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FP2

Further Pure Math 2

Hyperbolic Fuctions Notes and Revision

SP-20 | S-20 | W-20

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FP2

Hyperbolic functions - Notes and Revision.

- **Exponention forms of Hyperbolic functions:**

(i) $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$; $x \in R$ | (ii) $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$; $x \in R$

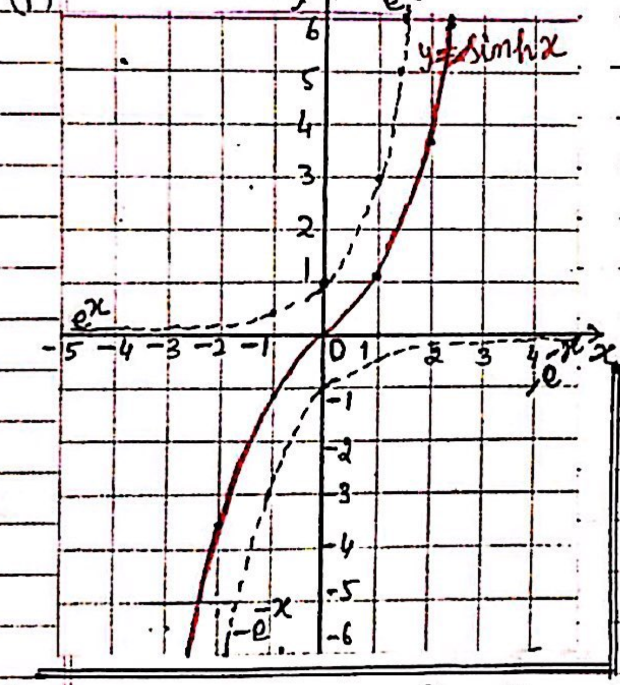
$\therefore e^{x} = \cosh x + \sinh x$ [from (i) and (ii)]

(iii) $\tanh x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$; $x \in R$ | (iv) $\coth x = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$; $x \neq 0$, $x \in R$

(v) $\operatorname{cosech} x = \dfrac{2}{e^{x} - e^{-x}}$, $x \neq 0, x \in R$ | (vi) $\operatorname{sech} x = \dfrac{2}{e^{x} + e^{-x}}$; $x \in R$

- Graph of : $y = \sinh x = \dfrac{e^{x} - e^{-x}}{2}$ (or $y = \dfrac{e^{x} + (-e^{-x})}{2}$) {average of e^{x} & $-e^{-x}$}

(i)



$y = \dfrac{e^{x} - e^{-x}}{2} = \sinh x$

Domain : $x \in R$

Range : $y \in R$

One-one function.

It is an odd function

$f(-x) = -f(x)$

(ii) $y = \cosh x = \dfrac{e^{x} + e^{-x}}{2}$

Domain : $x \in R$

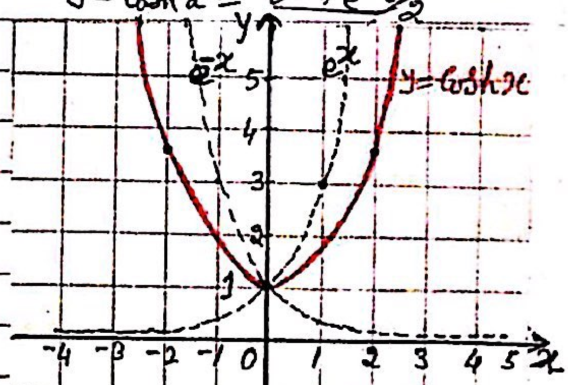
Range : $y = f(x) \geq 1$

It is an even function : $f(-x) = f(x)$

and not a one-one function

[For function be one-one,
Restrict, Domain : $x \geq 0$]

$y = \cosh x = \dfrac{e^{x} + e^{-x}}{2}$



- Graphs of hyperbolic functions:

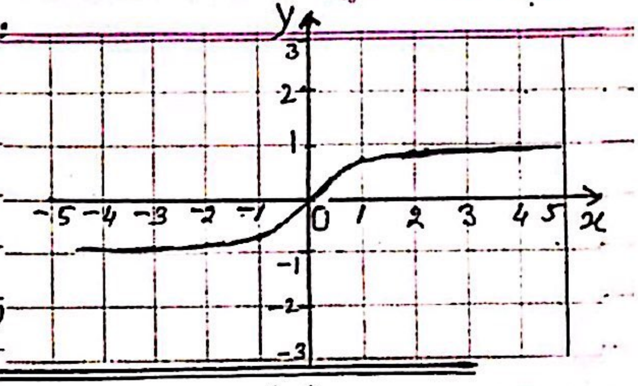
(iii) $y = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = f(x)$

Domain: $x \in R$

Range: $-1 < f(x) < 1$

It is one-one function.

Odd function: $f(-x) = -f(x)$



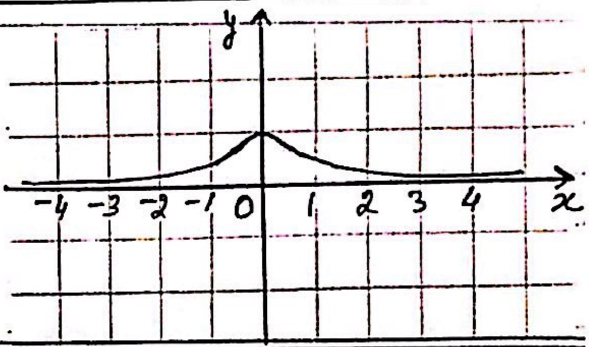
(iv) $y = f(x) = \operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

Domain: $x \in R$

Range: $0 < f(x) \leq 1$

Even function: $f(-x) = f(x)$.

It is not a one-one function.

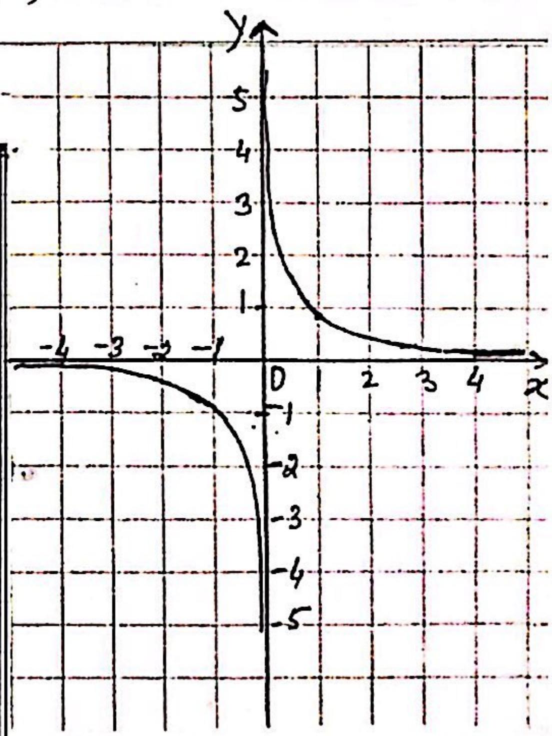


(v) $y = f(x) = \operatorname{cosech} x = \dfrac{2}{e^x - e^{-x}} = f(x)$

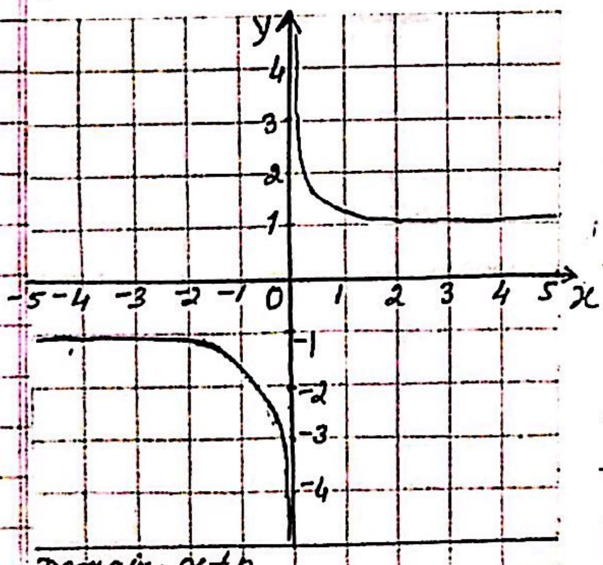
Domain: $x \neq 0$, $x \in R$

Range: $f(x) \neq 0$, $f(x) \in R$

It is odd function.



(vi) $y = f(x) = \coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$



Domain: $x \neq 0$

Range: $f(x) > 1$; $f(x) < -1$

It is an odd function.

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Example 1. Express the following hyperbolic functions in terms of (i) exponential functions (ii) Find their values to 3 decimal places.
(a) $\sinh 4$ (b) $\tanh 5$

Solution (a) (i) $\sinh 4 = \frac{e^4 - e^{-4}}{2}$ ✓ $\left(\because \sinh x = \frac{e^x - e^{-x}}{2}\right)$

or $= \frac{e^8 - 1}{2e^4}$ ✓

(ii) $\sinh 4 = \frac{e^8 - 1}{2e^4} = \frac{2980.957 - 1}{109.196} = 27.28998 = 27.290$ ✓ (3 d.p)

(b) (i) $\tanh 5 = \frac{e^5 - e^{-5}}{e^5 + e^{-5}} = \frac{e^{10} - 1}{e^{10} + 1}$ ✓

(ii) $\tanh 5 = \frac{e^{10} - 1}{e^{10} + 1} = 0.999909 = 1.000$ ✓ (3 d.p)

Example 2: Find the exact value of: $\operatorname{sech}(\ln 2)$.

Solution: $\operatorname{sech}(\ln 2) = \frac{2}{e^{\ln 2} + e^{-\ln 2}}$ $\left[\operatorname{sech} x = \frac{2}{e^x + e^{-x}}\right]$

$= \frac{2}{2 + \frac{1}{2}} = \frac{4}{5}$ ✓ $\left\{ e^{\ln x} = x \text{ and } e^{-\ln x} = e^{\ln(\frac{1}{x})} = \frac{1}{x} \right.$

Example 3: Find the value of x, to 3 decimal places: (i) $\cosh x = 1$
(ii) $\tanh x = \frac{1}{2}$

Solution (i) $\cosh x = 1 \Rightarrow x = \cosh^{-1} 1 = 0 \Rightarrow x = 0$ ✓

(ii) $\tanh x = \frac{1}{2} \Rightarrow x = \tanh^{-1} \frac{1}{2} = 0.549306 = 0.549$ ✓ (3 d.p)
(using Calculator).

Example 4 Solve $2\sinh x + 10\cosh x = 10$

Solution: $2\sinh x + 10\cosh x = 10$

$\Rightarrow 2\left(\frac{e^x - e^{-x}}{2}\right) + 10\left(\frac{e^x + e^{-x}}{2}\right) = 10$

$\Rightarrow 6e^x + 4e^{-x} - 10 = 0$

$\Rightarrow 6e^x + \frac{4}{e^x} - 10 = 0$

$6e^{2x} - 10e^x + 4 = 0$

$2(3e^{2x} - 2)(e^x - 1) = 0$

$e^x = 2/3$; $e^x = 1$

$\Rightarrow x = \ln 2/3$; $x = \ln 1$ ✓

$\Rightarrow x = -0.406465$ ✓ ; $x = 0$ ✓

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Hyperbolic identities:

1. $\cosh^2 x - \sinh^2 x = 1$ or $\cosh^2 x = 1 + \sinh^2 x$ or $\sinh^2 x = \cosh^2 x - 1$

2. $1 - \tanh^2 x = \text{sech}^2 x$ or $\text{sech}^2 x + \tanh^2 x = 1$ or $\tanh^2 x = 1 - \text{sech}^2 x$

3. $\coth^2 x - 1 = \text{cosech}^2 x$ or $\coth^2 x - \text{cosech}^2 x = 1$ or $\coth^2 x = 1 + \text{cosech}^2 x$

4. (i) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
   (ii) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

5. (i) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
   (ii) $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

6. $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \cdot \tanh y}$ ‖ $\tanh(x-y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x \cdot \tanh y}$

7. $\sinh 2x = 2 \sinh x \cosh x = \dfrac{2 \tanh x}{1 - \tanh^2 x}$

8. $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \dfrac{1 + \tanh^2 x}{1 - \tanh^2 x}$

9. $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

10. $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$

11. $\cosh 3x = 4 \cosh^3 x - 3 \cosh x$

12. $\tanh 3x = \dfrac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$

13. (i) $\cosh x + \sinh x = e^x$
    (ii) $\cosh x - \sinh x = e^{-x}$

14. (i) $\sinh(-x) = -\sinh x$
    (ii) $\cosh(-x) = \cosh x$
    (iii) $\tanh(-x) = -\tanh x$

Osborne's Rule: To move from trignometric identiti to a hyperbolic identity; change cos to cosh and change the sign of a product of sinhs.

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Example 5: Prove that: $\cosh^2 x - \sinh^2 x = 1$

Proof: $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$$= \frac{1}{4}\left[(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\right]$$

$$= \frac{1}{4} \times 4 = 1$$

$$\therefore \underline{\cosh^2 x - \sinh^2 x = 1}$$

Example 6: Starting from the definitions of $\tanh x$ and $\operatorname{sech} x$ in terms of exponentials, prove that: $1 - \tanh^2 x = \operatorname{sech}^2 x$ --[3]

[S-20/23/ Q6(a)]

Proof: $1 - \tanh^2 x = 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$

$$= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{e^x + e^{-x}}\right)^2 = \operatorname{sech}^2 x$$

$= R.H.S$ ✓

Example 7: The curves C_1: $y = \cosh x$ and C_2: $y = \sinh 2x$, intersect at the point $x = a$.

(a) Find the exact value of a, giving your answer in log. form. --[4]

(b) Sketch C_1 and C_2 on the same diagram. [S-20/21/Q5(a)(b)] --[2]

Solution: For the point of intersect of

(a) C_1 & C_2 $\Rightarrow$ $\cosh x = \sinh 2x$

$\Rightarrow \cosh x = 2 \sinh x . \cosh x$

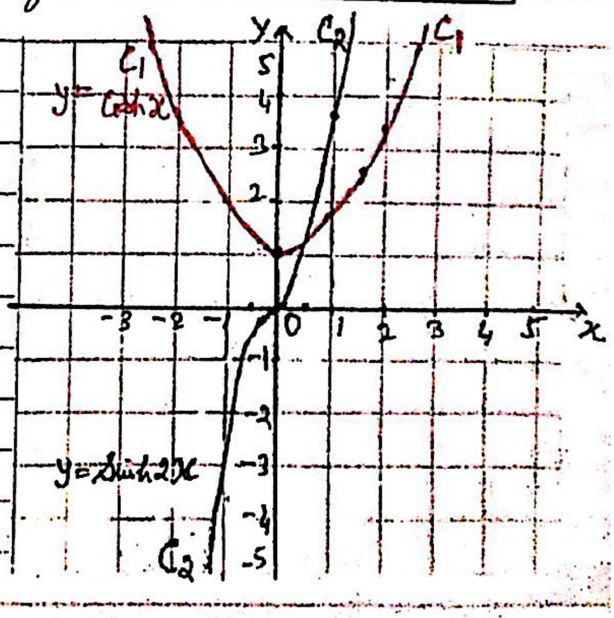
$\Rightarrow \sinh x = \frac{1}{2}$

$x = \sinh^{-1}\frac{1}{2} = a$

$\Rightarrow x = \ln\left[\frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + 1}\right]$

$\therefore a = \ln\left[\frac{1}{2} + \frac{1}{2}\sqrt{5}\right]$

$\left[\because \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)\right]$

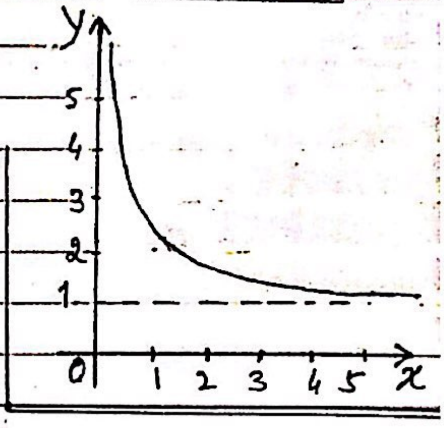


Example 8 (a) Sketch the graph of $y = \coth x$ for $x > 0$ and state the equations of the asymptotes. ---[2]

(b) Starting from the definitions of coth and cosech in term of exponentials, prove that: $\coth^2 x - \operatorname{cosech}^2 x = 1$. --[3]

[W-20/21/Q8(a)(b)]

Solution: (a) $y = \coth x$ for $x > 0$



asymptotes are: $x = 0$ ✓

and $y = 1$ ✓

(b) To Prove $\coth^2 x - \operatorname{cosech}^2 x = 1$.

L.H.S $\coth^2 x - \operatorname{cosech}^2 x$

$$= \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 - \left(\frac{2}{e^x - e^{-x}}\right)^2$$

$$= \frac{(e^{2x} + 2 + e^{-2x})}{(e^x - e^{-x})^2} - \frac{4}{(e^x - e^{-x})^2} = \frac{e^{2x} - 2 + e^{-2x}}{(e^x - e^{-x})^2} = \frac{(e^x - e^{-x})^2}{(e^x - e^{-x})^2}$$

$= 1$ ✓ $=$ R.H.S ✓

Example 9: Show that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

Solution: Consider R.H.S: $\cosh x \cosh y + \sinh x \sinh y$

$$= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right) \cdot \left(\frac{e^y - e^{-y}}{2}\right)$$

$$= \frac{e^{x+y} + \cancel{e^{x-y}} + \cancel{e^{y-x}} + e^{-(x+y)} + e^{x+y} - \cancel{e^{x-y}} - \cancel{e^{y-x}} + e^{-(x+y)}}{4}$$

$$= \frac{2e^{x+y} + 2e^{-(x+y)}}{4}$$

$$= \frac{e^{(x+y)} + e^{-(x+y)}}{2}$$

$= \cosh(x+y) =$ R.H.S ✓

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Example 10: Prove that: $\sinh 3x = 3\sinh x + 4\sinh^3 x$.

Proof:

L.H.S.

$$\sinh 3x = \sinh(x+2x)$$
$$= \sinh x \cdot \cosh 2x + \cosh x \cdot \sinh 2x$$
$$= \sinh x \cdot (1+2\sinh^2 x) + \cosh x \ast 2\sinh x \cosh x$$
$$= \sinh x(1+2\sinh^2 x) + 2\sinh x \cdot \cosh^2 x$$
$$= \sinh x(1+2\sinh^2 x) + 2\sinh x(1+\sinh^2 x)$$
$$= \sinh x + 2\sinh^3 x + 2\sinh x + 2\sinh^3 x$$
$$= 3\sinh x + 4\sinh^3 x = \text{R.H.S} \checkmark$$

- **Inverse hyperbolic functions:**

The graph of an inverse hyperbolic functions is a reflection of the corresponding hyperbolic function in the line $y=x$.

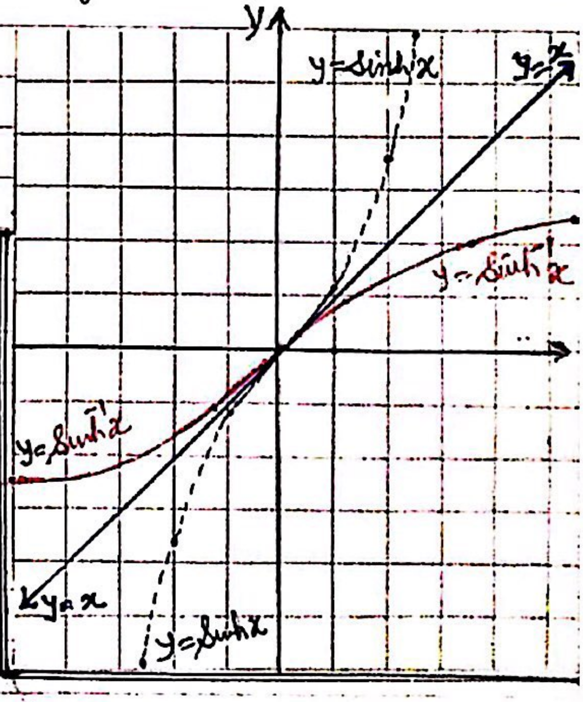
Inverse function is only defined if the given function is one-one.

(1) Graph of $\sinh^{-1}x$:

Domain of $\sinh^{-1}x$ is $x \in R$

Range of $\sinh^{-1}x$ is $x \in R$

both are one-one funtions.

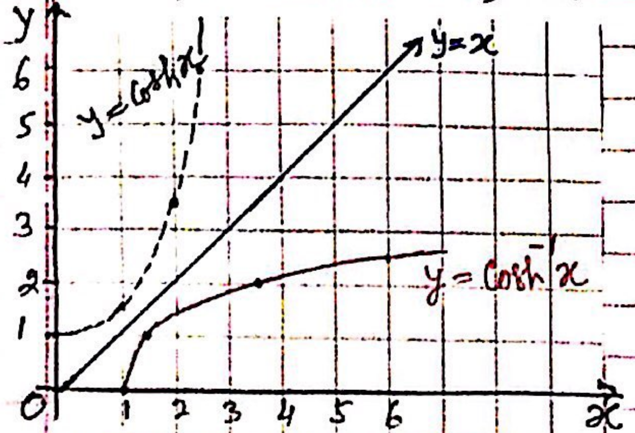


(2) Graph of $\cosh^{-1}x$:

$\cosh x$ is not a one-one function

$\cosh x$ is one-one for $x \geq 0$, $f(x) \geq 1$

$\therefore$ Domain of $\cosh^{-1}x$ is $x \geq 1$, $x \in R$



Range of $\cosh^{-1}x$; $f(x) \geq 0$

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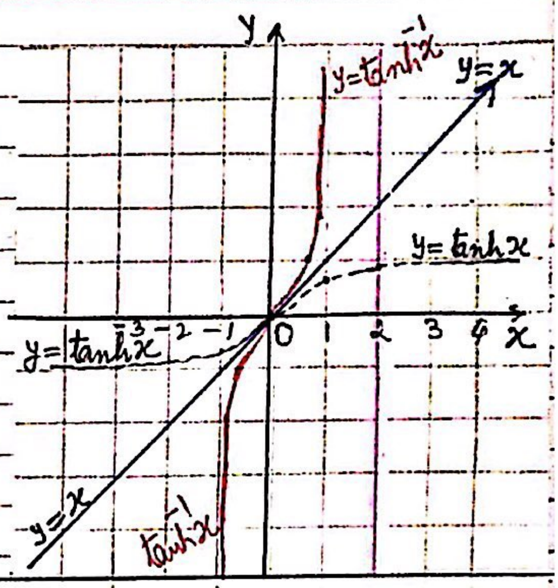
3. Graph of $y = \tanh^{-1}x$:

$y = \tanh^{-1}x$.
Domain: $-1 < x < 1$
Range: $f(x) \in R$



• **Logarithmic form of the inverse hyperbolic function:**

1. $\sinh^{-1}x = \ln(x+\sqrt{x^2+1})$
2. $\cosh^{-1}x = \ln(x+\sqrt{x^2-1}),\ x\geq 1$
3. $\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right);\ |x|<1$
(See Example 11/P-9)
4. $\coth^{-1}x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$
5. $\operatorname{sech}^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$
$0 < x \leq 1$
6. $\operatorname{cosech}^{-1}x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)$

7(i) $\begin{cases} \sinh^{-1}x = \operatorname{cosech}^{-1}\left(\frac{1}{x}\right) \\ \operatorname{cosech}^{-1}x = \sinh^{-1}\frac{1}{x} \end{cases}$

(ii) $\begin{cases} \cosh^{-1}(x) = \operatorname{sech}^{-1}\left(\frac{1}{x}\right) \\ \operatorname{sech}^{-1}(x) = \cosh^{-1}\left(\frac{1}{x}\right) \end{cases}$

(iii) $\begin{cases} \coth^{-1}x = \tanh^{-1}\left(\frac{1}{x}\right) \\ \tanh^{-1}\frac{1}{x} = \coth^{-1}(x) \end{cases}$

(1.) Find the log. form of $\sinh^{-1}x$

Solution: $y = \sinh^{-1}x$

$\Rightarrow x = \sinh y = \frac{e^y - e^{-y}}{2}$

$\Rightarrow x = \frac{e^{2y}-1}{2e^y} \Rightarrow e^{2y}-1 = 2e^y x$

$\Rightarrow e^{2y} - 2e^y x - 1 = 0$

$\Rightarrow (e^y - x)^2 - x^2 - 1 = 0$

$\Rightarrow (e^y - x)^2 = x^2+1$

$\Rightarrow e^y - x = \pm\sqrt{x^2+1}$

$\Rightarrow e^y = x \pm \sqrt{x^2+1}$

$\Rightarrow y = \ln(x+\sqrt{x^2+1})$

$\Rightarrow \sinh^{-1}x = \ln(x+\sqrt{x^2+1})$

$\because \ln x$ is defined only for positive values

8. $\sinh^{-1}x + \sinh^{-1}y = \sinh^{-1}\left(x\sqrt{1+y^2} + y\sqrt{1+x^2}\right)$

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Example 11: Starting from the definition of $\tanh x$ in terms of exponentials, prove that $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$; $|x|<1$. ---[3]

[SP-20|02|Q7]

Proof: Let $\tanh^{-1} x = u \Rightarrow x = \tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}}$

$\Rightarrow \frac{x}{1} = \frac{e^u - e^{-u}}{e^u + e^{-u}}$ using componendo and dividendo.

$\Rightarrow \frac{1-x}{1+x} = \frac{(e^u+e^{-u})-(e^u-e^{-u})}{(e^u+e^{-u})+(e^u-e^{-u})} = \frac{2e^{-u}}{2e^u}$

$\Rightarrow \frac{1-x}{1+x} = e^{-2u} \Rightarrow e^{2u} = \frac{1+x}{1-x}$

$\Rightarrow 2u = \ln\left(\frac{1+x}{1-x}\right) \Rightarrow u = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$\Rightarrow \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ ✓

Example 12: Solve the equation:

$\sinh^2 x - 5\sinh x + 4 = 0$ Give your answer in exact log. form.

Solution: $\sinh^2 x - 5\sinh x + 4 = 0$

$\Rightarrow (\sinh x - 4)(\sinh x - 1) = 0$

$\Rightarrow \sinh x = 4$; $\sinh x = 1$ $[\sinh^{-1} x = \ln(x + \sqrt{x^2+1})]$

$\Rightarrow x = \sinh^{-1} 4$; $x = \sinh^{-1} 1$

$= \ln(4 + \sqrt{4^2+1})$; $x = \ln(1 + \ln\sqrt{1^2+1})$

$x = \ln(4+\sqrt{17})$ ✓ ; $x = \ln(1+\sqrt{2})$ ✓

Example 13: Find the exact value of x that satisfy: $\cosh x = 2\sinh^2 x - 26$.

Solution: $\cosh x = 2\sinh^2 x - 26$

$\Rightarrow \cosh x = 2(\cosh^2 x - 1) - 26$

$\Rightarrow 2\cosh^2 x - \cosh x - 28 = 0$

$(2\cosh x + 7)(\cosh x - 4) = 0$

$\cosh x = 4$; $\cosh x = -\frac{7}{2}$ ✗

$\Rightarrow \frac{e^x + e^{-x}}{2} = 4$ $(\because \cosh x \geq 1)$

$\Rightarrow e^x + e^{-x} = 8$

$\Rightarrow e^{2x} + 1 = 8e^x$

$e^{2x} - 8e^x + 1 = 0$ $\{b^2 - 4ac = 8^2 - 4\times1\times1 = 60$

$e^x = \frac{8 \pm \sqrt{60}}{2} = 4 \pm \sqrt{15}$

$\Rightarrow x = \ln(4+\sqrt{15})$ or $x = \ln(4-\sqrt{15})$

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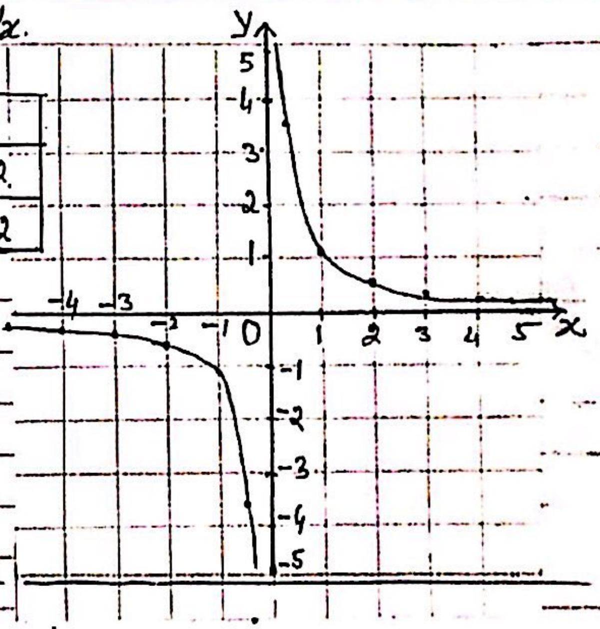
Example 14: Sketch the graph of $y = \operatorname{Cosech}^{-1} x$.
$x \neq 0$

x	0.5	1	2	3	4	5
$\frac{1}{x}$	2	1	0.5	0.333	0.25	0.2
$\operatorname{Cosech}^{-1} x$ $\left(= \sinh^{-1} \frac{1}{x}\right)$	3.6	1.17	0.52	0.34	0.25	0.2

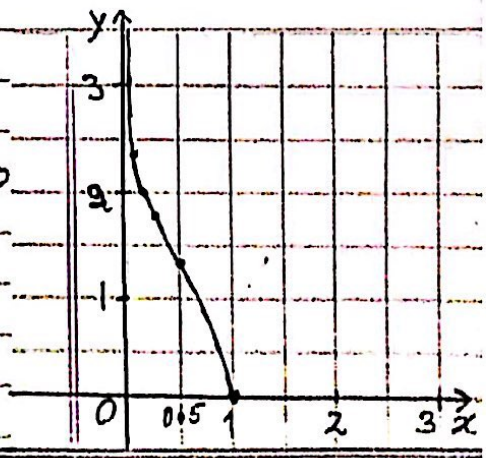


$y = \operatorname{Cosech}^{-1} x$; $x \neq 0$
Domain: $x \neq$, $x \in R$
Range: $f(x) \neq 0$, $f(x) \in R$.
asymptotes: $x = 0$; $y = 0$

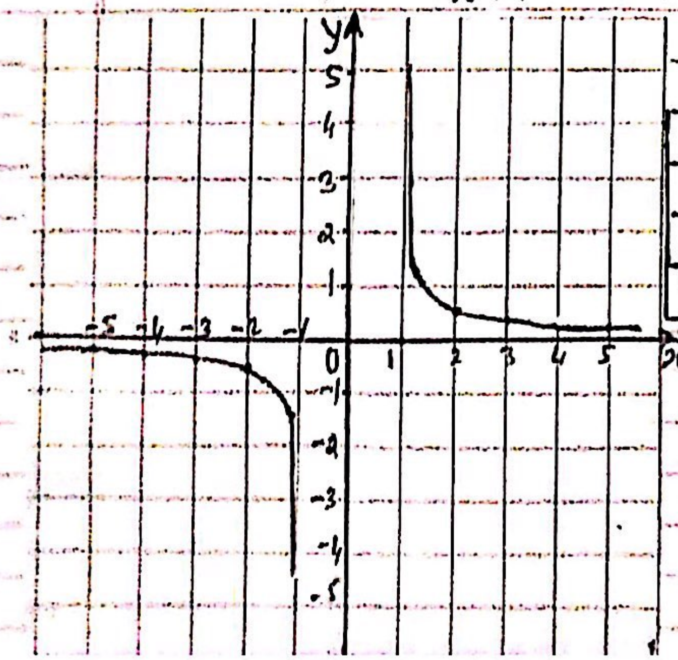
Example 15: Sketch the graph of $y = \operatorname{sech}^{-1} x$;
for $0 < x \leq 1$

x	.2	.25	.333	.5	1
$\frac{1}{x}$	5	4	3	2	1
$\operatorname{sech}^{-1} x$ $\left(\cosh^{-1} \frac{1}{x}\right)$	2.3	2	1.76	1.3	0

Range $f(x) \geq 0$
asymptote; $x = 0$



Example 16: Sketch the graph of $y = \coth^{-1} x$
$x < -1$ or $x > 1$



x	± 1.1	± 2	± 3	± 4	± 5
$\frac{1}{x}$	0.9	0.5	0.333	0.25	0.2
$\coth^{-1} x$ $= \tanh^{-1} \frac{1}{x}$	± 1.47	± 0.55	± 0.35	± 0.25	± 0.2

Range: $f(x) \neq$ $f(x) \in R$

Asymptotes: $\begin{cases} x = 1 \text{ and} \\ x = -1 \end{cases}$

Example 16: Starting from the definitions of Sinh and cosh in terms of exponentions, prove that: $2\sinh^2 x = \cosh 2x - 1$. ---[3]

[S-21/23/Q6(a)]

Solution: $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$

Consider L.H.S: $2\sinh^2 x = 2\left(\frac{e^x - e^{-x}}{2}\right)^2$

$= 2\left[\frac{e^{2x} + e^{-2x} - 2}{4}\right]$

$= \left(\frac{e^{2x} + e^{-2x}}{2}\right) - 1 = \cosh 2x - 1 = \text{R.H.S.}$ ✓

Example 17: Starting from the definition of cosh in terms of exponential, prove that: $2\cosh^2 A = \cosh 2A + 1$ ---[3]

[W-21/21/Q8(a)]

Solution: $\cosh A = \left(\frac{e^A + e^{-A}}{2}\right) \Rightarrow 2\cosh^2 A = 2 \times \left(\frac{e^A + e^{-A}}{2}\right)^2 = \frac{1}{2}\left(e^{2A} + 2 + e^{-2A}\right)$

$= \frac{1}{2}\left(e^{2A} + e^{-2A}\right) + 1 = \cosh 2A + 1$ ✓

Example 18: Starting from the definition of tanh and sech in terms of exponential, prove that: $1 - \tanh^2 x = \text{sech}^2 x$ ---[3]

[W-21/22/Q8(a)]

Solution: $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$; $\text{sech}\, x = \frac{2}{e^x + e^{-x}}$

Now $1 - \tanh^2 x = 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$

$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} + e^{-2x} + 2}{(e^x + e^{-x})^2}$

$= \frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{e^x + e^{-x}}\right)^2 = \text{sech}^2 x$ ✓

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