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FP-2

Further Pure Math - 2

Differentiation
Notes and Revision.

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FP-2

Differentiation - Notes and Revision

§ Differentiation of Implicit functions:

Implicit functions are of the type:

(i) $x^2 + 2xy + y^2 + 7 = 0$

(ii) $x \sin y + y \cos x = 0$ etc.

and here the instruction is to find $\frac{dy}{dx}$?

Let

$y = \sin^3 x$

$y = u^3$, $u = \sin x$

$\frac{dy}{du} = 3u^2$; $\frac{du}{dx} = \cos x$

$\boxed{\text{Using Chain rule}}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 3u^2 \cdot \cos x$

$= 3 \sin^2 x \cdot \cos x$

But what if ?

$z = y^3$

$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$

$= 3y^2 \cdot \frac{dy}{dx}$

Example 1. The equation of a curve is:

$x^3 + 3xy^2 - y^3 = 5$

Show that $\frac{dy}{dx} = \frac{x^2+y^2}{y^2-2xy}$

[9709/M-20/32/Q7]

Solution: $x^3 + 3xy^2 - y^3 = 5$

Diff. w.r.t x.

$3x^2 + 3(1 \cdot y^2 + x \cdot 2y\frac{dy}{dx}) - 3y^2\frac{dy}{dx} = 0$

Note: we don't write $\frac{dy}{dx} = ?$

$\Rightarrow 3x^2 + 3y^2 + 6xy\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$

$-3\frac{dy}{dx}(y^2 - 2xy) = -3(x^2 + y^2)$

$\therefore \frac{dy}{dx} = \frac{+3(x^2+y^2)}{+3(y^2-2xy)}$

$\therefore \frac{dy}{dx} = \frac{x^2+y^2}{(y^2-2xy)}$ ✓

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To find $\frac{d^2y}{dx^2}$ of implicit functions:

Example 2:

(i) A curve has equation $x^2 - 6xy + 25y^2 = 16$, Show that $\frac{dy}{dx} = 0$ at the point $(3,1)$ ---[4]

(ii) By finding $\frac{d^2y}{dx^2}$ at the point $(3,1)$, determine the nature of this turning point. ---[5]

S-15	11	Q6

Solution: $x^2 - 6xy + 25y^2 = 16$

Diff. w.r.t x

$$2x - 6\left[x\frac{dy}{dx} + y\times 1\right] + 25\cdot 2y\frac{dy}{dx} = 0$$

$$50\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 2x \Rightarrow \frac{dy}{dx} = \frac{2(3y-x)}{2(25-3x)}$$

$$\therefore \frac{dy}{dx} = \frac{(3y-x)}{(25-3x)} \quad ①$$

Now at the point $(3,1)$, $\left(\frac{dy}{dx}\right)_{(3,1)} = \frac{(3\times 1 - 3)}{(25-3\times 3)} = 0$ ✓

(ii) Diff. ① w.r.t x

$$\frac{d^2y}{dx^2} = \frac{(25-3x)\cdot\left(3\frac{dy}{dx} - 1\right) - (3y-x)(-3)}{(25-3x)^2}$$

$\therefore$ at $(3,1)$, $\left(\frac{d^2y}{dx^2}\right)_{(3,1)} = \frac{(25-9)(3\times 0 - 1) - 0}{(25-9)^2}$ $\quad \left[\left(\frac{dy}{dx}\right)_{(3,1)} = 0\right.$

$$= -\frac{1}{16} < 0 \Rightarrow \text{Max} ✓$$

$\therefore$ There is a Max at the point $(3,1)$ on curve.

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Example 3: The curve C has equation: $y^2+(xy+1)^2=5$

(a) Show that at the point (1,1) on C, $\frac{dy}{dx}=-\frac{2}{3}$ ---[3]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point (1,1). ---[5]

[W-20/22/Q5]

Solution: C: $y^2+(xy+1)^2=5$

(a) diff w.r.t. x

$$2y\frac{dy}{dx}+2(xy+1)\left(x\frac{dy}{dx}+y\times+0\right)$$

$$\Rightarrow y\frac{dy}{dx}+(xy+1)\left(x\frac{dy}{dx}+y\right)=0 \quad ---(i)$$

at the point (1,1) in (i)

$$\Rightarrow 1\times\frac{dy}{dx}+(1\times1+1)\left(1\times\frac{dy}{dx}+1\right)=0$$

$$\Rightarrow \frac{dy}{dx}+2\frac{dy}{dx}+2=0 \Rightarrow 3\frac{dy}{dx}=-2 \Rightarrow \left.\frac{dy}{dx}\right|_{(1,1)}=-2/3 \quad ---(2)$$

(b) Now to find $\frac{d^2y}{dx^2}$; diff (i) w.r.t. x

$$y.\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+(xy+1)\left(x\frac{d^2y}{dx^2}+\frac{dy}{dx}\times+\frac{dy}{dx}\right)+\left(x\frac{dy}{dx}+y\right)\left(x\frac{dy}{dx}+y\right)=0$$

$$\Rightarrow y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+(xy+1)\left[x\frac{d^2y}{dx^2}+2\frac{dy}{dx}\right]+\left(x.\frac{dy}{dx}+y\right)^2=0 \quad ---(3)$$

To find the value of $\frac{d^2y}{dx^2}$ at (1,1), Put $x=1, y=1, \frac{dy}{dx}=-\frac{2}{3}$ in (3)

$$1\times\frac{d^2y}{dx^2}+\left(-\frac{2}{3}\right)^2+(1\times1+1)\left[1.\frac{d^2y}{dx^2}+2\times\left(-\frac{2}{3}\right)\right]+\left(1\times-\frac{2}{3}+1\right)^2=0$$

$$\Rightarrow 3\frac{d^2y}{dx^2}-\frac{19}{9}=0$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{19}{27} \checkmark$$

§ The Maclaurin's expansion for the function $f(x)$ is given by:

$$f(x)=f(0)+\frac{f'(0)}{1!}.x+\frac{f''(0)}{2!}.x^2+\frac{f'''(0)}{3!}.x^3+----$$

Example 4: It is given that $y = 2^x$

(a) By differentiating $\ln y$ w.r.t x, Show that $\frac{dy}{dx} = 2^x \ln 2$ ---[3]

(b) Write down $\frac{d^2y}{dx^2}$ --[1]

(c) Hence find the first three terms in the Maclaurin's series for 2^x -- [3]

[S-20/21/Q2]

Solution: Given $y = 2^x$ ---- (1)

(a) $\Rightarrow \ln y = x \cdot \ln 2$

Diff. w.r.t $x \rightarrow \frac{1}{y}\frac{dy}{dx} = \ln 2 \times 1 \Rightarrow \frac{dy}{dx} = y \ln 2 \Rightarrow \frac{dy}{dx} = 2^x \cdot \ln 2$ --- (2)

$\left[\text{or } \frac{d}{dx} 2^x = \ln 2 \cdot 2^x\right]$

(b) Diff. (2) w.r.t $\frac{d^2y}{dx^2} = \ln 2 \cdot \frac{d}{dx} 2^x$

$= \ln 2 \times \ln 2 \cdot 2^x$

$\frac{d^2y}{dx^2} = (\ln 2) \cdot 2^x$ --- (3)

$y(0) = 2^0 = 1$ from (1)

$y'(0) = 2^0 \ln 2 = \ln 2$ } from (2)

$y''(0) = 2^0 \cdot (\ln 2)^2 = (\ln 2)^2$ from (3)

(c) Maclaurin's series:

$y = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!} \cdot x^2 + \cdots$

$y = 1 + (\ln 2)x + \frac{(\ln 2)^2}{2} \cdot x^2 + \cdots$ ✓

Example 5: A curve has equation $\cos y = x$ for $-\pi < x < \pi$

(i) Show that: $\frac{d^2y}{dx^2} = -\cot y \left(\frac{dy}{dx}\right)^2$ ---[4]

(ii) Find the exact value of $\frac{d^2y}{dx^2}$ at the point $\left(\frac{1}{3}, \frac{\pi}{3}\right)$ on C, --[2]

[S-19/11/Q1]

Solution: C: $\cos y = x$ ---- (1)

(i) Diff. w.r.t $x \rightarrow -\sin y \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = -(\sin y)^{-1}$ --- (2)

diff. (2) w.r.t x

$\left(\frac{d^2y}{dx^2}\right) = +(\sin y)^{-2} \cdot \cos y \frac{dy}{dx}$

$= -(\sin y)^{-1} \cdot \left(-\frac{\cos y}{\sin y}\right) \cdot \frac{dy}{dx}$

$\frac{d^2y}{dx^2} = -\cot y \left(\frac{dy}{dx}\right)^2$ --- (2)

$\left(\frac{dy}{dx}\right)_{\left(\frac{1}{3}, \frac{\pi}{3}\right)} = -\left(\sin \frac{\pi}{3}\right)^{-1} = -\frac{2}{\sqrt{3}}$ --- (3)

(ii) $\frac{d^2y}{dx^2}$ at $\left(\frac{1}{3}, \frac{\pi}{3}\right)$

from (2)

$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{1}{3}, \frac{\pi}{3}\right)} = -\cot \frac{\pi}{3} \times \left(-\frac{2}{\sqrt{3}}\right)^2$ [from (2) & (3)]

$= -\frac{1}{\sqrt{3}} \times \frac{4}{3} = \frac{-4}{3\sqrt{3}}$

$= -\frac{4\sqrt{3}}{9}$ ✓

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§ Differentiation of Parametric functions:

When x and y are both expressed as a function of third variable 't'. (called parameter)

$$y = f(t) \quad \text{and} \quad x = g(t)$$

Find $\frac{dy}{dt} = f'(t)$ and $\frac{dx}{dt} = g'(t)$

Then

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}} \checkmark$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} \quad \text{or} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{1}{\frac{dx}{dt}}}$$

Example 5: The curve is defin. parametrically by:

$x = 2\cos^3 t$ and $y = 2\sin^3 t$ for $0 < t < \frac{1}{2}\pi$

Show that $\frac{d^2y}{dx^2} = \frac{1}{6}\sec^4 t . \operatorname{cosec} t$. ---[4]

[W-15/11/Q1]

Solution: $y = 2\sin^3 t$

$\frac{dy}{dt} = 2 \times 3 \sin^2 t . \cos t = 6 \sin^2 t \cos t$ ---(i)

and $\frac{dx}{dt} = 2 \times 3 \cos^2 t \times (-\sin t) = -6\cos^2 t . \sin t$ ---(ii)

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{6\sin^2 t . \cos t}{-6\cos^2 t \sin t} = -\tan t$ ---(iii)

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}(-\tan t) = -\sec^2 t$ ---(iv) (diff (iii) w.r.t t)

Now $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$ $\quad \left[\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}\right]$

$= -\sec^2 t \times \frac{-1}{6\cos^2 t . \sin t}$ [from (ii) and (iv)]

$= \frac{1}{6}\sec^4 t . \operatorname{cosec} t$ ✓

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Example 6: The curve C has parametric equations;

$x = \frac{1}{2}t^2 - \ln t, \quad y = 2t+1, \quad$ for $\frac{1}{2} \leq t \leq 2$

Find $\frac{d^2y}{dx^2}$ in terms of t, simplifying your answer. ---[4]

[S-20 | 23 | Q5(b)]

Solution: $y = 2t+1$ and $x = \frac{1}{2}t^2 - \ln t$

$\frac{dy}{dt} = 2$ ----(i) & $\frac{dx}{dt} = t - \frac{1}{t} = \frac{t^2-1}{t}$ ----(ii)

$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2}{\frac{t^2-1}{t}} = \frac{2t}{t^2-1}$ ----(iii)

from (iii) $\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{2t}{t^2-1}\right) = \frac{(t^2-1)\times 2 - 2t \times 2t}{(t^2-1)^2} = \frac{2t^2-2-4t^2}{(t^2-1)^2}$

$= \frac{-2(1+t^2)}{(t^2-1)^2}$ ----(iv)

Now $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

$= -\frac{2(1+t^2)}{(t^2-1)^2} \times \frac{1}{\frac{(t^2-1)}{t}}$ $\left\{ \text{from (iv) \& (ii)} \quad \frac{dt}{dx} = \frac{1}{dx/dt} \right.$

$= \frac{2t(t^2+1)}{(t^2-1)^3}$ ✓

Example 7: The curve C is defined as: $x = 18t - t^2$ and $y = 8y^{3/2}$

Show that at all points of C; $\frac{d^2y}{dx^2} = \frac{3(9+t)}{2t^{1/2}(9-3)^3}$ (where $0 < t \leq 4$)

[W-18 | 12 | Q11(i)]

Solution: $x = 18t - t^2 \Rightarrow \frac{dx}{dt} = (18-2t)$ --(1)

$y = 8t^{3/2} \Rightarrow \frac{dy}{dt} = 12t^{1/2}$ --(2)

$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{12t^{1/2}}{(18-2t)} = \frac{6t^{1/2}}{(9-t)}$ --(3)

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{6t^{1/2}}{9-t}\right) = \frac{(9-t) \cdot 3t^{-\frac{1}{2}} - 6t^{\frac{1}{2}} \cdot (-1)}{(9-t)^2} = \frac{3t^{-\frac{1}{2}}[9-t+2t]}{(9-t)^2}$

$= \frac{3(9+t)}{t^{1/2}(9-t)^2}$ --(4)

from (1) & (4):

$\therefore \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{3(9+t)}{t^{1/2}(9-t)^2} \times \frac{1}{2(9-t)}$

$= \left[3(9+t) / 2t^{1/2} \cdot (9-t)^3\right]$ ✓

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§ Differentiation of Hyperbolic functions:

$\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\sinh x = \dfrac{e^x - e^{-x}}{2}$

$\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(i) $\dfrac{d}{dx}\cosh x = \dfrac{d}{dx}\left(\dfrac{e^x + e^{-x}}{2}\right) = \dfrac{1}{2}\left(e^x - e^{-x}\right) = \sinh x$ ✓

(ii) $\dfrac{d}{dx}\sinh x = \dfrac{d}{dx}\left(\dfrac{e^x - e^{-x}}{2}\right) = \dfrac{1}{2}\left(e^x + e^{-x}\right) = \cosh x$ ✓

(iii) $\dfrac{d}{dx}\tanh x$

$$= \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{\cosh x \cdot \frac{d}{dx}\sinh x - \sinh x \cdot \frac{d}{dx}\cosh x}{\cosh^2 x}$$

(Using quotient rule) $= \dfrac{\cosh x \cdot \cosh x - \sinh x \times \sinh x}{\cosh^2 x}$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \quad \left\{\because \cosh^2 x - \sinh^2 x = 1\right.$$

$$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x \checkmark$$

Formulae:

§ 1. $\dfrac{d}{dx}\cosh x = \sinh x$

2. $\dfrac{d}{dx}\sinh x = \cosh x$

3. $\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$

4. $\dfrac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \cdot \tanh x$

5. $\dfrac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \cdot \coth x$

6. $\dfrac{d}{dx}\coth x = -\operatorname{cosech}^2 x$

Derivative of Inverse function:

§ 1. $\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$

2. $\dfrac{d}{dx}\cos^{-1} x = \dfrac{-1}{\sqrt{1-x^2}}$

3. $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$

1. $\dfrac{d}{dx}\sinh^{-1} x = \dfrac{1}{\sqrt{1+x^2}}$

2. $\dfrac{d}{dx}\cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}}$

3. $\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2}$

(Proof on the next Page)

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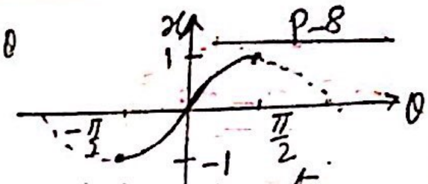
$$\begin{cases} x = \sin\theta \\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ -1 \le x \le 1 \end{cases}$$



one - one function

Example 8: Find the derivative of $\sin^{-1}x$:

$$\begin{cases} -1 \le x \le 1 \\ -\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2} \end{cases}$$

Solution: Let $y = \sin^{-1}x \Rightarrow x = \sin y$ --- (1)

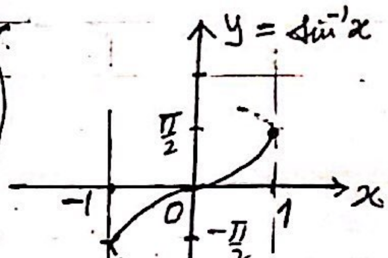
diff. w.r.t x

$$1 = \cos y \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\pm\sqrt{1-\sin^2 y}}$$

$$= \pm \frac{1}{\sqrt{1-x^2}} \text{ from (1)}$$

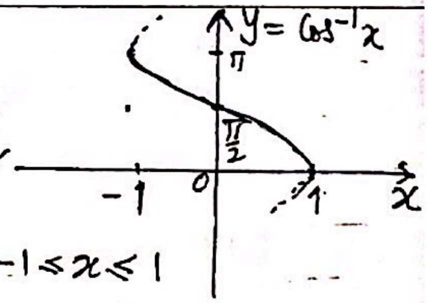
$$\therefore \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \checkmark$$



The graph of $\sin^{-1}x$ has a positive gradient.

$\frac{d}{dx}\sin^{-1}x$ is + (Increasing fn)

Example 9: Find the derivative of $\cos^{-1}x$.



Solution: $y = \cos^{-1}x \Rightarrow x = \cos y$ --- (1)

diff. w.r.t x.

$$1 = -\sin y \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y}$$

$$= \frac{-1}{\pm\sqrt{1-\cos^2 y}}$$

$$= \mp \frac{1}{\sqrt{1-x^2}} \text{ from (1)}$$

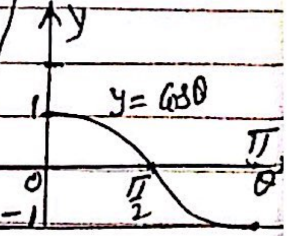
$$\therefore \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

$-1 \le x \le 1$

$0 \le \cos^{-1}x \le \pi$

The graph of $\cos^{-1}x$ has negative gradient (Decreasing)

$\therefore \frac{d}{dx}\cos^{-1}x$ is $-$ve



Example 10. To find the derivative of $\tan^{-1}x$.

Solution:

$y = \tan^{-1}x \Rightarrow x = \tan y$ --- (1)

diff. w.r.t x $\rightarrow$ $1 = \sec^2 y \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$$

$$\Rightarrow \frac{d}{dx}\tan^{-1}x = \frac{1}{(1+x^2)} \checkmark$$

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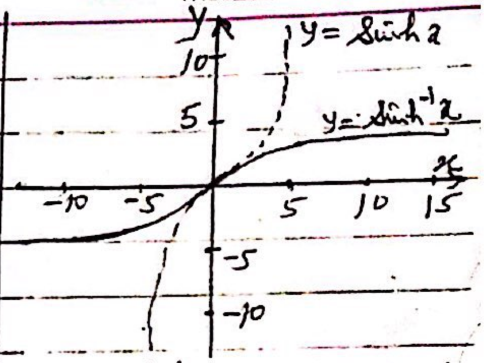
Example 11: To find $\frac{d}{dx}\sinh^{-1}x$.



$y = \sinh^{-1}x$ has a + grad.

Solution: let $y = \sinh^{-1}x \Rightarrow x = \sinh y$ --- (1)

diff w.r.t. x;

$$1 = \cosh y \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} = \pm\frac{1}{\sqrt{1+\sinh^2 y}}$$

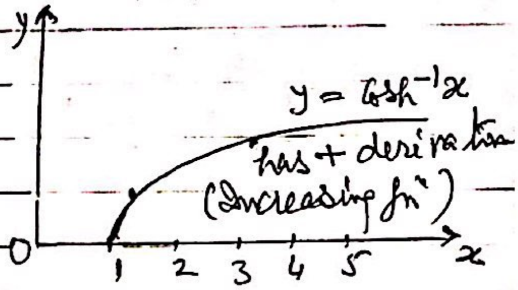
$\left\{ \because \cosh^2 y - \sinh^2 y = 1 \right.$
$\left. \cosh y = \pm\sqrt{1+\sinh^2 y} \right.$

$$\therefore \frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$$ ✓ {from (1) $\sinh y = x$

Example 12: Find the derivative of $\cosh^{-1}x$.

Solution: Let $y = \cosh^{-1}x \Rightarrow x = \cosh y$ --- (1)

diff. w.r.t x; $1 = \sinh y \frac{dy}{dx}$



$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y}$$

$$= \frac{\pm 1}{\sqrt{\cosh^2 y - 1}}$$ [$\sinh^2 y = \cosh^2 y - 1$

[$\frac{d}{dx}\cosh^{-1}x$ is +]

$$\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$$ [from (1) $\cosh y = x$]

Example 13: Find the derivative of $\tanh^{-1}x$.

Solution: Let $y = \tanh^{-1}x \Rightarrow x = \tanh y$ --- (1)

diff. w.r.t x → $1 = \operatorname{sech}^2 y \cdot \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1-\tanh^2 y}$$

$$\therefore \frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}$$ ✓ (from (1))

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14 (a) Starting from the definition of of tanh in terms of exponentials, prove that: $\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. ---[3]

(b) Given that $y = \tanh^{-1}\left(\frac{1-x}{2+x}\right)$, show that $(2x+1)\frac{dy}{dx}+1=0$ ---[4]

(C) Hence find the first three terms in the Maclaurins series for $\tanh^{-1}\left(\frac{1-x}{1+x}\right)$ in the form: $a\ln 3 + bx + cx^2$ where a, b, and c are constants to be determined. ---[5]

[SP-20/02/Q7]

Solution (a) Let $u = \tanh^{-1}x \Rightarrow x = \tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{e^{2u}-1}{e^{2u}+1}$

$\Rightarrow \frac{e^{2u}-1}{e^{2u}+1} = \frac{x}{1}$

Given $\frac{a}{b} = \frac{c}{d}$
using componendo & dividendo
$\Rightarrow \frac{a-b}{a+b} = \frac{c-d}{c+d}$

$\Rightarrow \frac{(e^{2u}-1)-(e^{2u}+1)}{(e^{2u}-1)+(e^{2u}+1)} = \frac{x-1}{x+1}$

$\Rightarrow \frac{-2}{2e^{2u}} = -\frac{(1-x)}{(1+x)}$

$\Rightarrow e^{2u}(1-x) = (1+x)$

$\Rightarrow e^{2u} = \frac{1+x}{1-x}$

$\Rightarrow 2u = \ln\left(\frac{1+x}{1-x}\right)$

$\Rightarrow \tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1-x}{1+x}\right)$ ✓

(b) $y = \tanh^{-1}\left(\frac{1-x}{2+x}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1-x}{2+x}}{1-\frac{1-x}{2+x}}\right)$

$y = \frac{1}{2}\ln\left(\frac{3}{2x+1}\right)$ ---(i)

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\times\left(\frac{2x+1}{3}\right)\cdot\frac{-6}{(2x+1)^2}$

$\frac{dy}{dx} = -\frac{1}{2x+1}$

$\Rightarrow (2x+1)\frac{dy}{dx} = -1$

$\Rightarrow (2x+1)\frac{dy}{dx}+1=0$ ✓ ---(ii)

(C) diff (ii)

$(2x+1)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ ---(iii)

from ① $y(0) = \frac{1}{2}\ln 3$. ✓

from ② $y'(0) = -1$ ✓

from ③ $(0+1)y''(0) + 2(-1) = 0$

$\Rightarrow y''(0) = 2$ ✓

Now Maclaurins theorem:

$y = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \cdots$

$\therefore y = \frac{1}{2}\ln 3 - x + x^2 + \cdots$ ✓

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15. The variables x and y are such that $\tanh y = \cos(x+\frac{\pi}{4})$ for $-\frac{1}{4}\pi < x \leq \frac{3\pi}{4}$.

(a) Show that:

$$\frac{dy}{dx} = -\operatorname{cosec}\left(x+\frac{\pi}{4}\right) \quad \text{--- [4]}$$

(b) Hence find the first three terms in the Maclaurin's series for $\tanh^{-1}(\cos(x+\frac{1}{4}\pi))$ in the form $\frac{1}{2}\ln a + bx + cx^2$, giving the exact values of the constants a, b and c. [S-20/23/Q6] --- [5]

Solution: $\tanh y = \cos(x+\frac{\pi}{4})$ ----(i)

(a) Diff. w.r.t x:

$$\operatorname{sech}^2 y \frac{dy}{dx} = -\sin\left(x+\frac{\pi}{4}\right)$$

$$\Rightarrow (1-\tanh^2 y)\frac{dy}{dx} = -\sin\left(x+\frac{\pi}{4}\right)$$

$$\Rightarrow \left(1-\cos^2\left(x+\frac{\pi}{4}\right)\right)\frac{dy}{dx} = -\sin\left(x+\frac{\pi}{4}\right)$$

$$\Rightarrow \sin^2\left(x+\frac{\pi}{4}\right)\frac{dy}{dx} = -\sin\left(x+\frac{\pi}{4}\right) \Rightarrow \frac{dy}{dx} = -\operatorname{cosec}\left(x+\frac{\pi}{4}\right) \checkmark \text{ --(ii)}$$

[$\operatorname{sech}^2 y = 1-\tanh^2 y = 1-\cos^2(x+\frac{\pi}{4})$ from (1) ; $\tanh y = \cos(x+\frac{\pi}{4})$]

(b) Diff (ii), $\frac{d^2y}{dx^2} = +\operatorname{cosec}\left(x+\frac{\pi}{4}\right)\cdot\cot\left(x+\frac{\pi}{4}\right)$ ---(iii)

from (1) $y = \tanh^{-1}\left(\cos(x+\frac{\pi}{4})\right) \Rightarrow y(0) = \tanh^{-1}\left(\cos\frac{\pi}{4}\right) = \tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)$ ✓

$$= \frac{1}{2}\ln\left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right) = \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = \frac{1}{2}\ln(3+2\sqrt{2})$$

and from (2) $y'(0) = -\operatorname{cosec}\frac{\pi}{4} = -\sqrt{2}$ ✓

from (3) $y''(0) = \operatorname{cosec}\left(0+\frac{\pi}{4}\right)\cdot\cot\left(0+\frac{\pi}{4}\right) = \sqrt{2}\times 1 = \sqrt{2}$ ✓

∴ Maclaurin's series:

$$y = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \cdots$$

$$\therefore y = \frac{1}{2}\ln(3+2\sqrt{2}) - x\sqrt{2} + \frac{1}{2}x^2\sqrt{2} + \cdots \checkmark$$

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16. It is given that $x = \sinh^{-1} t$, $y = \cos^{-1} t$; $-1 < t < 1$

(a) By diff. $\cos y$ w.r.t. t, show that: $\frac{dy}{dt} = -\frac{1}{\sqrt{1-t^2}}$ ---[4]

(b) Find $\frac{d^2y}{dx^2}$ in terms of t, simplify your answer. ---[5]

[W-20/21/Q5]

Solution: Given $y = \cos^{-1} t$

(a) $\Rightarrow \cos y = t$ ----(i)

diff. w.r.t t $\Rightarrow -\sin y \frac{dy}{dt} = 1 \Rightarrow \frac{dy}{dt} = \frac{-1}{\sin y}$

$\left\{ \begin{array}{l} -1<t<1 \\ 0 < \cos^{-1} t < \pi \\ \sin y > 0 \end{array} \right.$

$$\Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dt} = -\frac{1}{\sqrt{1-t^2}} \checkmark \quad \text{---(ii)}$$

(b) Given $x = \sinh^{-1} t$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1+t^2}} \quad \text{---(iii)}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = -\frac{1}{\sqrt{1-t^2}} \Big/ \frac{1}{\sqrt{1+t^2}} = -\sqrt{\frac{1+t^2}{1-t^2}} \quad \text{---(iv)}$$

from (iv)

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -\frac{\left[(1-t^2)^{1/2} \times \frac{1}{2}(1+t^2)^{-1/2} \times 2 - \sqrt{1+t^2} \times \frac{1}{2}(1-t^2)^{-1/2} \times (-2t)\right]}{(1-t^2)}$$

$$= -\frac{[t[(1-t^2) + (1+t^2)]]}{(1-t^2)(1-t^2)^{1/2}(1+t^2)^{1/2}}$$

$$= \frac{-2t}{(1-t^2)^{3/2}(1+t^2)^{1/2}} \quad \text{---(v)}$$

Now $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

$\left[\begin{array}{l} \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} \end{array} \right.$

$$= \frac{-2t}{(1-t^2)^{3/2} \cdot (1+t^2)^{1/2}} \times \sqrt{1+t^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-2t}{(1-t^2)^{3/2}} \checkmark$$

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17 The curve C has equation $y = \ln(\coth \frac{1}{2}x)$ for $x > 0$

Show that: $\frac{dy}{dx} = -\operatorname{cosech} x$ ---[3]

[W-20/21/Q8(c)]

Solution: $y = \ln(\coth \frac{1}{2}x)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\coth(\frac{1}{2}x)} \times \frac{1}{2} \times -\operatorname{cosech}^2 \frac{x}{2} = \frac{-1}{2 \cdot \frac{\cosh(\frac{1}{2}x)}{\sinh(\frac{1}{2}x)} \cdot \sinh^2 \frac{x}{2}}$$

$$\frac{dy}{dx} = \frac{-1}{2\sinh \frac{x}{2} \cdot \cosh \frac{x}{2}} = \frac{-1}{\sinh x} = -\operatorname{cosech} x$$

18, (a) It is given that $y = \operatorname{sech}^{-1}(x + \frac{1}{2})$

Express $\cosh y$ in terms of x and hence show that, $\sinh y \frac{dy}{dx} = \frac{-1}{(x+\frac{1}{2})^2}$ ---[3]

(b) Find the first three terms in the Maclaurin's series, for $\operatorname{sech}^{-1}(x + \frac{1}{2})$ in the form: $\ln a + bx + cx^2$, where a, b and c are constants to be determined. ---[7]

[S-21/21/Q7]

Solution: Given $y = \operatorname{sech}^{-1}(x + \frac{1}{2})$ ----(i)

(a) $\Rightarrow \operatorname{sech} y = (x + \frac{1}{2}) \Rightarrow \cosh y = (x + \frac{1}{2})^{-1}$ ---(ii)

Diff. w.r.t x, $\sinh y \cdot \frac{dy}{dx} = \frac{-1}{(x+\frac{1}{2})^2}$ ---(iii)

(b) Diff. (iii) w.r.t $x \Rightarrow \sinh y \frac{d^2y}{dx^2} + \cosh y \cdot \frac{dy}{dx} \cdot \frac{dy}{dx} = 2(x + \frac{1}{2})^{-3}$ ---(iv)

from (i) $y(0) = \operatorname{sech}^{-1}\frac{1}{2} = \cosh^{-1} 2 = \ln[2 + \sqrt{3}]$

from (iii) $y'(0) = \frac{1}{\sinh y} \times \frac{-1}{(0+\frac{1}{2})^2} = -\frac{4}{\sqrt{3}}$

$\cosh^{-1}x = \ln[x + \sqrt{x^2 - 1}]$

from (ii) $x = 0$, $\cosh^2 y = 4$

$\sinh y = \sqrt{\cosh^2 y - 1}$

$= \sqrt{4 - 1} = \sqrt{3}$

from (iv) $\sqrt{3} \cdot y''(0) + 2\left(\frac{-4}{\sqrt{3}}\right)^2 = 2 \times 8$

$\sqrt{3} \cdot y''(0) = 16 - \frac{32}{3} \Rightarrow y''(0) = \frac{16}{3\sqrt{3}}$

$\therefore$ Maclaurin's Series:

$$y = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \cdots$$

or $y = \ln(2 + \sqrt{3}) - \frac{4}{\sqrt{3}}x + \frac{8}{3\sqrt{3}}x^2 + \cdots$

19. The curve C has parametric equations:

$x = 2\cosh t$, $y = \frac{3}{2}t - \frac{1}{4}\sinh 2t$, for $0 \le t \le 1$

find $\frac{dx}{dt}$ and show that $\frac{dy}{dt} = 1 - \sinh^2 t$ --[3]

[S-21/21/Q8(a)]

Solution: $x = 2\cosh t \Rightarrow \frac{dx}{dt} = 2\cosh t$

and $y = \frac{3}{2}t - \frac{1}{4}\sinh 2t \Rightarrow \frac{dy}{dt} = \frac{3}{2} - \frac{1}{4} \times 2\cosh 2t = \frac{3}{2} - \frac{1}{2}\cosh 2t$

$= \frac{1}{2}(3 - \cosh 2t) = \frac{1}{2}[3 - (1 + 2\sinh^2 t)]$

$= \frac{1}{2}[2 - 2\sinh^2 t]$

$\therefore \frac{dy}{dt} = 1 - \sinh^2 t$

20. Find the Maclaurin's series for $\ln\cosh x$ upto and including in terms in x^4. ---[7]

[S-21/23/Q2]

Solution: Let $f(x) = \ln\cosh x$ ----(i)

diff. $f'(x) = \frac{1}{\cosh x} \times \sinh x$

$\therefore f'(x) = \tanh x$ ---(ii)

diff (ii) $\rightarrow f''(x) = \operatorname{sech}^2 x$ ---(iii)

Maclaurin's series:

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \cdots$

or

$f(x) = 0 + 0 + \frac{1}{2}x^2 - \frac{1}{12}x^4 + \cdots$

diff (iii) $\rightarrow f'''(x) = 2\operatorname{sech} x \cdot (-\operatorname{sech} x \tanh x)$

or $f'''(x) = -2\tanh x \cdot \operatorname{sech}^2 x$ --(iv)

diff (iv) using product rule

$f''''(x) = -2[\operatorname{sech}^2 x \cdot \operatorname{sech}^2 x + \tanh x \times (2\operatorname{sech} x \cdot -\operatorname{sech} x \tanh x)]$

$= -2[\operatorname{sech}^4 x - \tanh^2 x \cdot \operatorname{sech}^2 x]$ --(v)

Now from (i) $f(0) = \ln\cosh 0 = \ln 1 = 0$

from (ii) $f'(0) = \tanh 0 = 0$

from (iii) $f''(0) = \operatorname{sech}^2(0) = 1$

from (iv) $f'''(0) = 0$

from (v) $f''''(0) = -2[1 - 0] = -2$

21. Find the Maclaurin's series for $e^x \cdot \tan x$ from first principles upto and including the term in x^2. [W-21/21/Q1]---[5]

Solution: $y = e^x \cdot \tan x$ -------(i) $\Rightarrow y(0) = 0$ ---(i)'

Diff. $\frac{dy}{dx} = e^x \cdot \sec^2 x + e^x \tan x$

$y' = \frac{dy}{dx} = e^x(\tan x + \sec^2 x)$ ---(ii) $\Rightarrow y'(0) = 1$ ---(ii)'

Diff.

$\frac{d^2y}{dx^2} = e^x(\tan x + \sec^2 x) + e^x(\sec^2 x + 2\sec x \cdot \sec x \tan x)$

or $y'' = e^x(\tan x + 2\sec^2 x + 2\sec^2 \tan x)$ ---(iii) $\Rightarrow y''(0) = 2$ ---(iii)'

Maclaurin's theorem: $y = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \cdots$

$\therefore y = 0 + \frac{1}{1!}x + \frac{1}{2!} \times 2 \times x^2$ {from (i)', (ii)' & (iii)'}

$y = x + x^2 + \cdots$ ✓

22 It is given that $y = \sinh(x^2) + \cosh(x^2)$

Use standard results from the list of formulae (MF19) to find the Maclaurin's series for y in terms of x upto and including the terms of x^4. [W-21/22/Q1]---[2]

Solutions: $y = \sinh(x^2) + \cosh(x^2)$ -----(i) $\Rightarrow y(0) = 1$ ✓

$y' = 2x(\cosh x^2 + \sinh x^2)$ ---(ii) $\rightarrow y'(0) = 0$ ✓

$y'' = 2x \cdot 2x(\sinh x^2 + \cosh x^2) + 2(\sinh x^2 + \cosh x^2)$

$\Rightarrow y'' = (\sinh x^2 + \cosh x^2)(2 + 4x^2)$ ---(iii) $\rightarrow y''(0) = 2$ ✓

$y''' = 2x(\sinh x^2 + \cosh x^2)(2 + 4x^2) + (\sinh x^2 + \cosh x^2)(8x)$

$\Rightarrow y''' = (\sinh x^2 + \cosh x^2)(12x + 8x^3)$ ---(iv) $\rightarrow y'''(0) = 0$ ✓

$y'''' = (\sinh x^2 + \cosh x^2)(12 + 24x^2) + 2x(\sinh x^2 + \cosh x^2)(12 + 24x^2)$

$\Rightarrow y'''' = (\sinh x^2 + \cosh x^2)(12 + 24x + 24x^2 + 48x^3)$ ---(v) $\rightarrow y''''(0) = 12$ ✓

Now Maclaurin's Theorem: $y = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y''''(0)}{4!}x^4 + \cdots$

$y = 1 + 0 + \frac{2}{2}x^2 + 0 + \frac{12}{24}x^4 + \cdots$

$y = 1 + x^2 + \frac{1}{2}x^4 + \cdots$ ✓

Q3. The curve C has equation; $xy^3 - 4x^3y = 3$

(a) Show that, at the point $(-1,1)$ on C, $\frac{dy}{dx} = 11$ --- [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $(-1,1)$. --- [5]

W-21/21/Q3

Solution (a) $xy^3 - 4x^3y = 3$

Diff. w.r.t x; $x \cdot 3y^2\frac{dy}{dx} + y^3 - \left(4x^3\frac{dy}{dx} + y \cdot 12x^2\right) = 0$

$\Rightarrow \frac{dy}{dx}(3xy^2 - 4x^3) = -y^3 + 12yx^2$

$\Rightarrow \frac{dy}{dx} = \frac{-(-y^3 + 12yx^2)}{(3xy^2 - 4x^3)}$ ---- ①

at $(-1,1)$, $\frac{dy}{dx} = \frac{(-1+12)}{-3+4} = \frac{11}{1} = 11$

(b) from ① $(3xy^2 - 4x^3) \cdot y' = (12yx^2 - y^3)$

Diff. w.r.t:

$$(3xy^2 - 4x^3)\frac{d^2y}{dx^2} + \frac{dy}{dx}\left[3x \cdot 2y\frac{dy}{dx} + 3y^2 - 12x^2\right] = 24xy + 12x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx}$$

Now at $(-1,1)$, $\frac{dy}{dx} = 11$

$\Rightarrow (-3+4)\frac{d^2y}{dx^2} + 11[-3 \times 2 \times 11 + 3 - 12] = -24 + 12 \times 11 - 3 \times 11$

$\Rightarrow \frac{d^2y}{dx^2} - 825 = 75$

$\Rightarrow \frac{d^2y}{dx^2} = 900$