sqrt{3}i$$

Giving your answers in exponential form.

Solution: Given $z^4 = 8+8\sqrt{3}i = r(\cos\theta + i\sin\theta)$

$r\cos\theta = 8$

$r\sin\theta = 8\sqrt{3}$

$r = \sqrt{8^2+(8\sqrt{3})^2} = 16$ ✓

$\tan\theta = \frac{8\sqrt{3}}{8} = \sqrt{3}$

$\Rightarrow \theta = \frac{\pi}{3}$ ✓

$\Rightarrow z^4 = 16\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 16\left[\cos\left(2\pi k + \frac{\pi}{3}\right) + i\sin\left(2\pi k + \frac{\pi}{3}\right)\right]$

$z = \left[16\left(\cos\left(\frac{6\pi k +}{3}\right) + i\sin\left(\frac{6\pi k + \pi}{3}\right)\right]^{\frac{1}{4}}$

$= 16^{\frac{1}{4}}\left[\cos\left(\frac{6\pi k + \pi}{12}\right) + i\left(\frac{6\pi k + \pi}{12}\right)\right]$; $k = 0, 1, 2, 3$

$z = 2e^{\left(\frac{6\pi k + \pi}{12}\right)i}$; $k = 0, 1, 2, 3$

Req. roots are $= 2e^{\frac{\pi}{12}i}, 2e^{\frac{7\pi}{12}i}, e^{\frac{13\pi}{12}i}, e^{\frac{19\pi}{12}i}$ ✓

Example 18: $z = (\cos\theta + i\sin\theta)$ and $(2z-1)^2$ is a real number. Find the possible value of θ.

Solution: $z = (\cos\theta + i\sin\theta) \Rightarrow z^2 = (\cos 2\theta + i\sin 2\theta)$ ---(i)

Consider $(2z-1)^2 = 4z^2 - 4z + 1$

$= 4(\cos\theta + i\sin\theta)^2 - 4(\cos\theta + i\sin\theta) + 1$

$= 4(\cos 2\theta + i\sin 2\theta) - 4(\cos\theta + i\sin\theta) + 1$

$= (4\cos 2\theta - 4\cos\theta + 1) + i(4\sin 2\theta - 4\sin\theta)$

Given $(2z-1)^2$ is real $\Rightarrow$ imaginary part of it $= 0$

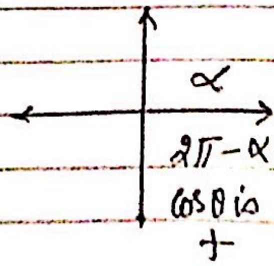
$\Rightarrow 4\sin 2\theta - 4\sin\theta = 0$

$4 \times 2\sin\theta\cos\theta - 4\sin\theta = 0$

$\Rightarrow 4\sin\theta(2\cos\theta - 1) = 0$

$\Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2}$

$\theta = 0$; $\theta = \frac{\pi}{3}, \left(2\pi - \frac{\pi}{3}\right)$



$\therefore \theta = 0, \frac{\pi}{3}, \frac{5\pi}{3}$ ✓

Scanned with CamScanner

Example 19 (a) Find a and b such that:

$$z^8 - iz^5 - z^3 + i = (z^5 - a)(z^3 - b) \quad \text{---[1]}$$

(b) Hence find the roots of

$$z^8 - iz^5 - z^3 + i = 0 \quad \text{---[6]}$$

giving your answer in the form $re^{i\theta}$, where $r > 0$ and $0 \le \theta < 2\pi$

[S-21/23/Q1]

Solution (a) $z^8 - iz^5 - z^3 + i = (z^5 - a)(z^3 - b)$

$\Rightarrow z^5(z^3 - i) - 1(z^3 - i) = ----$

$\Rightarrow (z^5 - 1)(z^3 - i) = (z^5 - a)(z^3 - b)$

$\Rightarrow \underline{a = 1 \text{ and } b = i}$ ✓

(b) To find the roots of $z^8 - iz^5 - z^3 + i = 0$

$\Rightarrow (z^5 - 1)(z^3 - i) = 0$

$\Rightarrow z^5 = 1$ and $z^3 = i$

$\Rightarrow z^5 = e^{\frac{2k\pi}{5}i}, \; k = 0, 1, 2, 3, 4$ ✓ ; $z^3 = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$z = 1, e^{\frac{2\pi}{5}i}, e^{\frac{4\pi i}{5}}, e^{\frac{6\pi i}{5}}, e^{\frac{8\pi i}{8}}$; $z^3 = \left[\cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]$

$$\left\{\begin{aligned} z &= \left[\cos\left(\frac{4k\pi + \pi}{2}\right) + i\sin\left(\frac{4k\pi + \pi}{2}\right)\right]^{\frac{1}{3}} \\ z &= \cos\frac{(4k+1)\pi}{6} + i\sin\frac{(4k+1)\pi}{6} \\ z &= e^{\frac{(4k+1)\pi i}{6}} \quad k = 0, 1, 2 \end{aligned}\right.$$

$\underline{z = 1, e^{\frac{2\pi}{5}i}, e^{\frac{4\pi i}{5}}, e^{\frac{6\pi i}{5}}, e^{\frac{8\pi i}{5}} \; ; \text{ or } z = e^{\pi i/6}, e^{\frac{5\pi i}{6}}, e^{9\pi i/6} \left(\text{or } e^{\frac{i3\pi}{2}}\right)}$

Example 20: Find all the roots of the equation $(w+1)^6 = 1$, giving your answer in the form $(x+iy)$, where x and y are real and exact. ---[4]

[W-20/22/Q3]

Solution: $(w+1)^6 = 1 = (\cos 0 + i\sin 0) = (\cos(2k\pi) + i\sin(2k\pi))$

$\Rightarrow w + 1 = (\cos 2k\pi + i\sin 2k\pi)^{1/6} = \cos\left(\frac{2k\pi}{6}\right) + i\sin\left(\frac{2k\pi}{6}\right)$

$w + 1 = 1, \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right), \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$; $k = 0, 1, 2, 3, 4, 5$

$(\cos\pi + i\sin\pi), \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right), \left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

$w + 1 = 1, \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), -1, \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right), \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$\underline{w = 0, \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), \left(-\frac{3}{2} + \frac{\sqrt{3}}{2}\right), -2, \left(-\frac{3}{2} - \frac{\sqrt{3}}{2}i\right), \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)}$ ✓

Scanned with CamScanner

§ Finite Series of trigonometric ratios:

Given $z = (\cos\theta + i\sin\theta)$, then $\begin{cases} z+1 = 2z^{1/2}\cos\theta/2 \\ \text{and} \\ z-1 = 2iz^{1/2}\sin\theta/2 \end{cases}$

Proof: $z = \cos\theta + i\sin\theta$

$\Rightarrow z+1 = 1+\cos\theta + i\sin\theta$

$= 2\cos^2\frac{\theta}{2} + i\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\Rightarrow z+1 = 2\cos\theta/2\,(\cos\theta/2 + i\sin\theta/2)$

$= 2\cos\theta/2\,(\cos\theta + i\sin\theta)^{1/2}$

$\left\{\begin{array}{l} \cos 2\theta = 2\cos^2\theta - 1 \\ \Rightarrow 1+\cos 2\theta = 2\cos^2\theta \\ \Rightarrow 1+\cos\theta = 2\cos^2\theta/2 \\ \text{and } \sin 2\theta = 2\sin\theta\cos\theta \end{array}\right.$

$\Rightarrow z+1 = 2z^{1/2}\cdot\cos\theta/2$ ✓ ---- (i)

Again $z-1 = \cos\theta + i\sin\theta - 1$

$= -(1-\cos\theta) + i\sin\theta$

$= -2\sin^2\theta/2 + i\cdot 2\sin\theta/2\cos\theta/2$

$= 2\sin\theta/2\,[-\sin\theta/2 + i\cos\theta/2]$

$= 2\sin\theta/2\,[i\cos\theta/2 + i^2\sin\theta/2]$ $\{-1 = i^2$

$= 2i\sin\theta/2\,[\cos\theta/2 + i\sin\theta/2]$

$= 2i\sin\theta/2\,(\cos\theta + i\sin\theta)^{1/2}$

$\left\{\begin{array}{l} \cos 2\theta = 1-2\sin^2\theta \\ 1-\cos 2\theta = 2\sin^2\theta \\ \Rightarrow 1-\cos\theta = 2\sin^2\theta/2 \end{array}\right.$

$\Rightarrow z-1 = 2iz^{1/2}\sin\theta/2$ ✓ ——— (ii)

Scanned with CamScanner

Example 21 (a): State the sum of the series: $z+z^2+z^3+\cdots+z^n$ for $z\neq 1$ --[1]

(b) Given that z is the n_{th} root of unity and $z\neq 1$, deduce that $1+z+z^2+\cdots+z^{n-1}=0$ --[2]

(c) Given instead that $z=\frac{1}{3}(\cos\theta+i\sin\theta)$ use de Moivre's theorem; To show that:

$$\sum_{m=1}^{\infty} 3^{-m}\cos m\theta = \frac{3\cos\theta-1}{10-6\cos\theta} \quad ---[7]$$

[S-21/21/Q5]

Solution (a) $z+z^2+z^3+\cdots+z^n$

$$= \frac{z(1-z^n)}{1-z}$$

$$= \frac{z-z^{n+1}}{1-z}$$

[Geometric Series $a+ar+ar^2+\cdots ar^{n-1}=\frac{a(1-r^n)}{1-r}$]

(b) z is the n_{th} root of unity $\Rightarrow z^n=1$ and $z\neq 1$ $(1-z\neq 0)$

Now $1+z+z^2+\cdots z^{n-1} = \frac{1\cdot(1-z^n)}{1-z}$ $\quad\left[\frac{a(1-r^n)}{1-r}\right.$

$$= 0 \quad [\because z^n=1 \text{ and } z\neq 1]$$

(c) $\sum_{m=1}^{\infty} z^m = \frac{z}{1-z}$ $\quad\left\{ a+ar+ar^2+\cdots\infty \quad S_\infty=\frac{a}{1-r}\right.$

$$= \frac{\frac{1}{3}(\cos\theta+i\sin\theta)}{1-\frac{1}{3}(\cos\theta+i\sin\theta)} = \frac{\cos\theta+i\sin\theta}{3-\cos\theta-i\sin\theta}$$

$$= \frac{(\cos\theta+i\sin\theta)}{(3-\cos\theta)-i\sin\theta}\times\frac{(3-\cos\theta)+i\sin\theta}{((3-\cos\theta)+i\sin\theta)}$$

$$= \frac{3\cos\theta-\cos^2\theta-\sin^2\theta+i\sin\theta\cos\theta+i\sin\theta(3-\cos\theta)}{(3-\cos\theta)^2+\sin^2\theta}$$

$$\Rightarrow \sum_{m=1}^{\infty}\left(\frac{1}{3}(\cos\theta+i\sin\theta)\right)^m = \frac{3\cos\theta-1+i\cdot 3\sin\theta}{9-6\cos\theta+\sin^2\theta+\cos^2\theta}$$

$$\Rightarrow \sum_{m=1}^{\infty} 3^{-m}(\cos m\theta+i\sin m\theta) = \frac{(3\cos\theta-1)+3i\sin\theta}{10-6\cos\theta}$$

Equating the real parts:

$$\sum_{m=1}^{\infty} 3^{-m}\cos m\theta = \frac{3\cos\theta-1}{10-6\cos\theta}$$

Scanned with CamScanner

Example 22 (a) Show that $\sum_{r=1}^{n} z^{2r} = \frac{z^{2n+1} - z}{z - z^{-1}}$ for $z \neq 0, 1, -1$. --- [2]

(b) By letting $z = \cos\theta + i\sin\theta$, show that, if $\sin\theta \neq 0$

$$1 + 2\sum_{r=1}^{n} \cos(2r\theta) = \frac{\sin(2n+1)\theta}{\sin\theta} \quad \dots [5]$$

[W-20/22/Q7]

Solution (a)

$$\sum_{r=1}^{n} z^{2r} = z^2 + z^4 + z^6 + \dots + z^{2n}$$

$$= z^2 + z^2 \cdot z^2 + z^2 \cdot (z^2)^2 + \dots + z^2 \cdot (z^2)^{n-1}$$

Geo. Series: $a + ar + \dots + ar^{n-1}$, $S_n = \frac{a(r^n - 1)}{r-1}$

$$= \frac{z^2((z^2)^n - 1)}{z^2 - 1} = \frac{z^2(z^{2n} - 1)}{(z^2 - 1)}$$

$$= \frac{z^2(z^{2n} - 1)}{(z^2 - 1)} \times \frac{z^{-1}}{z^{-1}}$$

$$= \frac{z(z^{2n} - 1)}{z - z^{-1}} = \frac{z^{2n+1} - z}{z - z^{-1}} \checkmark$$

(b) Consider

$$\frac{z^{2n+1} - z}{z - z^{-1}} = \frac{\cos(2n+1)\theta + i\sin(2n+1)\theta - \cos\theta - i\sin\theta}{2i\sin\theta}$$

$[\because z - z^{-1} = 2i\sin\theta]$

from Part (a)

$$\Rightarrow \sum_{r=1}^{n} z^{2r} = \frac{i(\sin(2n+1)\theta - \sin\theta) + (\cos(2n+1)\theta - \cos\theta)}{i \times 2\sin\theta}$$

$$\Rightarrow \sum_{r=1}^{n} (\cos\theta + i\sin\theta)^{2r} = \frac{\sin(2n+1)\theta - \sin\theta}{2\sin\theta} + \frac{1}{i}\frac{(\cos(2n+1)\theta - \cos\theta)}{2\sin\theta}$$

$$\Rightarrow \sum_{r=1}^{n} \left(\cos(2r\theta) + i\sin(2r\theta)\right) = \frac{\sin(2n+1)\theta - \sin\theta}{2\sin\theta} - i\frac{(\cos(2n+1)\theta - \cos\theta)}{2\sin\theta}$$

Equating real parts:

$$\sum_{r=1}^{n} \cos(2r\theta) = \frac{\sin(2n+1)\theta}{2\sin\theta} - \frac{1}{2}$$

$$\Rightarrow 2\sum_{r=1}^{n} \cos(2r\theta) = \frac{\sin(2n+1)\theta}{\sin\theta} - 1$$

$$\Rightarrow 1 + 2\sum_{r=1}^{n} \cos(2r\theta) = \frac{\sin(2n+1)\theta}{\sin\theta} \checkmark$$

Scanned with CamScanner

Example 23: By letting $z = \frac{1}{2}(\cos\theta + i\sin\theta)$, use de-Moivre's the to deduce that: $\sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m \cdot \sin m\theta = \frac{2\sin\theta}{5-4\cos\theta}$

S-19	11	Q8(ii)

Solution: $z = \frac{1}{2}(\cos\theta + i\sin\theta) \Rightarrow |z| = \frac{1}{2} < 1$

$\Rightarrow 1 + z + z^2 + \cdots \infty = \frac{1}{1-z}$

G.P, $a, ar, ar^2, \cdots \infty$
$S_\infty = \frac{1}{1-r}$; $|r| < 1$

$$\Rightarrow \sum_{r=0}^{\infty} z^m = \frac{-1}{(z-1)}$$

$$\Rightarrow \sum_{r=0}^{\infty} \left[\frac{1}{2}(\cos\theta + i\sin\theta)\right]^n = \frac{-1}{\frac{1}{2}(\cos\theta + i\sin\theta) - 1}$$

$$\Rightarrow \sum_{r=0}^{\infty} \left[2^{-m}(\cos m\theta + i\sin m\theta)\right] = \frac{-1}{\left(\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta - 1\right)} \times \frac{\left(\frac{1}{2}\cos\theta - 1 - \frac{1}{2}i\sin\theta\right)}{\left(\frac{1}{2}\cos\theta - 1 - \frac{1}{2}i\sin\theta\right)}$$

$$\Rightarrow \sum_{r=1}^{\infty} \left[2^{-m}(\cos m\theta + i\sin m\theta)\right] = \frac{-\left[\frac{1}{2}\cos\theta - 1 - \frac{1}{2}i\sin\theta\right]}{\left(\frac{1}{2}\cos\theta - 1\right)^2 + \left(\frac{1}{2}\sin\theta\right)^2}$$

$$= \frac{-\left(\frac{1}{2}\cos\theta - 1\right) + i\cdot\frac{1}{2}\sin\theta}{\frac{1}{4}\cos^2\theta - \cos\theta + 1 + \frac{1}{4}\sin^2\theta}$$

$$= \frac{-\left(\frac{1}{2}\cos\theta - 1\right) + i\cdot\frac{1}{2}\sin\theta}{\frac{1}{4} + 1 - \cos\theta}$$

$$= \frac{-4\left(\frac{1}{2}\cos\theta - 1\right) + i\cdot 2\sin\theta}{5 - 4\cos\theta}$$

Equating imaginary parts:

$$\sum_{r=1}^{\infty} 2^{-m}\sin m\theta = \frac{2\sin\theta}{5-4\cos\theta}$$

Scanned with CamScanner

Example 24 (i) By considering $\sum_{r=1}^{n} z^{2r-1}$, where $z = \cos\theta + i\sin\theta$, show that, if $\sin\theta \neq 0$

$$\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta} \quad \text{--[7]}$$

(ii) deduce that;

$$\sum_{r=1}^{n} (2r-1)\cos\left[\frac{(2r-1)\pi}{2n}\right] = -\operatorname{cosec}\left(\frac{\pi}{2n}\right)\cdot\cot\left(\frac{\pi}{2n}\right) \text{--[4]}$$

[S-15 / 11 / Q 8]

Solution (i) $\sum_{r=1}^{r=n} z^{2r-1} = z + z^3 + z^5 + \dots + z^{2r-1} = \frac{z[1-(z^2)^n]}{1-z^2} = \frac{1-z^{2n}}{z^{-1}-z}$

$$\sum_{r=1}^{n}(\cos(2r-1)\theta + \sin(2r-1)\theta) = \frac{1-(\cos 2n\theta + i\sin 2n\theta)}{(\cos\theta - i\sin\theta)-(\cos\theta + i\sin\theta)} = \frac{(1-\cos 2n\theta) - i\sin 2n\theta}{-2i\sin\theta}$$

$$= \frac{2\sin^2 n\theta - i\sin 2n\theta}{2\sin\theta} \times \frac{1}{-i} \times \frac{i}{i} = \frac{+\sin 2n\theta + 2i\sin^2 n\theta}{2\sin\theta}$$

Equating imaginary parts;

$$\Rightarrow \sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta} \quad \text{---(i)}$$

(ii) Differentiating (i)

$$\sum_{r=1}^{n} (2r-1)\cdot\cos(2r-1)\theta = 2\sin n\theta\cdot\cos n\theta\times n\times\operatorname{cosec}\theta - \sin^2 n\theta\times\operatorname{cosec}\theta\cot\theta$$

Put $\theta = \frac{\pi}{2n}$

$$\Rightarrow \sum_{r=1}^{n} (2r-1)\cos(2r-1)\frac{\pi}{2n} = 2n\sin\frac{\pi}{2}\cdot\cos\frac{\pi}{2}\cdot\operatorname{cosec}\left(\frac{\pi}{2n}\right) - \sin^2\frac{\pi}{2}\times\operatorname{cosec}\left(\frac{\pi}{2n}\right)\cot\left(\frac{\pi}{2n}\right)$$

$$\Rightarrow \sum_{r=1}^{n} (2r-1)\cos\left(\frac{(2r-1)\pi}{2n}\right) = -\operatorname{cosec}\left(\frac{\pi}{2n}\right)\cdot\cot\left(\frac{\pi}{2n}\right) \quad \left\{\begin{array}{l}\cos\frac{\pi}{2}=0\\ \sin\frac{\pi}{2}=0\end{array}\right.$$

Scanned with CamScanner

Example 25 (i) Let $z = (\cos\theta + i\sin\theta)$, use binomial expansion of $(1+z)^n$, where n is a positive integer, to show that:
$\binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + --- + \binom{n}{n}\cos n\theta = 2^n\cos^n(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}n\theta) - 1$ ...[7]
(ii) Find $\binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + -- \binom{n}{n}\sin n\theta$. --[2]

$\boxed{S\text{-}15/13/Q6}$

Solution: $(1+z)^n = 1 + \binom{n}{1}z + \binom{n}{2}z^2 + -- \binom{n}{n}z^n$ [Using Binomial theorem]

$\Rightarrow [1 + (\cos\theta + i\sin\theta)]^n = 1 + \binom{n}{1}(\cos\theta + i\sin\theta) + \binom{n}{2}(\cos 2\theta + i\sin 2\theta) + ---$
$+ -- + \binom{n}{n}(\cos n\theta + i\sin n\theta)$

$\Rightarrow [(1+\cos\theta) + i\sin\theta]^n = ----$

$\Rightarrow [2\cos^2\theta/_2 + 2i\sin\theta/_2\cos\theta/_2]^n = ----$

$\Rightarrow [2\cos\theta/_2(\cos\theta/_2 + i\sin\theta/_2)]^n = ---$

$\Rightarrow 2^n\cdot\cos^n(\theta/_2)[\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}] = ---$ (i)

Equating the real parts on both side of (i)

$2^n\cdot\cos^n(\frac{\theta}{2})\cos(\frac{1}{2}n\theta) = 1 + \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + -- + \binom{n}{n}\cos n\theta$

$\Rightarrow \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + -- + \binom{n}{n}\cos n\theta = 2^n\cos^n(\tfrac{1}{2}\theta)\cdot\cos(\frac{n\theta}{2}) - 1$ ✓

(ii) Equating the imaginary parts on both sides of (i)

$\binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + --- + \binom{n}{n}\sin n\theta = 2^n\cdot\cos^n(\tfrac{1}{2}\theta)\cdot\sin(\tfrac{1}{2}n\theta)$ ✓

Scanned with CamScanner

Date 30.12.21.

FP-2

Further Pure Maths_2

Complex Numbers
Notes and Revision

SP-20	S-21	S-19	S-17	S-15	W-19
	S-20	S-18	S-16	W-20	W-18

Suresh Goel
(Former Director)
Alliance World School,
Noida_Delhi_NCR.
INDIA.

(+91 9810444804)

Scanned with CamScanner

26.(a) Use de Moivre's theorem to show that:

$$\text{Cosec}\,5\theta = \frac{\text{Cosec}^5\theta}{5\,\text{Cosec}^4\theta - 20\,\text{Cosec}^2\theta + 16} \qquad \text{---[6]}$$

(b) Hence obtain the roots of the equation:

$$x^5 - 10x^4 + 40x^2 - 32 = 0 \qquad \text{---[4]}$$

in the form $\text{Cosec}(q\pi)$, where q is rational $\boxed{W\text{-}21/21/Q6}$

Solution (a) Consider $(\cos\theta + i\sin\theta)^5 = (\cos 5\theta + i\sin 5\theta)$ ---(i) using de Moivre's theorem.

Now $(c+is)^5 = c^5 + 5c^4(is) + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5$

$= (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^3s^2 + s^5)$ ---(ii)

Comparing imaginary parts from (i) and (ii)

$\sin 5\theta = s^5 - 10c^2s^3 + 5c^4s$

$= s^5 - 10(1-s^2)s^3 + 5(1-s^2)^2 s$

$\sin 5\theta = 16s^5 - 20s^3 + 5s$

$\Rightarrow \text{Cosec}\,5\theta = \dfrac{1}{16s^5 - 20s^3 + 5s} \times \dfrac{\text{Cosec}^5\theta}{\text{Cosec}^5\theta}$

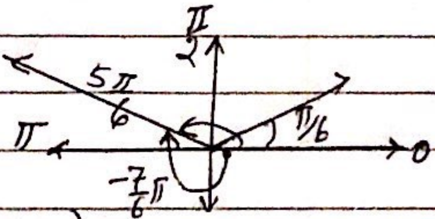
$\text{Cosec}\,5\theta = \dfrac{\text{Cosec}^5\theta}{5\,\text{Cosec}^4\theta - 20\,\text{Cosec}^2\theta + 16}$ ---(iii)

(b) Now to solve:

$x^5 - 10x^4 + 40x^2 - 32 = 0$

$\Rightarrow x^5 = 2(5x^4 - 20x^2 + 16)$

$\Rightarrow \dfrac{x^5}{5x^4 - 20x^2 + 16} = 2$ ---(iv)



Now let $x = \text{Cosec}\,\theta$ ✓

from (iii) $\Rightarrow$ $\text{Cosec}\,5\theta = 2$ (from (iii) & (iv))

$\therefore$ $\sin 5\theta = \frac{1}{2}$

$5\theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{7}{6}\pi, -\frac{11}{6}\pi, \frac{13}{6}\pi)$

$\theta = \frac{\pi}{30}, \ , \ , \ , \ )$

$x = \text{Cosec}\,\theta = \text{Cosec}\left(\frac{\pi}{30}\right), \ , \ , \ , \ )$

$x = \text{Cosec}\left(\frac{1}{30}\pi\right), \text{Cosec}\left(\frac{5}{30}\pi\right), \text{Cosec}\left(-\frac{7}{30}\pi\right), \text{Cosec}\left(-\frac{11}{30}\pi\right), \text{Cosec}\left(\frac{13}{30}\pi\right)$ ✓

27. (a) Write all the roots of the equation, $x^5 - 1 = 0$ ---[2]
(b) Use de Moivre's theorem to show: $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ ---[4]
(c) Use the results of part (a) and (b) to express each real root of the equation: $8x^9 - 8x^7 + x^5 - 8x^4 + 8x^2 - 1 = 0$ ---[4]
in the form $\cos k\pi$, where k is a rational. [W-21 / 22 / Q4]

Solution (a) To solve: $x^5 - 1 = 0$

$\Rightarrow x^5 = 1$ (To find the fifth root of Unity)

$\Rightarrow x^5 = (\cos 0 + i \sin 0)$

$x = [\cos(2k\pi + 0) + i\sin(2k\pi + 0)]^{\frac{1}{5}}$

$\Rightarrow x = \left(\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}\right)$; $k = 0, 1, 2, 3, 4$

or $x = e^{\frac{2}{5}\pi k i}$; $k = 0, 1, 2, 3, 4$ $\quad [x = e^{\frac{2}{5}k\pi i}]$

(b) Consider, $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ --- ①

Now L.H.S $(c + is)^4 = c^4 + 4c^3(is) + 6c^2(is)^2 + 4c(is)^3 + (is)^4$

$= (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3)$ ---- ②

Comparing the real parts in ① & ②

$\cos 4\theta = c^4 - 6c^2s^2 + s^4 = c^4 - 6c^2(1 - c^2) + (1 - c^2)^2$

$\Rightarrow \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

(c) $8x^9 - 8x^7 + x^5 - 8x^4 + 8x^2 - 1 = 0$

$\Rightarrow (x^5 - 1)(8x^4 - 8x^2 + 1) = 0$

$\Rightarrow 8x^4 - 8x^2 + 1 = 0$ $\quad$ [for $x = \cos\theta$]

$8\cos^4\theta - 8\cos^2\theta + 1 = 0$

from part (b)

$\cos 4\theta = 0$

$\Rightarrow 4\theta = \frac{(2k+1)\pi}{2}$ $\quad k = 0, 1, 2, 3$

$4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5}{8}\pi, \frac{7\pi}{8}$

or $x^5 - 1 = 0$ from Part (a)

$\Rightarrow x = e^{\frac{2}{5}k\pi i}$

The only real root is for $k = 0 \Rightarrow x = \cos 0 = 1$ --③

$\therefore x = \cos\frac{\pi}{8}, \cos\frac{3}{8}\pi, \cos\frac{5\pi}{8}, \cos\frac{7}{8}\pi$ --④

from ③ & ④ Required real roots are $x = \cos 0, \cos\frac{1}{8}\pi, \cos\frac{3}{8}\pi, \cos\frac{5}{8}\pi, \cos\frac{7}{8}\pi$.

Scanned with CamScanner