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FP-2

Further Pure Maths - 2

Differential Equations Notes and Revision.

SP-20	S-20	W-20	S-21

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FP.2

Differential Equations

§ Solution of differential equations:

§ (1) Variable separable method:

To solve: $\frac{dy}{dx} = f(y)\cdot g(x)$

$$\Rightarrow \int \frac{1}{f(y)}\,dy = \int g(x)\,dx$$

Example 1: The variables x and y satisfy the differential equation:

$\frac{dy}{dx} = \frac{1+4y^2}{e^x}$; It is given that $y=0$ when $x=1$.

(a) Solve the differential equation, obtaining an expression for y in terms of x. ---[7]

(b) State what happens to the value of y as x tends to infinity. --[1]

[9709/M-20/32/Q6] → A-level

Solution: $\frac{dy}{dx} = \frac{1+4y^2}{e^x}$

(a)

$$\Rightarrow \int \frac{1}{1+4y^2}\,dy = \int e^{-x}\,dx$$

$$\Rightarrow \frac{1}{4}\int \frac{1}{\frac{1}{4}+y^2}\,dy = -e^{-x}+c$$

$$\Rightarrow \frac{1}{4}\int \frac{1}{(\frac{1}{2})^2+y^2}\,dy = -e^{-x}+c \quad \left[\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$\Rightarrow \frac{1}{4}\left[\frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{y}{\frac{1}{2}}\right)\right] = -e^{-x}+c$$

$$\Rightarrow \frac{1}{2}\tan^{-1}(2y) = -e^{-x}+c \quad \text{--(i)}$$

Now for $y=0$, $x=1$ from (1)

$\Rightarrow \frac{1}{2}\tan^{-1}0 = -e^{-1}+c \Rightarrow c = e^{-1}$

from (i) $\frac{1}{2}\tan^{-1}(2y) = -e^{-x}+e^{-1}$

$\Rightarrow 2y = \tan(2e^{-1}-2e^{-x})$

or $y = \frac{1}{2}\tan(2e^{-1}-2e^{-x})$ --(2)

is the required solution.

(b) when $x\to\infty$, $e^{-x}\to 0$

$\therefore$ from (2)

$y = \frac{1}{2}\tan(2e^{-1})$ ✓

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§2. First-order linear differential equations:

Solution using "Integrating factor".

Consider: $\frac{dy}{dx} + P(x)\cdot y = Q(x)$ ---- (i)

Find the Integrating factor $(I.F) = e^{\int P(x)dx}$

multiply both sides of (i) by 'I.F'

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}\, P(x)\cdot y = Q(x)\cdot e^{\int P(x)dx}$$

$$\Rightarrow \frac{d}{dx}\left(y\cdot e^{\int P(x)dx}\right) = Q(x)\cdot e^{\int P(x)dx}$$

$$\Rightarrow y\cdot e^{\int P(x)dx} = \int Q(x)\cdot e^{\int P(x)dx}\, dx \quad \left[I.F = e^{\int P(x)dx}\right.$$

$$\boxed{y\cdot(I.F) = \int Q(x)(I.F)dx}$$ ✓

Example 2: Find the solution of the differential equation:

$$x\frac{dy}{dx} + 3y = \frac{\sin x}{x}$$

for which $y=0$ when $x = \frac{\pi}{2}$.

Give your answer in the form $y = f(x)$. ---[8]

[SP-20/02/Q3]

Solution: Given $x\frac{dy}{dx} + 3y = \frac{\sin x}{x}$

$$\Rightarrow \frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^2} \quad \text{----(1)}$$

$\therefore$ Integrating factor $I.F = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

multiplying (i) by $I.F = x^3$

$$x^3\cdot\frac{dy}{dx} + 3x^2\cdot\frac{dy}{dx} = x\sin x \Rightarrow \frac{d}{dx}(yx^3) = x\sin x$$

$$\Rightarrow \text{Req Solu. } y\cdot x^3 = \int x\sin x\, dx \quad \left[\, y\cdot(IF) = \int Q(x)(IF)dx \,\right]$$

$$\Rightarrow yx^3 = x\cdot\int \sin x\, dx - \int\left(\frac{d}{dx}x\cdot\int \sin x\, dx\right)dx$$

$$= x\cdot(-\cos x) - \int 1\times(-\cos x)dx$$

$$\Rightarrow yx^3 = -x\cos x + \sin x + c \quad \text{---(ii)}$$

In (ii) for $x = \frac{\pi}{2}$, $y=0 \Rightarrow 0 = 0 + 1 + C \Rightarrow C = -1 \Rightarrow yx^3 = -x\cos x + \sin x - 1$

$\therefore$ Req Solu. $\rightarrow y = \frac{-x\cos x + \sin x - 1}{x^3}$ ✓

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Example 3: Find the solution of the differential equation, $\frac{dy}{dx} + 5y = e^{-7x}$ for which $y=0$ when $x=0$.
Give your answers in the form: $y = f(x)$ ---[6]

[S-20|21|Q1]

Solution: Given, $\frac{dy}{dx} + 5y = e^{-7x}$

$$I.F = e^{\int 5dx} = e^{5x}$$

$$\therefore \frac{d}{dx}(y.e^{5x}) = e^{-7x}.e^{5x}$$

$$y.e^{5x} = \int e^{-2x}dx$$

$$\Rightarrow ye^{5x} = -\frac{1}{2}e^{-2x} + c \quad \text{---(i)}$$

for $y=0, x=0$ in (i):

$$0 = -\frac{1}{2} + c \Rightarrow c = \frac{1}{2}$$

$\therefore$ from (i) $ye^{5x} = -\frac{1}{2}e^{-2x} + \frac{1}{2}$

$$\Rightarrow y = \frac{1}{2}e^{-5x} - \frac{1}{2}e^{-7x}$$

Example 4: Find the solution of the differention equation:
$\frac{dy}{d\theta} + y\cot\theta = \sin^3\theta$, for which $y=0$ when $\theta = \frac{\pi}{2}$ ---[6]

[W-20|21|Q6(b)]

Solution: Given $\frac{dy}{dx} + y\cot\theta = \sin^3\theta$ ---(i)

$$I.F = e^{\int \cot\theta d\theta} = e^{\ln \sin\theta} = \sin\theta.$$

$$\frac{d}{d\theta}(y.\sin\theta) = \sin^3\theta.\sin\theta$$

$$\Rightarrow y\sin\theta = \int \sin^4\theta\, d\theta$$

$$\Rightarrow y\sin\theta = \int \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)d\theta$$

$$\Rightarrow y.\sin\theta = \frac{1}{8}\left[\frac{\sin 4\theta}{4} - 4\frac{\sin 2\theta}{2} + 3\theta\right] + c \quad (2)$$

Now $y=0$ and $\theta = \frac{\pi}{2}$ in (2)

$$\Rightarrow 0 = \frac{1}{8}\left[0 - 0 + \frac{3\pi}{2}\right] + c \Rightarrow c = -\frac{3\pi}{16}$$

Consider:

$$\sin^4\theta = (\sin^2\theta)^2$$
$$= \left(\frac{1-\cos 2\theta}{2}\right)^2$$
$$= \frac{1}{4}[\cos^2 2\theta - 2\cos 2\theta + 1]$$
$$= \frac{1}{4}\left[\left(\frac{1+\cos 4\theta}{2}\right) - 2\cos 2\theta + 1\right]$$
$$= \frac{1}{8}[\cos 4\theta - 4\cos 2\theta + 3]$$

May be proved using (De-Moivre's Theorem)

from (2) Req. Solution; $y.\sin\theta = \frac{1}{8}\left[\frac{1}{4}\sin 4\theta - 2\sin 2\theta + 3\theta - \frac{3}{2}\pi\right]$

Example 5: Find the solution of the differential equation: $x\frac{dy}{dx}+2y=e^{x}$ ...[8]
for which $y=3$ when $x=1$, give your answer in the form $y=f(x)$

[W-20/22/Q4]

Solution: Given $x\frac{dy}{dx}+2y=e^{x} \Rightarrow \frac{dy}{dx}+\frac{2}{x}y=\frac{1}{x}e^{x}$ ...(i)

$I.F = e^{\int \frac{2}{x}dx}$

$I.F = e^{2\ln x} = e^{\ln x^{2}} = x^{2}$

from (i) $\frac{d}{dx}(y.x^{2}) = \frac{1}{x}e^{x}.x^{2} \Rightarrow yx^{2} = \int xe^{x}dx$

$\Rightarrow yx^{2} = x.\int e^{x}dx - \int\left(\frac{d}{dx}x.\int e^{x}dx\right)dx$

$\Rightarrow yx^{2} = xe^{x} - \int 1xe^{x}dx$

$\Rightarrow yx^{2} = xe^{x} - e^{x} + c$ ---(2)

for $y=3$, $x=1$ in (2) $\Rightarrow 3 = e - e + c \Rightarrow c=3$

$\therefore$ from (2) $yx^{2} = e^{x}(x-1)+3$

$\therefore$ Req. solution: $y = \frac{e^{x}(x-1)+3}{x^{2}}$ ✓

Given:
$\frac{dy}{dx}+P(x).y=Q(x)$
$\Rightarrow y.(I.F) = \int Q(x)(I.F)dx$

Example 6: Find the solution of the differential equation: $\sin\theta\frac{dy}{d\theta}+y=\tan\frac{1}{2}\theta$.
where $0<\theta<\pi$, given that $y=1$ when $\theta=\frac{1}{2}\pi$. ---[9]
Give your answer in the form $y=f(\theta)$ [Use $\int \operatorname{cosec}\theta\, d\theta = \ln\tan\frac{1}{2}\theta$]

[S-21/21/Q4]

Solution: Given $\sin\theta\frac{dy}{d\theta}+y=\tan\frac{1}{2}\theta$

$\Rightarrow \frac{dy}{dx}+y.\operatorname{cosec}\theta = \frac{\tan\theta/2}{\sin\theta}$ ---(i)

$I.F = e^{\int \operatorname{cosec}\theta\, d\theta} = e^{\ln\tan\theta/2} = \tan\theta/2$

$\therefore$ Solution is

$y.(I.F) = \int \tan\theta/2.(I.F)d\theta$

from (i) $\Rightarrow y.\tan\theta/2 = \int\frac{\tan\theta/2}{\sin\theta}.\tan\theta/2\, d\theta$

$\Rightarrow y\tan\theta/2 = \int \tan\theta/2 \times \frac{\sin\theta/2}{\cos\theta/2}\times\frac{1}{2\sin\theta/2\cos\theta/2}d\theta = \int \tan\theta/2 \times \frac{1}{2}\sec^{2}\theta/2\, d\theta$

$y\tan\theta/2 = \int u\,du$ $\left[\begin{array}{l} u=\tan\theta/2 \\ du=\frac{1}{2}\sec^{2}\theta/2\, d\theta\end{array}\right.$

$= \frac{u^{2}}{2}+c$

$y\tan\theta/2 = \frac{1}{2}\tan^{2}\theta/2 + c$ ---(2)

for $\theta=\frac{\pi}{2}$, $y=1 \Rightarrow 1 = \frac{1}{2}+c \Rightarrow c=\frac{1}{2} \Rightarrow$

from (2) $y\tan\theta/2 = \frac{1}{2}\tan^{2}\theta/2 + \frac{1}{2}$

$\Rightarrow y = \frac{1}{2}(\tan\theta/2 + \cot\theta/2)$ ✓

Given:
$\frac{dy}{dx}+P(x).y=Q(x)$
Solution is
$y.(I.F) = \int Q(x).(I.F)dx$

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Example 7: Find the solution of the differential equation:

$$\frac{dy}{dx} + y\coth x = 4\sinh x \; ; \text{ for which } y=1 \text{ when } x=\ln 3.$$

[S-21 | 23 | 6(b)] ---[7]

Solution: $\frac{dy}{dx} + y\coth x = 4\sinh x$ --- (i)

from (i) $\quad I.F = e^{\int \coth x\,dx} = e^{\ln \sinh x} = \sinh x$

$\frac{d}{dx}(y.\sinh x) = 4\sinh x.\sinh x$

$\Rightarrow y\sinh = \int 4\sinh^2 x\,dx \qquad [\because 2\sinh^2 x = \cosh 2x - 1]$

$= 2\int 2\sinh^2 x\,dx = 2\int(\cosh 2x - 1)dx = 2\left(\frac{\sinh 2x}{2} - x\right) + c$

$\Rightarrow y\sinh x = \sinh 2x - 2x + c$ --- (ii)

Now $y=1$, $x=\ln 3 \Rightarrow 1\times\sinh(\ln 3) = \sinh(2\ln 3) - 2\ln 3 + c$

$\Rightarrow \sinh(\ln 3) = \sinh(\ln 9) - 2\ln 3 + c$ --- (iii)

$\Rightarrow \frac{4}{3} = \frac{40}{9} - 2\ln 3 + c$

$\Rightarrow c = 2\ln 3 - \frac{28}{9}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\sinh(\ln 3) = \frac{e^{\ln 3} - e^{-\ln 3}}{2} = \frac{3 - \frac{1}{3}}{2} = \frac{4}{3}$

$\sinh(\ln 9) = \frac{e^{\ln 9} - e^{-\ln 9}}{2} = \frac{9 - \frac{1}{9}}{2} = \frac{40}{9}$

hence from (ii)

Required Solution:

$y\sinh x = \sinh 2x - 2x + 2\ln 3 - \frac{28}{3}$ ✓

Example 8: Find the general solution for the differential equation,

$$\frac{dy}{dx} + y\tanh x = 4x$$

Solution: Given $\frac{dy}{dx} + y\tanh x = 4x$ --- (i)

$I.F = e^{\int \tanh x} = e^{\ln\cosh x} = \cosh x$

$\therefore \frac{d}{dx}(y\cosh x) = 4x\cdot\cosh x \Rightarrow y\cosh x = \int 4x\cosh x\,dx$

$\Rightarrow y\cosh x = 4\left[x.\int\cosh x\,dx - \int\left(\frac{d}{dx}x.\int\cosh x\,dx\right)dx\right]$

$= 4\left[x\sinh x - \int 1x\sinh x\,dx\right]$

$\Rightarrow y\cosh x = 4x\sinh x - 4\cosh x + c$

$\Rightarrow y = 4x\tanh x - 4 + c\,\mathrm{sech}\,x$ ✓ (Dividing by $\cosh x$)

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§ Second Order differential equations: The homogeneous linear:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = 0 \quad \text{----(i)} \quad ; \; a, b, c \in R \text{ (Constants)}$$

we try $y = Ae^{\lambda x}$ in (i) $\Rightarrow a\lambda^2 + b\lambda + c = 0$ (Auxiliary equation)

We solve the auxiliary equation (quadratic equation).

Case I: If the solution of $a\lambda^2 + b\lambda + c = 0$ has two distinct real roots λ_1, λ_2 ($b^2 - 4ac > 0$) ; $\lambda_1 \neq \lambda_2$

Then the complementary function

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x} \checkmark$$

Case II: If the equation $a\lambda^2 + b\lambda + c = 0$ has two real and equal roots $\lambda_1 = \lambda_2 = \lambda$ ($b^2 - 4ac = 0$)

Then the complementary solution is:

$$y = (Ax + B)e^{\lambda x} \checkmark$$

Case III: If the auxiliary equation $a\lambda^2 + b\lambda + c = 0$ has two conjugate complex roots: $(\alpha + i\beta)$ and $(\alpha - i\beta)$ $\left(\begin{array}{c} b^2 - 4ac < 0 \\ \text{and } b \neq 0 \end{array}\right)$

Then the complementary solution is: $\checkmark$

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

Case IV: If the equation $a\lambda^2 + b\lambda + c = 0$ has two purely imaginary roots $\pm\beta i$ $\left(\begin{array}{c} b^2 - 4ac < 0 \\ \text{and } b = 0 \end{array}\right)$

Then the complementary solution is:

$$y = A\cos\beta x + B\sin\beta x \checkmark$$

Also on Page-17 foot Note

§ For Particular Integral:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

(i) $f(x) = kx^2 + mx + n$ then try $y = \alpha x^2 + \beta x + \gamma$

(ii) $f(x) = ke^{mx} \longrightarrow$ Try $y = \alpha e^{mx}$ (*)

(iii) $f(x) = k_1\cos mx + k_2\sin mx \longrightarrow$ Try $y = \alpha\cos mx + \beta\sin mx$

(*) If one term in Complementary fun^n is also $e^{mx} \rightarrow$ Then $y = \alpha x e^{mx}$)

(*) If Complementary fun^n $= (A + Bx)e^{mx} \longrightarrow$ Then $y = \alpha x^2 e^{mx}$ (*)

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§ Second Order non-homogeneous linear differential equation:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

(i) Solve $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

and find the complementary solution.

(ii) Find the particular solution of $y = f(x)$

Then the general solution
= Complementary function + the particular integral

Example 9: Find the general solution of the differential equation:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 7 - 2t^2$$ --[6]

[SP-20/02/Q1]

Solution: $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 7 - 2t^2$ --(i)

for $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0$

The auxiliary equation is:

$\lambda^2 + 4\lambda + 4 = 0$

$\Rightarrow (\lambda+2)^2 = 0 \Rightarrow \lambda = -2, -2$
real - equal roots

$\therefore$ complementary solution is

$x = (At + B)e^{-2t}$ --(ii)
(Case II)

Now to find the particular integral

Let $x = P + Qt + Rt^2$ --(iii)

diff w.r.t 't' $\Rightarrow \frac{dx}{dt} = Q + 2Rt$ --(IV)

diff again. $\frac{d^2x}{dt^2} = 2R$ ---(V)

from (iii), (IV) and (V) in (i)

$2R + 4(Q + 2Rt) + 4(P + Qt + Rt^2) = 7 - 2t^2$

Comparing the coeff of $t^2 \Rightarrow 4R = -2$
$R = -\frac{1}{2}$ ✓

Comparing the constant term:

$2R + 4Q + 4P = 7$

$-1 + 4P + 4Q = 7$ $(R = -1/2)$

$4P + 4Q = 8$ ---(VI)

Comparing the coeff of t:

$8R + 4Q = 0$

$-4 + 4Q = 0$ $[R = -\frac{1}{2}]$

$\Rightarrow Q = 1$ ✓

Put $Q = 1$ in equation (VI)

$4P + 4\times 1 = 8 \Rightarrow P = 1$ ✓

$\therefore$ The Particular Integral is:

$x = 1 + t - \frac{1}{2}t^2$ --(VII)

$\therefore$ General Solution of equation (i)

$x = (At+B)e^{-2t} + 1 + t - \frac{1}{2}t^2$ ✓

(x = Comp. Funct + Particular soln)

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Example 10: The variables x and y are related by differential equation:

$$9\frac{d^2y}{dx^2}+6\frac{dy}{dx}+y=3x^2+30x$$

(a) Find the general solution for y in terms of x. ---[6]

(b) State an approximate solution for large positive values of x. ---[1]

[W-20/21/Q2]

Solution: $9\frac{d^2y}{dx^2}+6\frac{dy}{dx}+y=3x^2+30x$ ----(i)

(a) Consider the auxilary equation: $9\lambda^2+6\lambda+1=0$

$(3\lambda+1)^2=0 \Rightarrow \lambda=-\frac{1}{3}$ repeated real root.

$\therefore$ complementary function for (i) is:

$$y=e^{-\frac{1}{3}x}(Ax+B) \quad \text{----(ii)}$$

Now for particular integral of equation (i),

Consider; $y=p+qx+rx^2$ ----(iii)

diff. w.r.t x: $\frac{dy}{dx}=q+2rx$ --(iv)

diff. again $\frac{d^2y}{dx^2}=2r$ ----(v)

from (iii), (iv) and (v) in (i)

$\Rightarrow 9\times 2r+6(q+2rx)+p+qx+rx^2=3x^2+30$

$\Rightarrow (18r+6q+p)+(12r+6q)x+rx^2=3x^2+30$ ---(vi)

Comparing the coeff. of x^2 on both sides of (vi) $\Rightarrow r=3$ ✓

comparing coeff of x $\Rightarrow 12r+6q=0$

$12\times 3+6q=0 \Rightarrow q=-6$ ✓ $(r=3)$

Comparing the constant term: $18r+6q+p=30$

$18\times 3+6\times(-6)+p=30 \Rightarrow p=-18$ ✓

$\therefore$ from (iii) particular integral is: $y=-18-6x+3x^2$ --(vii)

Hence General solution = Complementary function + Particular Integral.

$$y=e^{-\frac{1}{3}x}\cdot(Ax+B)+3x^2-6x-18 \checkmark \quad \text{[from (ii) and (vii)]}$$

(b) When x is large $(x\to\infty) \Rightarrow e^{-\frac{1}{3}x}\to 0$

$\therefore$ Solution is: $y=3x^2-6x-18$ ✓

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Example 11: Find the particular solution of the differential equation:

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 15x = 102\cos 3t$$

given that when $t=0$, $x=1$ and $\frac{dx}{dt}=0$ --[11]

W-20/22/Q6

Solution: Given $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 15x = 102\cos 3t$ ---(i)

Consider the auxiliary equation: $\lambda^2+8\lambda+15=0 \Rightarrow \lambda=-3,-5$

$\therefore$ Complementary function: $x = Ae^{-5t}+Be^{-3t}$ ----(ii)

Now consider particular integral: $x = P\sin 3t + Q\cos 3t$ ---(iii)

Diff w.r.t $t \rightarrow \frac{dx}{dt} = 3P\cos 3t - 3Q\sin 3t$ --(iv)

Diff again $\rightarrow \frac{d^2x}{dt^2} = -9P\sin 3t - 9Q\cos 3t$ --(v)

from (iii), (iv) and (v) put in (i)

$-9P\sin 3t - 9Q\cos 3t + 8(3P\cos 3t - 3Q\sin 3t) + 15(P\sin 3t + Q\cos 3t) = 102\cos 3t$

$\Rightarrow (-9P-24Q+15P)\sin 3t + (-9Q+24P+15Q)\cos 3t = 102\cos 3t$ ---(vi)

Comparing the coeff of $\sin 3t \Rightarrow 6P-24Q=0 \Rightarrow P=4Q$ --(vii)

Comparing the coeff of $\cos 3t \Rightarrow 24P+6Q=102 \Rightarrow 4P+Q=17$ --(viii)

from (vii) and (viii) $\rightarrow P=4,\ Q=1$

$\therefore$ The particular integral of (i) is $x = 4\sin 3t + \cos 3t$ —— (ix)

$\therefore$ General solution is: $x = Ae^{-5t}+Be^{-3t}+4\sin 3t+\cos 3t$ ---(x)

Diff (x) w.r.t 't',

$\frac{dx}{dt} = -5Ae^{-5t} - 3Be^{-3t} + 12\cos 3t - 3\sin 3t$ --(xi)

Put $t=0$ & $x=1$ in (ii) $\Rightarrow 1 = A+B$ --(xii)

Put $\frac{dx}{dt}=0$ at $t=0 \Rightarrow 0 = -5A-3B+12 \rightarrow 5A+3B=12$ --(xiii)

from (xii) & (xiii) $\Rightarrow A=6,\ B=-6$ ✓

from X $\rightarrow$ Required particular solution is:

$$x = 6e^{-5t} - 6e^{-3t} + 4\sin 3t + \cos 3t$$ ✓

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Example 12: The variables x and y are related by the differential equation: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 4e^{-x}$

(a) Find the value of the constant k such that $y = kxe^{-x}$ is a particular integral of the differential equation. ---[4]

(b) Find the solution of the differential equation for which:
$y = \frac{dy}{dx} = \frac{1}{2}$ when $x = 0$ ---[6]

[S-21/23/Q5]

Solution: Given $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 4e^{-x}$ ---(i)

(a)

Note: Here the term e^{-x} is same as in the complementary function $\therefore$ Particular Int is $y = kxe^{-x}$

For particular integral: $y = kxe^{-x}$ --(ii)

Diff: $\frac{dy}{dx} = k[1\times e^{-x} - xe^{-x}] = ke^{-x}(1-x)$ --(iii)

Diff again $\frac{d^2y}{dx^2} = k[-1e^{-x} - (1-x)e^{-x}]$
$= ke^{-x}(x-2)$ ---(iv)

Put from (ii), (iii) & (iv) in (i)

$\Rightarrow ke^{-x}(x-2) - 2(ke^{-x}(1-x)) - 3kxe^{-x} = 4e^{-x}$

$\Rightarrow x(k+2k-3k) - 2k - 2k = 4 \Rightarrow -4k = 4 \Rightarrow k = -1$ ✓

from (ii) the particular integral is $y = -xe^{-x}$ ---(v)

(b) Now to find the complementary function of (i),
Consider the auxiliary equation of (i): $\lambda^2 - 2\lambda - 3 = 0 \Rightarrow \lambda = 3, -1$
$\therefore$ complementary function of (1) is $y = Ae^{-x} + Be^{3x}$ --- (vi)

$\therefore$ General Solution of (1) is: $y = Ae^{-x} + Be^{3x} - xe^{-x}$
$y = e^{-x}(A-x) + Be^{3x}$ ---(vii)

Now diff (vii):

$\frac{dy}{dx} = -e^{-x}(A-x) - e^{-x} + 3Be^{3x}$ ---(viii)

Given $y = \frac{1}{2}$; $\frac{dy}{dx} = \frac{1}{2}$ when $x = 0$

from (vii) $\rightarrow A + B = \frac{1}{2}$ --(ix)
from (viii) $\rightarrow \frac{1}{2} = -A - 1 + 3B$ -(x)

Solving (ix) and (x) $\Rightarrow A = 0$; $B = \frac{1}{2}$

$\therefore$ Required particular solution is:

$y = \frac{1}{2}e^{3x} - xe^{-x}$ ✓ (for $A = 0$ and $B = \frac{1}{2}$) (from (vii))

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Example 13: The variables x and y are related by the differential equation:

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=2x+1$$

(a) Find the general solution for y in terms of x, ---[6]
(b) State an approximate solution for large positive values of x. ---[1]

$\boxed{S\text{-}21/21/Q\,2}$

Solution: Given $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=2x+1$ ---(i)

Consider auxiliary equation: $\lambda^2+3\lambda+2=0 \Rightarrow \lambda=-2,-1$

$\therefore$ Complementary function: $y=Ae^{-x}+Be^{-2x}$ ------(ii)

for particular integral consider: $y=P+Qx$ -----(iii)

Diff. w.r.t $x \rightarrow \frac{dy}{dx}=Q$ ---(iv)

Diff again $\longrightarrow \frac{d^2y}{dx^2}=0$ ---(V)

from (iii), (iv) and (V) put in (i) $\rightarrow$

$$0+3Q+2(P+Qx)=2x+1$$

$$(2P+3Q)+2Qx=2x+1 \quad ---(VI)$$

Comparing the coeff of x in (vi) $\rightarrow 2Q=2 \Rightarrow Q=1$ ✓

and comparing the constant term: $2P+3Q=1$

$\Rightarrow P=-1$ ✓ $[Q=1]$

$\therefore$ from (iii) Particular integral is: $y=-1+x$ --(vi)

$\therefore$ General solution = Complementary function + Particular integral

$$y=Ae^{-x}+Be^{-2x}+x-1 \quad (\text{from (ii) \& (vi)})$$

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Example 14: Find the general solution of the differential equation:

$$\frac{d^2x}{dt^2} - 8\frac{dx}{dt} - 9x = 9e^{8t}$$ ---[6]

[S-20|23|Q1]

Solution: Given $\frac{d^2x}{dt^2} - 8\frac{dx}{dt} - 9x = 9e^{8t}$ ----(i)

The auxiliary equation is: $\lambda^2 - 8\lambda - 9 = 0 \Rightarrow \lambda = -1, 9$

$\therefore$ Complementary function is: $x = Ae^{-t} + Be^{9t}$ ------(ii)

Now for particular integral consider $x = ke^{8t}$ ------(iii)

diff w.r.t. 't' $\rightarrow \frac{dx}{dt} = 8ke^{8t}$ ---(iv)

Again diff $\longrightarrow \frac{d^2x}{dt^2} = 64ke^{8t}$ ---(v)

from (iii), (iv) and (v) put in (i)

$\Rightarrow 64ke^{8t} - 8\cdot 8ke^{8t} - 9\cdot ke^{8t} = 9e^{8t}$

$\Rightarrow -9k = 9 \Rightarrow k = -1$ ✓

$\therefore$ from (iii) Particular integral is: $y = -e^{8t}$ ---(vi)

$\therefore$ General Solution = complementary function + Particular integral.

$= \underline{Ae^{-t} + Be^{9t} - e^{8t}}$ (from (ii) & (vi))

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Example 15: Find the general Solution of the differential equation: --- [7]

(a) $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4e^{2x}$$

(b) Find the particular solution, such that $y=3$, and $\frac{dy}{dx} = -2$ for $x=0$ --- [4]

$\boxed{S\text{-}16/11/Q9}$

Solution: Given $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4e^{2x}$ -----(i)

(a)

Consider the auxiliary equation: $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda-2)^2 = 0 \Rightarrow \lambda = +2, +2$

$\therefore$ Complementary function is: $y = (A+Bx)e^{2x}$ --(ii) (Two equal roots)

In complementary function we have a term Bxe^{2x} same as $y = f(x) = e^{2x}$

$\therefore$ Particular Int. $y = Kx^2e^{2x}$

Now let the Particular integral is:

$y = Kx^2e^{2x}$ ---(iii)

Diff. w.r.t x:

$$\frac{dy}{dx} = K[2xe^{2x} + 2x^2e^{2x}] = 2K(x+x^2)e^{2x} \text{ ---(iv)}$$

diff again $$\frac{d^2y}{dx^2} = 2K[(1+2x)e^{2x} + 2(x+x^2)e^{2x}]$$
$$= 2Ke^{2x}[1+4x+4x^2] \text{ ----(v)}$$

from (iii), (iv) and (v) in (i)

$$2Ke^{2x}(1+4x+4x^2) - 4[2Ke^{2x}(x+x^2)] + 4Kx^2e^{2x} = 4e^{2x}$$

$$K[2+\cancel{8x}+4x^2-\cancel{8x}-8x^2+4x^2] = 4 \Rightarrow 2K = 4 \Rightarrow K = 2$$

from (iii) Particular integral $= 2x^2e^{2x}$ -------(vi)

$\therefore$ General Solution = Complementary function + Particular integral.

or $y = (A+Bx)e^{2x} + 2x^2e^{2x}$ [from (ii) and (vi)] (vii)

(b) diff. (vii) $$\frac{dy}{dx} = 2(A+Bx)e^{2x} + Be^{2x} + 4x^2e^{2x} + 4xe^{2x} \text{ -- (viii)}$$

from (vii) and $y=3$ for $x=0 \Rightarrow 3 = A + 0 \Rightarrow A = 3$

and $\frac{dy}{dx} = -2$ for $x=0$ in (viii) $\Rightarrow -2 = 2A + B \Rightarrow B = -8$

$\therefore$ from vii the required particular solution is:

$$y = (3-8x)e^{2x} + 2x^2e^{2x}$$

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Example 16: It is given that $x = t^3 y$ and

$$t^3\frac{d^2y}{dt^2} + (4t^3+6t^2)\frac{dy}{dt} + (13t^3+12t^2+6t)y = 61e^{\frac{1}{2}t} \quad ⊗$$

(a) Show that $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = 61e^{\frac{1}{2}t}$ ---[4]

(b) Find the general solution for y in terms of t. ---[7]

[S-20/21/Q7]

Solution: Given $x = t^3 y$ ----(i)

diff $\frac{dx}{dt} = t^3\frac{dy}{dt} + 3t^2 y$ --(ii)

diff. again: $\frac{d^2x}{dt^2} = t^3\frac{d^2y}{dt} + 3t^2\frac{dy}{dt} + 3\left(t^2\frac{dy}{dt} + 2 t y\right)$

or $\frac{d^2x}{dt^2} = t^3\frac{d^2y}{dt} + 6t^2\frac{dy}{dt} + 6ty$ ---(iii)

$\therefore \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = t^3\frac{d^2y}{dt^2} + 6t^2\frac{dy}{dt} + 6ty + 4\left(t^3\frac{dy}{dt} + 3t^2y\right) + 13t^3y$ [from (i), (ii), (iii)]

$= t^3\frac{d^2y}{dt^2} + (4t^3+6t^2)\frac{dy}{dt} + (13t^3+12t^2+6t)y$

$\Rightarrow \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = 61e^{\frac{1}{2}t}$ ---(iv) (Given in the question ⊗)

Now the auxiliary equation is: $\lambda^2+4\lambda+13=0 \Rightarrow \lambda = -2 \pm 3i$

$\therefore$ complementary function is:

$x = e^{-2t}(A\cos 3t + B\sin 3t)$ --(V)

[Case III (Conjugate complex roots) (Page-5)
$\lambda = \alpha \pm i\beta \Rightarrow x = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$]

Now the particular Integral is: $x = ke^{\frac{1}{2}t}$ --- (VI)

$\frac{dx}{dt} = \frac{1}{2}ke^{\frac{1}{2}t}$ --(VII)

$\frac{d^2x}{dt^2} = \frac{1}{4}ke^{\frac{1}{2}t}$ --(VIII)

from (VI), (VII) and (VIII) in (iv)

$\frac{1}{4}ke^{\frac{1}{2}t} + 4\times\frac{1}{2}ke^{\frac{1}{2}t} + 13\cdot ke^{\frac{1}{2}t} = 61e^{\frac{1}{2}t} \Rightarrow \frac{61}{4}k = 61 \Rightarrow k = 4$ ✓

$\therefore$ Particular integral is: $x = 4e^{\frac{1}{2}t}$ ---(IX)

$\therefore$ General Solution = complementary function + Particular integral.

or $x = e^{-2t}(A\cos 3t + B\sin 3t) + 4e^{\frac{1}{2}t}$

$\Rightarrow t^3 y = e^{-2t}(A\cos 3t + B\sin 3t) + 4e^{\frac{1}{2}t}$ [$x = t^3y$ from (i)]

$y = t^{-3}e^{-2t}(A\cos 3t + B\sin 3t) + 4t^{-3}e^{\frac{1}{2}t}$ ✓

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Example 17 (a) Show an approximate integrating factor for:

$$(x^2+1)\frac{dy}{dx} + y\sqrt{x^2+1} = x^2 - x\sqrt{x^2+1} \quad \text{is } x+\sqrt{x^2+1} \quad \text{-- [4]}$$

(b) Hence find the solution of the differential equation:

$$(x^2+1)\frac{dy}{dx} + y\sqrt{x^2+1} = x^2 - x\sqrt{x^2+1}, \text{ for which } y = \ln 2,$$

when $x=0$; give your answer in the form $y=f(x)$. -- [7]

[S-20/23/Q7]

Solution: Given $(x^2+1)\frac{dy}{dx} + y\sqrt{x^2+1} = x^2 - x\sqrt{x^2+1}$

(a)

$$\Rightarrow \frac{dy}{dx} + \frac{1}{\sqrt{x^2+1}} = \frac{x}{(x^2+1)}(x-\sqrt{x^2+1} \quad \text{---(i)}$$

$$I.F = e^{\int \frac{1}{\sqrt{x^2+1}}dx} = e^{\sinh^{-1}x} = e^{\ln(x+\sqrt{x^2+1})}$$

$$\Rightarrow I.F = (x+\sqrt{x^2+1}) \checkmark$$

$$\left\{ \int \frac{1}{\sqrt{x^2+a^2}}dx = \sinh^{-1}\frac{x}{a} = \ln(x+\sqrt{x^2+1}) \right.$$

(b) Required solution of diff. equation (i) is:

$$y.I.F = \int \frac{x}{\sqrt{x^2+1}}(x-\sqrt{x^2+1}) \times (x+\sqrt{x^2+1})\,dx$$

$$\Rightarrow y.(x+\sqrt{x^2+1}) = \int \frac{x(x^2-(x^2+1))}{\sqrt{x^2+1}}\,dx$$

$$= \int \frac{-x}{(x^2+1)}\,dx \qquad \left\{ \begin{array}{l} x^2+1=u \\ \text{diff } 2x\,dx = du \end{array} \right.$$

$$= -\frac{1}{2}\int \frac{2x}{(x^2+1)}\,dx$$

$$\Rightarrow y.(x+\sqrt{x^2+1}) = -\frac{1}{2}\ln(x^2+1) + C \quad \text{---(ii)}$$

Now given $y=\ln 2$ for $x=0$; put in (ii)

$$\Rightarrow \ln 2 = 0 + C \Rightarrow C = \ln 2 \checkmark$$

$\therefore$ Required soln. is:

from (ii) $\rightarrow$ $y.(x+\sqrt{x^2+1}) = -\frac{1}{2}\ln(x^2+1) + \ln 2$

or
$$y = \frac{\ln 2 - \ln\sqrt{x^2+1}}{(x+\sqrt{x^2+1})} \times \frac{(x-\sqrt{x^2+1})}{(x-\sqrt{x^2+1})}$$

$$y = -(x-\sqrt{x^2+1})(\ln 2 - \ln\sqrt{x^2+1})$$

$$\Rightarrow y = (x-\sqrt{x^2+1}).\ln\left(\frac{1}{2}.\sqrt{x^2+1}\right) \checkmark$$

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Example 18(a) Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=x+e^{2x}$$

Solution: (a) Given $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y= x+e^{2x}$ ---(i)

The auxiliary equation is: $\lambda^2+6\lambda+9=0 \Rightarrow (\lambda+3)^2=0$

repeated root; $\lambda=-3$

$\therefore$ Complementary function is:

$$y=(Ax+B)e^{-3x} \text{ ----(ii)}$$

Let the Particular integral is:

$$y= ae^{2x}+bx+c \text{ ---(iii)}$$

diff. w.r.t x; $\frac{dy}{dx}=2ae^{2x}+b$ ---(iv)

diff. again: $\frac{d^2y}{dx^2}=4ae^{2x}$ ---(V)

from (iii), (iv) and (V) in (i) $\rightarrow$

$$4ae^{2x}+6(2ae^{2x}+b)+9(ae^{2x}+bx+c)=x+e^{2x}$$

$$\Rightarrow (4a+12a+9a)e^{2x}+9bx+(6b+9c)=x+e^{2x}$$

Comparing the coeff of e^{2x}; $25a=1 \Rightarrow a=\frac{1}{25}$ ✓

comparing the coeff of x : $9b=1 \Rightarrow b=\frac{1}{9}$ ✓

Comparing the constants; $6b+9c=0 \Rightarrow 6\times\frac{1}{9}+9c=0$

$\Rightarrow c=-\frac{2}{27}$ ✓

$\therefore$ from (iii) the particular integral $=\frac{1}{25}e^{2x}+\frac{1}{9}x-\frac{2}{27}$ --(Vi)

$\therefore$ General Solution = Complementary function + Particular Integral

or $\quad y=(Ax+B)e^{-3x}+\frac{1}{25}e^{2x}+\frac{1}{9}x-\frac{2}{27}$ ✓

Example 19. Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + 9y = 10xe^x.$$

Solution: Given $\frac{d^2y}{dx^2} + 9y = 10xe^x$ ---(i)

The auxiliary equation is: $\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i$ ---(ii)

$\therefore$ The complementary function is: $y = (A\cos 3x + B\sin 3x)$

($\because$ Case(IV)/Page 6)

Consider the particular integral:

$$y = (ax+b)e^x \quad \text{---(iii)}$$

Diff. w.r.t $x \rightarrow \frac{dy}{dx} = (ax+b)e^x + ae^x = (ax+a+b)e^x$ ---(iv)

Diff again $\longrightarrow \frac{d^2y}{dx^2} = (ax+a+b)e^x + ae^x$ ---(v)

from (iii), (iv) and (v) in (i) $\Rightarrow (ax+a+b)e^x + ae^x + 9((ax+b)e^x) = 10xe^x$

$\Rightarrow 10a\cdot xe^x + (2a+10b)e^x = 10xe^x$ ---(vi)

Comparing the coefficient of $xe^x \rightarrow 10a = 10 \Rightarrow a = 1$

Comparing the coefficient of $e^x \longrightarrow 2a + 10b = 0 \Rightarrow b = -\frac{1}{5}$ ($\because a = 1$)

$\therefore$ from (iii) the particular integral is:

$$y = \left(x - \frac{1}{5}\right)e^x \quad \text{---(vii)}$$

$\therefore$ Required general soln is:

y = complementary function + particular integral.

$\Rightarrow y = (A\cos 3x + B\sin 3x) + \left(x - \frac{1}{5}\right)e^x$ [from (ii) and (vii)]

Note: Given $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$; with auxiliary equation:

(For Particular Integral) $a\lambda^2 + b\lambda + c = 0$ ---(1)

Form of $f(x)$	Form of Particular Integral.
(i) Polynomial of order n (Example $2x^2+5x+6$)	Polynomial of degree n. $\rightarrow (\alpha x^2 + \beta x + \gamma)$
(ii) Trigonometric function $P\cos\beta x + Q\sin(\beta x)$	(i) $m\cos(\beta x) + n\sin(\beta x)$ if (βi) is not a root of (1) (ii) $x(m\cos\beta x + n\sin\beta x)$ if (βi) is a root of (1)
(iii) Exponential $e^{\alpha x}$	(i) $ae^{\alpha x}$ if 'α' is not a root of (1) (ii) $axe^{\alpha x}$ if α is non-repeating root of (1) (iii) $ax^2e^{\alpha x}$ if α is a repeating root of (1)

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Example 20. Find the particular solution to the linear differential equation:

$$\frac{d^2y}{dx^2} + y = 10\cos x$$

given $y = 3$ and $\frac{dy}{dx} = 2$ for $x = 0$.

Solution: Given $\frac{d^2y}{dx^2} + y = 10\cos x$ ---(i)

The auxiliary equation is: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

$\therefore$ Complementary function is: $y = (A\cos x + B\sin x)$ ---(ii)

Now let the Particular function is: $y = x(a\cos x + b\sin x)$ ---(iii)

diff (iii) $\frac{dy}{dx} = (a\cos x + b\sin x) + x(b\cos x - a\sin x)$ ---(iv)

diff again $\frac{d^2y}{dx^2} = (b\cos x - a\sin x) + (b\cos x - a\sin x) + x(-b\sin x - a\cos x)$ ---(v)

{ here multiple of angle is (1) same $\lambda = \pm(1)i$ $\therefore$ see Page 17 → (II)→(ii) case.

from (iii), (iv) and (v) put in (i)

$2b\cos x - 2a\sin x + x(-b\sin x - a\cos x) + x(a\cos x + b\sin x) = 10\cos x$

Comparing the coeff of $\cos x \rightarrow 2b = 10 \Rightarrow b = 5$ ✓

Comparing the coeff of $\sin x \rightarrow 2a = 0 \Rightarrow a = 0$ ✓

$\therefore$ from (ii) Particular integral is: $y = x(0 + 5\sin x)$

or $y = 5x\sin x$ ---(vi)

$\therefore$ General solution = complementary function + Particular integral

$\therefore$ $y = (A\cos B + B\sin x) + 5x\sin x$ ---(vii)

diff $\frac{dy}{dx} = -A\sin x + B\cos x + 5\sin x + 5x\cos x$ ---(viii)

Now $y = 3$ for $x = 0$ in (vii) $\Rightarrow 3 = A \Rightarrow A = 3$ ✓

and $\frac{dy}{dx} = 2$ for $x = 0$ in (viii) $\Rightarrow 2 = 0 + B + 0 \Rightarrow B = 2$ ✓

$\therefore$ Required Particular Solution is:

$$y = 3\cos x + 2\sin x + 5x\sin x$$ ✓

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Example 21 (a) Show that an appropriate integrating factor for,

$\sqrt{x^2-1}\,\frac{dy}{dx} + y = x^2 - x\sqrt{x^2-1}$ is $x+\sqrt{x^2-1}$ ---(4)

(b) Hence find the solution of the differential equation,

$\sqrt{x^2-1}\,\frac{dy}{dx} + y = x^2 - x\sqrt{x^2-1}$ for which

$y = 1$ when $x = \frac{5}{4}$, Give your answer in the form $y = f(x)$ ---[7]

[W-21/21/Q7]

Solution: (a) $\frac{dy}{dx} + \frac{1}{\sqrt{x^2-1}}\,y = \frac{x^2 - x\sqrt{x^2-1}}{\sqrt{x^2-1}}$ ----(i)

$I.F = e^{\int \frac{1}{\sqrt{x^2-1}}dx}$

$= e^{\cosh^{-1}x} = e^{\ln(x+\sqrt{x^2-1})} = \underline{x+\sqrt{x^2-1}}$ ✓

(b) Solution of diff equation (i)

$y.(I.F) = \int \frac{(x^2 - x\sqrt{x^2-1})}{\sqrt{x^2-1}} \times (x+\sqrt{x^2-1})\,dx$

$\Rightarrow y(x+\sqrt{x^2-1}) = \int \frac{x}{\sqrt{x^2-1}}dx$

Put $x^2-1 = u$
$2x\,dx = du$
$x\,dx = \frac{1}{2}du$

$= \frac{1}{2}\int u^{-\frac{1}{2}}du$

$= \frac{1}{2} \times \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = u^{1/2}$

$\therefore\ y(x+\sqrt{x^2-1}) = \sqrt{x^2-1} + c$ ---(ii)

Now $y=1$ for $x = \frac{5}{4}$ in (ii)

$\Rightarrow\ 2 = \frac{3}{4} + c \Rightarrow c = \frac{5}{4}$

from (ii) $y = \frac{\sqrt{x^2-1} + 5/4}{x+\sqrt{x^2-1}}$ ✓

Example 22: Find the solution of the differential equation:

$$\frac{dy}{dx} + \frac{4x^3y}{x^4+5} = 6x$$

---[7]

for which $y=1$ when $x=1$, Give your answer in the form $y=f(x)$.

[W-21/22/Q2]

Solution:

$$\frac{dy}{dx} + \frac{4x^3\,y}{x^4+5} = 6x \quad \text{---(i)}$$

$$I.F = e^{\int \frac{4x^3}{x^4+5}dx} = e^{\ln(x^4+5)} = (x^4+5)$$

$\therefore$ Solution of (i) is

$$y\cdot(I.F) = \int 6x\cdot(I.F)\,dx$$

$$\Rightarrow y\times(x^4+5) = \int 6x(x^4+5)\,dx$$

$$= \int (6x^5+30x)\,dx$$

$$\Rightarrow y\cdot(x^4+5) = x^6+15x^2+C \quad \text{----(ii)}$$

Now $y=1$ for $x=1$

$$\Rightarrow 6 = 16+C \Rightarrow C=-10$$

from (ii)

$$y\cdot(x^4+5) = x^4+15x^2-10$$

$$\therefore \quad y = \frac{x^4+15x^2-10}{(x^4+5)}$$

Example 23: Find the particular solution of the differential equation:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 4\cos x \text{ ; given that; } x=0,\ y=-4 \text{ and } \frac{dy}{dx}=3$$

$\boxed{W-21|21|Q5}$ --- [11]

Solution: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 4\cos x$ ---(i)

The auxiliary equation is: $\lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda-1)^2 = 0 \Rightarrow \lambda = 1$

$\therefore$ complementry solution is $y = e^x(ax+b)$ --(ii) (one real root) $(y = e^{\lambda x}(ax+b))$

Now consider particular integral:

$y = p\sin x + q\cos x$ ---(iii)

diff. w.r.t. x ; $\frac{dy}{dx} = p\cos x - q\sin x$ --(iv)

Again diff. $\frac{d^2y}{dx^2} = -p\sin x - q\cos x$ ---(v)

from (iii) and (iv) and (v) put in (i)

$\Rightarrow (-p\sin x - q\cos x) - 2(p\cos x - q\sin x) + (p\sin x + q\cos x) = 4\cos x$

$\Rightarrow (\cancel{-p} + 2q + \cancel{p})\sin x + (\cancel{-q} - 2p + \cancel{q})\cos x = 4$

$\Rightarrow 2q = 0$ and $-2p = 4 \Rightarrow q = 0 ; p = -2$ ✓

from (iii) particular integral is $y = -2\sin x$ ---(vi)

$\therefore$ General Solution of (1) is

$y = e^x(ax+b) - 2\sin x$ -(vii) {from (ii) and (vi)}

diff. w.r.t x

$\frac{dy}{dx} = ae^x + e^x(ax+b) - 2\cos x$ --(viii)

Now put $x=0$, $y=-4$ and $\frac{dy}{dx} = 3$

from (vii); $-4 = b \Rightarrow b = -4$ ✓

from (viii) $3 = a - 4 - 2 \Rightarrow a = 9$ ✓

from (vii), the required particular solution is:

$$y = e^x(9x-4) - 2\sin x$$ ✓

Example 24: It is given that; $y = x^2 w$ and

$$x^2 \frac{d^2w}{dx^2} + 4x(x+1)\frac{dw}{dx} + (5x^2+8x+2)w = 5x^2+4x+2$$

(a) Show that: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 5x^2+4x+2$ --- [4]

(b) Find the general solution for w in terms of x. --- [7]

[W-21/22/Q7]

Solution: (a) Given $y = x^2 w$ ----------- (i)

Diff. w.r.t $x \rightarrow \frac{dy}{dx} = x^2\frac{dw}{dx} + 2xw$ --- (ii)

Diff (ii) $\rightarrow \frac{d^2y}{dx^2} = x^2\frac{d^2w}{dx^2} + 2x\frac{dw}{dx} + 2x\frac{dw}{dx} + 2w$

$\Rightarrow \frac{d^2y}{dx^2} = x^2\frac{d^2w}{dx^2} + 4x\frac{dw}{dx} + 2w$ --- (iii)

$\downarrow$ from (i), (ii) and (iii)

Now consider, $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = x^2\frac{d^2w}{dx^2} + 4x\frac{dw}{dx} + 2w + 4x^2\frac{dw}{dx} + 8xw + 5x^2w$

$\Rightarrow x^2\frac{d^2w}{dx^2} + (4x^2+4x)\frac{dw}{dx} + (5x^2+8x+2)w = 5x^2+4x+2$ ✓

(b) To solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 5x^2+4x+2$ --- (iv)

The auxiliary equation is: $\lambda^2+4\lambda+5=0$; $b^2-4ac=-4$

$$\lambda = \frac{-4 \pm \sqrt{-4}}{2} = (-2 \pm i)$$

$\therefore$ Complementary function is:

$y = e^{-2x}(a\cos x + b\sin x)$ --- (V) $\left\{ \text{for } \alpha \pm i\beta \quad y = e^{\alpha x}(a\cos\beta x + b\sin\beta x) \right\}$

Now let the particular integral is $y = px^2+qx+r$ --- (VI)

$\frac{dy}{dx} = 2px+q$ and $\frac{d^2y}{dx^2} = 2p$

from (iv) $\rightarrow 2p + 4(2px+q) + 5(px^2+qx+r) = 5x^2+4x+2$

$\Rightarrow 5p.x^2 + (8p+5q)x + (2p+4q+5r) = 5x^2+4x+2$

$\Rightarrow 5p=5,\ 8p+5q=4;\ 2p+4q+5r=2 \} \Rightarrow p=1,\ q=-\frac{4}{5},\ r=\frac{16}{25}$

Hence, the general solution of (iv) is

$y = e^{-2x}(a\cos x + b\sin x) + x^2 - \frac{4}{5}x + \frac{16}{25}$ {from (V) and (VI) in (iv)

or $x^2 w =$ ---- --- + ----

$\therefore\ w = (xe^{x})^{-2}(a\cos x + b\sin x) + 1 - \left(\frac{5}{4}x\right)^{-1} + \left(\frac{5}{4}x\right)^{-2}$ ✓