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FP.2

Further Pure Math_2

Integration
Notes and Revision

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§ Integration of Hyperbolic functions:

(i) $\int \sinh x\,dx = \int \frac{(e^x - e^{-x})}{2}\,dx$

$= \frac{1}{2}[e^x + e^{-x}] = \cosh x + c$

(ii) $\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}\,dx$ {Put $\sinh x = u$, diff $\cosh x\,dx = du$

$= \int \frac{1}{u}\,du$

$= \ln u$

$= \ln|\sinh x| + c$

(iii) $\int \operatorname{sech} x\,dx = \int \frac{2}{e^x + e^{-x}}\,dx = 2\int \frac{e^x\,dx}{1+(e^x)^2}$

$= 2\int \frac{1}{1+u^2}\,du$ {Put $e^x = u$, diff $e^x dx = du$

$= 2\tan^{-1} u = 2\tan^{-1}(e^x) + c$

(iv) $\int \frac{1}{x^2 - a^2}\,dx$ {Put $x = a\tanh u$ ⊛, $dx = a\operatorname{sech}^2 u\,du$, ⊛ $u = \tanh^{-1} x/a$

$= \int \frac{1 \times a\operatorname{sech}^2 u\,du}{a^2\tanh^2 u - a^2}$

$= \int \frac{a\operatorname{sech}^2 u\,du}{-a^2(1-\tanh^2 u)}$ ($\because \operatorname{sech}^2 u = 1 - \tanh^2 u$)

$= -\frac{1}{a}\int 1\,du = -\frac{1}{a}u = -\frac{1}{a}\tanh^{-1}\frac{x}{a} + c$

(v) $\int \frac{1}{\sqrt{x^2 + a^2}}\,dx$ {Put $x = a\sinh u$ ⊛, diff $dx = a\cosh u\,du$, ⊛ $u = \sinh^{-1} x/a$

$= \int \frac{1 \times a\cosh u\,du}{\sqrt{a^2 + a^2\sinh^2 u}}$

$= \int \frac{\cancel{a}\cosh u\,du}{\cancel{a}\sqrt{1+\sinh^2 u}}$ [$\because 1 + \sinh^2 u = \cosh^2 u$]

$= \int \frac{\cosh u}{\sqrt{\cosh^2 u}}\,du = \int 1\,du = u + c$

$= \sinh^{-1}\frac{x}{a} + c$

(i) $\int \sinh x\,dx = \cosh x + c$

(ii) $\int \cosh x\,dx = \sinh x + c$

(iii) $\int \operatorname{sech}^2 x\,dx = \tanh x + c$

(iv) $\int \operatorname{cosech} x \coth x\,dx = -\operatorname{cosech} x + c$

(v) $\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + c$

(vi) $\int \operatorname{cosech}^2 x\,dx = -\coth x + c$

(i) $\int \tanh x\,dx = \ln|\cosh x| + c$

(ii) $\int \coth x\,dx = \ln|\sinh x| + c$

(iii) $\int \operatorname{sech} x\,dx = 2\tan^{-1}(e^x) + c$

(iv) $\int \operatorname{cosech} x\,dx = -2\coth(e^x) + c$

(i) $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(ii) $\int \frac{1}{x^2 - a^2}\,dx = -\frac{1}{a}\tanh^{-1}\frac{x}{a} + c$ (or $\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$)

(iii) $\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{a}\tanh^{-1}\frac{x}{a}$; $x^2 < a^2$ (or $\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$)

(i) $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}$; $|x| \le a$ (or $-\cos^{-1} x/a$)

(ii) $\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \sinh^{-1} x/a$ (or $\ln|x + \sqrt{x^2 + a^2}|$)

(iii) $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \cosh^{-1} x/a$ (or $\ln|x + \sqrt{x^2 - a^2}|$)

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Integral of Inverse hyperbolic functions:

(i) $\int \sinh^{-1}x\,dx = x\sinh^{-1}x - \sqrt{1+x^2} + C$

(ii) $\int \cosh^{-1}x\,dx = x\cosh^{-1}x - \sqrt{x^2-1} + C$

(iii) $\int \tanh^{-1}x\,dx = x\tanh^{-1}x + \frac{1}{2}\ln|1-x^2| + C$

(iv) $\int \operatorname{cosech}^{-1}x\,dx = x\operatorname{cosech}^{-1}x + \ln|x+\sqrt{x^2+1}|$

(v) $\int \operatorname{sech}^{-1}x\,dx = x\operatorname{sech}^{-1}x + \ln|x+\sqrt{1-x^2}|$

(vi) $\int \coth^{-1}x\,dx = x\coth^{-1}x + \frac{1}{2}\ln|x^2-1| + C$

(i) $\int \sinh^{-1}x\,dx = \int \sinh^{-1}x \times 1\,dx$ { Integration by parts $\int u\cdot v\,dx = u\cdot\int v\,dx - \int\left(\frac{du}{dx}\cdot\int v\,dx\right)dx$

$= \sinh^{-1}x\cdot\int 1\,dx - \int\left(\frac{d}{dx}\sinh^{-1}x\cdot\int 1\,dx\right)dx$

$= (\sinh^{-1}x)\cdot x - \int \frac{1}{\sqrt{1+x^2}}\cdot x\,dx$

$= x\cdot\sinh^{-1}x - \frac{1}{2}\int \frac{1\,du}{\sqrt{u}}$ (Put $1+x^2 = u$; diff $2x\,dx = du$

$= x\cdot\sinh^{-1}x - \frac{1}{2}\cdot 2\sqrt{u} + C$ $\quad \int \frac{1}{\sqrt{u}}du = \int u^{-\frac{1}{2}}du = 2\sqrt{u}$

$= x\sinh^{-1}x - \sqrt{1+x^2} + C$ ✓

(vi) $\int \coth^{-1}x\,dx$

$= \int \coth^{-1}x \times 1\,dx$ (Integration by parts)

$= \coth^{-1}x\cdot\int 1\,dx - \int\left(\frac{d}{dx}\coth^{-1}x\cdot\int 1\,dx\right)dx$

$= x\cdot\coth^{-1}x - \int \frac{-1}{(x^2-1)}\cdot x\,dx$ $\quad \left\{ \frac{d}{dx}\coth^{-1}x = \frac{-1}{(x^2-1)};\ x^2>1 \right.$

$= x\coth^{-1}x + \frac{1}{2}\int \frac{2x}{(x^2-1)}dx$

$= x\coth^{-1}x + \frac{1}{2}\ln|x^2-1| + C$ ✓ $\quad$ { for $f(x) = x^2-1$, $f'(x) = 2x$, $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|$

(vii) $\int \tan^{-1}x\,dx = \int \tan^{-1}x \times 1\,dx$ By Parts

$= \tan^{-1}x\cdot\int 1\,dx - \int\left(\frac{d}{dx}\tan^{-1}x\cdot\int 1\,dx\right)dx$

$= x\cdot\tan^{-1}x - \int \frac{1}{(1+x^2)}\cdot x\,dx$

$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$

$= x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + C$ ✓

(i) $\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2}$

(ii) $\int \cos^{-1}x\,dx = x\cos^{-1}x - \sqrt{1-x^2}$

(iii) $\int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\ln|1+x^2|$

(iv) $\int \cot^{-1}x\,dx = x\cot^{-1}x + \frac{1}{2}\ln|1+x^2|$

(v) $\int \sec^{-1}x\,dx = x\sec^{-1}x - \ln|x+\sqrt{x^2-1}|$

(vi) $\int \operatorname{cosec}^{-1}x\,dx = x\operatorname{cosec}^{-1}x + \ln|x+\sqrt{x^2-1}|$

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Example 1: Determine the integral. $\int \frac{-2}{\sqrt{25-9x^2}}\,dx$

Solution: $\int \frac{-2}{5\sqrt{1-\left(\frac{3x}{5}\right)^2}}\,dx = -\frac{2}{5}\int \frac{1\,dx}{\sqrt{1-\left(\frac{3x}{5}\right)^2}}$

Put $\frac{3}{5}x = \sin\theta$ $\Rightarrow$ or $\theta = \sin^{-1}\left(\frac{3x}{5}\right)$

diff $\frac{3}{5}dx = \cos\theta\, d\theta$

$dx = \frac{5}{3}\cos\theta\, d\theta$

$= -\frac{2}{5}\times\frac{5}{3}\int \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$

$= -\frac{2}{3}\int \frac{\cos\theta\, d\theta}{\cos\theta} = -\frac{2}{3}\int 1\cdot d\theta$

$= -\frac{2}{3}\theta + c = -\frac{2}{3}\sin^{-1}\left(\frac{3x}{5}\right) + c$ ✓

Alternate method:

$-\frac{2}{3}\int \frac{1}{\sqrt{\left(\frac{5}{3}\right)^2 - x^2}}\,dx$

Formula $= -\frac{2}{3}\sin^{-1}\left(\frac{x}{\frac{5}{3}}\right)$

$= -\frac{2}{3}\sin^{-1}\left(\frac{3x}{5}\right)$

2. $\int x\cosh x\,dx$ (using by parts)

$= x\cdot\int \cosh x\,dx - \int\left(\frac{d}{dx}x\cdot\int\cosh x\,dx\right)dx$

$= x\sinh x - \int 1\times\sinh x\,dx$

$= x\sinh x - \cosh x + c$ ✓

3. Find $\int_2^3 \frac{1}{\sqrt{x^2+6x}}\,dx = \int_2^3 \frac{1}{\sqrt{(x+3)^2-9}}\,dx$

$\left\{ \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1} x/a \right.$

$= \left[\cosh^{-1}\left(\frac{x+3}{3}\right)\right]_2^3$

$= \left(\cosh^{-1}2 - \cosh^{-1}\frac{5}{3}\right)$

$= 1.3169 - 1.0986 = 0.218$ ✓

4. $I = \int\sqrt{a^2+x^2}\,dx = \int\sqrt{a^2+x^2}\times 1\,dx$

$= x\cdot\sqrt{a^2+x^2} - \int\left(\frac{d}{dx}\sqrt{a^2+x^2}\cdot\int 1\,dx\right)dx$

$I = x\sqrt{a^2+x^2} - \int\left(\frac{1\times 2x}{2\sqrt{a^2+x^2}}\times x\right)dx$

$= x\sqrt{a^2+x^2} - \int\frac{(a^2+x^2)-a^2}{\sqrt{a^2+x^2}}$

$= x\sqrt{a^2+x^2} - \int\sqrt{a^2+x^2}\,dx + a^2\int\frac{1\,dx}{\sqrt{a^2+x^2}}$

$I = x\sqrt{a^2+x^2} - I + a^2\times\sinh^{-1}x/a$

(i) $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

(ii) $\int\sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$

or $= \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|$

(iii) $\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}x/a$

(or $\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|$

$\Rightarrow 2I = x\sqrt{a^2+x^2} + a^2\sinh^{-1}x/a \Rightarrow I = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c$ ✓

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5. $\int_3^4 \frac{1}{\sqrt{3x^2-9x+7}}dx = \frac{1}{\sqrt{3}}\int_3^4 \frac{1}{\sqrt{x^2-3x+7/3}}dx$

$= \frac{1}{\sqrt{3}}\int_3^4 \frac{1}{\sqrt{(x-3/2)^2+\frac{1}{12}}}dx =$ $\left\{ \because \int \frac{1}{\sqrt{x^2+a^2}}dx = \sinh^{-1}\left(\frac{x}{a}\right) \right.$

$= \frac{1}{\sqrt{3}}\left[\sinh^{-1}\left(\frac{x-1.5}{\sqrt{\frac{1}{12}}}\right)\right]_3^4$

$= \frac{1}{\sqrt{3}}\left[\sinh^{-1}(2.5\times\sqrt{12}) - \sinh^{-1}(1.5\times\sqrt{12})\right]$

$= \frac{1}{\sqrt{3}}[2.855 - 2.350] = 0.292$ ✓

6. Find the exact value of $\int_0^1 \frac{1}{\sqrt{3+4x-4x^2}}dx$ --- [6]

SP-20/02/Q2

Solution: $\int_0^1 \frac{1}{\sqrt{3+4x-4x^2}}dx$

$$\left\{\begin{aligned} & -4x^2+4x+3 \\ &= -4\left(x^2-x-\tfrac{3}{4}\right) \\ &= -4\left[\left(x-\tfrac{1}{2}\right)^2-\tfrac{1}{4}-\tfrac{3}{4}\right] \\ &= -4\left[\left(x-\tfrac{1}{2}\right)^2-1\right] \\ &= 4\left[1-\left(x-\tfrac{1}{2}\right)^2\right] \end{aligned}\right.$$

$= \int_0^1 \frac{1}{\sqrt{4[1-(x-\frac{1}{2})^2]}}dx$

$= \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-(x-\frac{1}{2})^2}}dx$

$\because \int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x$

$= \frac{1}{2}\left[\sin^{-1}\left(x-\frac{1}{2}\right)\right]_0^1$

$= \frac{1}{2}\left[\sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{6}+\frac{\pi}{6}\right] = \frac{\pi}{6}$ ✓

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§ Reduction formulae:

Example: 6 (a) Find $\int_0^1 x e^x dx$

(b) Find a reduction formula for $I_n = \int_0^1 x^n e^x dx$

(c) Use this to find $\int_0^1 x^4 e^x dx$.

Solution (a) $\int_0^1 x e^x dx = x \cdot \int e^x dx - \int \left(\frac{d}{dx} x \cdot \int e^x dx \right) dx$ $\left[\begin{array}{l} \text{Integration by parts:} \\ \int uv\,dx = u \cdot \int v\,dx - \int \left(\frac{du}{dx} \cdot \int v\,dx \right) dx \end{array} \right.$

$$= x \cdot e^x - \int 1 \times e^x dx$$

$$= \left[x e^x - e^x \right]_0^1 = \left[e^x (x-1) \right]_0^1$$

$$= 0 - (-1) = 1 \quad \text{---- (i)}$$

(b) $\int_0^1 x^n e^x dx$

$$I_n = x^n \cdot \int e^x dx - \int_0^1 \left(\frac{d}{dx} x^n \cdot \int e^x dx \right) dx$$

$$= \left[x^n \cdot e^x \right]_0^1 - \int_0^1 n x^{n-1} e^x dx$$

$$I_n = e - n \int_0^1 x^{n-1} e^x dx$$

Reduction formula: $I_n = e - n \cdot I_{n-1}$ ------------ (ii)

(c) $I_4 = \int_0^1 x^4 e^x dx = e - 4 \cdot I_3 = e - 4 \left[e - 3 I_2 \right] = -3e + 12 I_2$

$$= -3e + 12 \left(e - 2 I_1 \right)$$

$$= 9e - 24 I_1 \quad \left[\begin{array}{l} \text{from (i)} \\ I_1 = 1 \end{array} \right.$$

$$= 9e - 24 \times 1$$

$$= \underline{0.465} \checkmark$$

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Example 7: The integral I_n, where n is an integral, is defined by:

$$I_n = \int_0^{3/2} (4+x^2)^{-\frac{1}{2}n}\,dx$$

(a) Find the exact value of I_1, expressing your answers in logarithmic form. --- [3]

(b) By considering $\frac{d}{dx}\left(x(4+x^2)^{-\frac{1}{2}n}\right)$, or otherwise, show that

$$4n \cdot I_{n+2} = \frac{3}{2}\left(\frac{2}{5}\right)^n + (n-1)I_n. \quad \text{---[5]}$$

(c) Find the value of I_5. ---[3]

$\boxed{\text{S-21/23/Q 7}}$

Solution (a) $I_1 = \int_0^{3/2} (4+x^2)^{-\frac{1}{2}}\,dx = \int_0^{3/2} \frac{1}{\sqrt{2^2+x^2}}\,dx$ $\left\{\because \int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\frac{x}{a}\right\}$

$$= \left[\sinh^{-1}\frac{x}{2}\right]_0^{3/2}$$

$$= (\sinh^{-1} 3/4 - \sinh^{-1} 0)$$

$$= \ln\left(\frac{3}{4}+\frac{5}{4}\right) - \ln 1 \qquad \left(\sinh^{-1}x = \ln(x+\sqrt{x^2+1})\right)$$

$$= \underline{\ln 2} \quad \text{----(i)}$$

(b) $\frac{d}{dx}\left(x(4+x^2)^{-n/2}\right) = x\cdot\left(-\frac{n}{2}\right)\cdot(4+x^2)^{-n/2-1}\times 2x + (4+x^2)^{-n/2}\times 1$

$$= -nx^2(4+x^2)^{-n/2-1} + (4+x^2)^{-n/2}$$

$$= -n(4+x^2-4)(4+x^2)^{-n/2-1} + (4+x^2)^{-n/2}$$

$$= -n(4+x^2)^{-n/2} + 4n(4+x^2)^{-n/2-1} + (4+x^2)^{-n/2}$$

$$= -n(4+x^2)^{-n/2} + 4n(4+x^2)^{-\frac{(n+2)}{2}} + (4+x^2)^{-n/2}.$$

$$\therefore \left[x(4+x^2)^{-n/2}\right]_0^{3/2} = -nI_n + 4n\cdot I_{n+2} + I_n$$

$$\Rightarrow \frac{3}{2}\left(\frac{2}{5}\right)^n = (1-n)I_n + 4n\,I_{n+2}. \quad \Rightarrow 4n\cdot I_{n+2} = \frac{3}{2}\left(\frac{2}{5}\right)^n + (n-1)I_n$$

------(ii)

(c) for $n=1$ in (ii) $\Rightarrow$ $4I_3 = \frac{3}{2}\times\left(\frac{2}{5}\right)^1 + 0 \Rightarrow I_3 = \frac{3}{20}$ ---(iii)

for $n=3$, in (ii) $\Rightarrow$ $12I_5 = \frac{3}{2}\times\frac{8}{125} + 2I_3$

$$= \frac{12}{125} + 2\times\frac{3}{20} \qquad (\text{from iii) } I_3 = 3/20)$$

$$= \frac{12}{125} + \frac{3}{10} = \frac{99}{250}$$

$$I_5 = \frac{99}{12\times 250} = \frac{33}{1000} = \underline{0.033}$$

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Example 8: The integral I_n, where n is an integer, is defined by:

$$I_n = \int_0^{1/2} (1-x^2)^{-\frac{1}{2}n}\, dx$$

(a) Find the value of I_1. ---[2]

(b) By considering $\frac{d}{dx}\left(x\cdot(1-x^2)^{-\frac{n}{2}}\right)$ or otherwise, show that:

$$n\cdot I_{n+2} = 2^{n-1}\cdot 3^{-n/2} + (n-1)I_n \quad \text{---[5]}$$

(c) Find the exact value of I_5 giving your answer in the form $k\sqrt{3}$, where k is a rational number to be determined. ---[3]

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Solution (a) $I_1 = \int_0^{\frac{1}{2}} (1-x^2)^{-1/2}\, dx = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}$

$= \left[\sin^{-1}x\right]_0^{1/2} = \left(\sin^{-1}\tfrac{1}{2} - \sin^{-1}0\right) = \pi/6$ ✓ --(i)

(b) Consider: $I_n = \int_0^{1/2} (1-x^2)^{-n/2}\, dx$ (using by parts) (Alternate method)

$$I_n = \int_0^{1/2} (1-x^2)^{-n/2}\cdot 1\, dx = \left[(1-x^2)^{-n/2}\cdot x\right]_0^{1/2} - \int_0^{1/2} -\frac{n}{2}(1-x^2)^{-\frac{n}{2}-1} \times -2x \times x\, dx$$

$$I_n = \frac{1}{2}\left(1-\frac{1}{4}\right)^{-n/2} + \int_0^{1/2} -n\cdot x^2 (1-x^2)^{-\frac{n}{2}-1}\, dx$$

$$= \frac{1}{2}\left(\frac{3}{4}\right)^{-\frac{n}{2}} + \int_0^{\frac{1}{2}} \left\{ n\left((1-x^2)-1\right)(1-x^2)^{-\frac{n}{2}-1} \right. dx$$

$$= \frac{1}{2}\cdot 2^n\cdot 3^{-n/2} + \int_0^{1/2}\left\{ n(1-x^2)^{-n/2} - n(1-x^2)^{-\frac{n}{2}-1}\right\} dx$$

$$I_n = 2^{n-1}\cdot 3^{-n/2} + n\int_0^{1/2}(1-x^2)^{-n/2}dx - n\int (1-x^2)^{\frac{-(n+2)}{2}} dx$$

$$I_n = 2^{n-1}\cdot 3^{-n/2} + n\cdot I_n - n\cdot I_{n+2}$$

$$\Rightarrow n\cdot I_{n+2} = 2^{n-1}\cdot 3^{-n/2} + (n-1)I_n \ \checkmark \text{ ---- (ii)}$$

(c) from (ii) for $n=1 \Rightarrow \quad I_3 = 3^{-1/2}$ ------(iii)

form (ii) for $n=3 \Rightarrow \quad 3\cdot I_5 = 2^2\cdot 3^{-3/2} + 2I_3$

$= \frac{4}{3\sqrt{3}} + 2\times 3^{-1/2}$ $\left[\text{from (iii) } I_3 = 3^{-1/2}\right]$

$3\cdot I_5 = \frac{4}{3\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{10}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{10}{9}\sqrt{3}$

$\Rightarrow I_5 = \frac{10}{27}\sqrt{3}$ ✓

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Example 9: Let $I_n = \int_0^{\frac{\pi}{2}} x^n \cdot \sin x \, dx$

(i) Prove that for $n \geq 2$, $I_n = n(n-1) I_{n-2} = n(\frac{1}{2}\pi)^{n-1}$ ---[4]

(ii) Calculate the exact value of I_1 and deduce the exact value of I_3. ---[3]

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Solution (i) $I_n = \int_0^{\frac{\pi}{2}} x^n \cdot \sin x \, dx$ ---(i) (using Int. by parts)

$$= x^n \cdot \int \sin x \, dx - \int_0^{\pi/2} \left(\frac{d}{dx} x^n \cdot \int \sin x \, dx\right) dx$$

$$I_n = \left[-x^n \cdot \cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} n x^{n-1} \cdot (-\cos x) \, dx$$

$$= 0 + n \int_0^{\frac{\pi}{2}} x^{n-1} \cdot \cos x \, dx$$

$$= n \left[x^{n-1} \cdot \int \cos x \, dx \right]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} \left((n-1) x^{n-2} \cdot \int \cos x \, dx \right) dx \Big] \quad \text{(by parts again)}$$

$$I_n = n \left[x^{n-1} \cdot \sin x \right]_0^{\pi/2} - n \int_0^{\pi/2} (n-1) \cdot x^{n-2} \cdot \sin x \, dx$$

$$I_n = n \cdot \left(\frac{\pi}{2}\right)^{n-1} - n(n-1) \cdot I_{n-2}$$

$$\Rightarrow \quad I_n + n(n-1) I_{n-2} = n \cdot \left(\frac{\pi}{2}\right)^{n-1} \checkmark \quad \text{----(ii)}$$

(ii) from (i) $I_1 = \int_0^{\pi/2} x \cdot \sin x \, dx$ (Int by parts)

$$= \left[x \cdot \int \sin x \, dx\right]_0^{\pi/2} - \int_0^{\pi/2} \left(\frac{d}{dx} x \cdot \int \sin x \, dx\right) dx$$

$$= \left[-x \cos x\right]_0^{\pi/2} - \int_0^{\pi/2} 1 \times (-\cos x) \, dx$$

$$\Rightarrow I_1 = 0 + \left[\sin x\right]_0^{\pi/2} = 1 \checkmark \quad \text{------(iii)}$$

Now:

for $n = 3$ in (ii) $\Rightarrow$ $I_3 + 3(3-1) \cdot I_1 = 3 \cdot \left(\frac{\pi}{2}\right)^{3-1}$ $\left(\begin{array}{c}\text{from (iii)} \\ I_1 = 1\end{array}\right)$

$$I_3 + 3 \times 2 \times 1 = 3 \times \frac{\pi^2}{4}$$

$$I_3 = \frac{3}{4}\pi^2 - 6 \checkmark$$

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Example 10. Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \cdot \sin^2 x\, dx$ for $n \geq 0$. By differentiating $\cos^{n-1} x \cdot \sin^3 x$ with respect to x, prove that:

(i) $(n+2) I_n = (n-1) \cdot I_{n-2}$ for $n \geq 2$ ---[5]

(ii) Hence find the exact value of I_4 ---[4]

S-16/11/Q5

Solution:

(i) $\frac{d}{dx}(\cos^{n-1} x \cdot \sin^3 x) = -(n-1)\cos^{n-2} x \cdot \sin x \cdot \sin^3 x + \cos^{n-1} x \cdot 3\sin^2 x \cdot \cos x$

$= -(n-1)\cos^{n-2} x \cdot \sin^4 x + 3\cos^n x \cdot \sin^2 x$

$\Rightarrow \left[\cos^{n-1} x \cdot \sin^3 x\right]_0^{\frac{\pi}{2}} = \int_0^{\frac{\pi}{2}} \left[-(n-1)\cos^{n-2} x \cdot \sin^4 x + 3\cos^n x \cdot \sin^2 x\right] dx$

$\Rightarrow 0 = -(n-1)\int_0^{\pi/2} \cos^{n-2} x \cdot \sin^2 x (1-\cos^2 x) dx + 3\int_0^{\frac{\pi}{2}} \cos^n x \cdot \sin^2 x\, dx$

$\Rightarrow 0 = -(n-1)\int_0^{\frac{\pi}{2}} (\cos^{n-2} x \sin^2 x - \cos^n x \sin^2 x\, dx) dx + 3I_n$

$0 = -(n-1) I_{n-2} + (n-1) I_n + 3I_n$

$\Rightarrow (n+2) I_n = (n-1) I_{n-2}$ ----- (i)

(ii) Given $I_n = \int_0^{\pi/2} \cos^n x \cdot \sin^2 x\, dx$

$\Rightarrow I_0 = \int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2x}{2}\right) dx$

$= \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi/2}$

$I_0 = \frac{\pi}{4} - 0 \quad = \frac{\pi}{4}$ ---(ii)

Now put $n=2$ in (i) $\Rightarrow (2+2) I_2 = I_0 = \frac{\pi}{4}$ from (ii)

$\Rightarrow I_2 = \frac{\pi}{16}$ ——— (iii)

Put $n=4$ in (i) $\Rightarrow 6I_4 = 3I_2$

$= 3 \times \frac{\pi}{16}$ (from (iii))

$\Rightarrow I_4 = \frac{\pi}{32}$ ✓

Example 11: Let $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin 2n\theta}{\cos\theta}\, d\theta$, where n is a non-negative integer.

(i) Use the identity $\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$ to show that:
$I_n + I_{n-1} = \frac{2}{2n-1}$, for all positive integer n. ---[5]

(ii) Find the exact value of $\int_0^{\frac{\pi}{2}} \frac{\sin 8\theta}{\cos\theta}\, d\theta$ ---[4]

$\boxed{S\text{-}15/13/Q5}$

Solution: Given $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin 2n\theta}{\cos\theta} d\theta$ ---(i) $\therefore$ $I_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\sin 2(n-1)\theta}{\cos\theta} d\theta$ ---(ii)

(i)

$\therefore I_n + I_{n-1} = \int_0^{\frac{\pi}{2}} \left(\frac{\sin 2n\theta + \sin(2n-2)\theta}{\cos\theta} \right) d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta} \left\{ 2\sin\left(\frac{2n\theta + (2n-2)\theta}{2}\right) \cos\left(\frac{2n\theta - (2n-2)\theta}{2}\right) \right\} d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{2\sin(2n-1)\theta \cdot \cancel{\cos\theta}}{\cancel{\cos\theta}} d\theta$

$= \left[-2\frac{\cos(2n-1)\theta}{(2n-1)} \right]_0^{\pi/2} = 0 - \left(\frac{-2}{(2n-1)}\right) = \frac{2}{(2n-1)}$ ✓ ---(iii)

(ii) from (i) $I_1 = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\cos\theta} d\theta = \int_0^{\frac{\pi}{2}} 2\sin\theta\, d\theta = \left[-2\cos\theta\right]_0^{\pi/2} = 0 - (-2)$
$= 2$ ---(IV)

from (iii) for $n = 2$, $I_2 + I_1 = \frac{2}{2\times 2 - 1}$

$I_2 = \frac{2}{3} - 2 = -\frac{4}{3}$ ✓ $\left[\text{for } I_1 = 2\right]$

for $n = 3$, $I_3 + I_2 = \frac{2}{5} \Rightarrow I_3 = \frac{2}{5} - \left(-\frac{4}{3}\right) = \frac{26}{15}$ ✓

for $n = 4$, $I_4 + I_3 = \frac{2}{7} \Rightarrow I_4 = \frac{2}{7} - \frac{26}{15} = -\frac{152}{105}$ ---(V)

for $n = 4$, from (i) $I_4 = \int_0^{\frac{\pi}{2}} \frac{\sin 8\theta}{\cos\theta} d\theta = -\frac{152}{105}$ ✓ (from (V))

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Example 12 (i) Show that: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x}\cdot \cos x\,dx = \frac{1}{2}\left(e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}\right)$ ---[4]

(ii) It is given that, for $n \geq 0$, $I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\cdot \cos^n x\,dx$

Show that, for $n \geq 2$

$$4I_n = n(n-1)\int_{-\pi/2}^{\frac{\pi}{2}} e^{2x}\cdot \sin^2 x\cdot \cos^{n-2} x\,dx - nI_n$$

and deduce the reduction formula:

$$(n^2+4)I_n = n(n-1)\cdot I_{n-2}$$ ---[6]

[S-18/13/Q11(i)]

Solution: (i) Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x}\cos x\,dx$ (using integ. by parts)

$$= \left[\cos x\cdot \int e^x dx\right]_0^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{d}{dx}\cos x\cdot \int e^x dx\right)dx$$

$$I = \left[\cos x\cdot e^x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\sin x\cdot e^x\,dx$$

$$I = 0 + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin x\,dx = \left[\sin x\cdot \int e^x dx\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\pi/2}\left(\frac{d}{dx}\sin x\cdot \int e^x dx\right)dx$$

$$\Rightarrow I = \left[\sin x\cdot e^x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\cdot e^x\,dx$$

$$I = \left(e^{\frac{\pi}{2}} - (-e^{-\frac{\pi}{2}})\right) - I \Rightarrow 2I = e^{\pi/2} + e^{-\frac{\pi}{2}}$$

$$\Rightarrow I = \frac{1}{2}\left(e^{\pi/2} + e^{-\frac{\pi}{2}}\right) \checkmark$$

(ii) $I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\cdot \cos^n x\,dx$ ----(i) (Integral by parts)

$$= \left[\cos^n x\cdot \int e^{2x}dx\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{d}{dx}\cos^n x\cdot \int e^{2x}dx\right)dx$$

$$I_n = \left[\cos^n x\cdot \frac{e^{2x}}{2}\right]_{-\pi/2}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} n\cdot \cos^{n-1} x\cdot(-\sin x)\cdot \frac{e^{2x}}{2}\cdot dx$$

$$I_n = 0 + \frac{n}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-1} x\cdot \sin x\cdot e^{2x}\,dx$$

$$= \frac{n}{2}\left[\left(\cos^{n-1} x\sin x\cdot \int e^{2x}dx\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{d}{dx}(\cos^{n-1}x\cdot \sin x)\cdot \int e^{2x}dx\right)dx$$

$$I_n = \frac{n}{2}\left\{\cos^{n-1}x\sin x\cdot \frac{e^{2x}}{2}\right\}_{-\pi/2}^{\pi/2} - \frac{n}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\begin{matrix}(n-1)\cdot\cos^{n-2}x\cdot(-\sin x)\cdot\sin x \\ +\cos^{n-1}x\cdot\cos x\end{matrix}\right\}\cdot\frac{e^{2x}}{2}\,dx$$

$$I_n = 0 - \frac{n}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\left(\cos^n x - (n-1)\cos^{n-2}x\,\sin^2 x\right)dx$$

$$4I_n = n(n-1)\int e^{2x}\sin^2 x\cdot\cos^{n-2}x - nI_n$$

$$(n+4)I_n = n(n-1)\int_{-\pi/2}^{\frac{\pi}{2}} e^{2x}(1-\cos^2 x)\cdot\cos^{n-2}x\,dx$$

$$\Rightarrow (n+4)I_n = n(n-1)I_{n-2} - n(n-1)I_n$$

$$\Rightarrow (n^2+4)I_n = n[n-1]I_{n-2} \checkmark$$

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Example 13. Let $I_n = \int_0^{\frac{\pi}{4}} \sec^n x\, dx$ for $n > 0$

(i) Find the value of I_2 ---[2]

(ii) Show that:

$$(n-1)\cdot I_n = 2^{\frac{1}{2}n-1} + (n-2)\cdot I_{n-2} \quad \text{for } n > 2. \quad \text{---[5]}$$

[W-17/11/Q8]

Solution: $I_n = \int_0^{\frac{\pi}{4}} \sec^n x\, dx$ for $n > 0$ ---(i)

(i)

$$\therefore I_2 = \int_0^{\frac{\pi}{4}} \sec^2 x\, dx = \left[\tan x\right]_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0 = 1$$

(ii) $I_n = \int_0^{\frac{\pi}{4}} \sec^n x\, dx = \int_0^{\frac{\pi}{4}} \sec^{n-2} x \cdot \sec^2 x\, dx$

(Integrating by parts)

$$= \left[\sec^{n-2} x \cdot \int \sec^2 x\, dx\right]_0^{\frac{\pi}{4}} - \int_0^{\pi/4} \left(\frac{d}{dx}\sec^{n-2} x \cdot \int \sec^2 x\, dx\right) dx$$

$$= \left[\sec^{n-2} x \cdot \tan x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (n-2)\sec^{n-3} x\,(\sec x \tan x)\cdot \tan x\, dx$$

$$= (\sqrt{2})^{n-2} - \int_0^{\pi/4} (n-2)\cdot \sec^{n-2} x \cdot \tan^2 x\, dx$$

$$= 2^{\left(\frac{n}{2}-1\right)} - (n-2)\int_0^{\frac{\pi}{4}} \sec^{n-2} x\,(\sec^2 x - 1)\, dx$$

$$= 2^{\left(\frac{n}{2}-1\right)} - (n-2)\int_0^{\frac{\pi}{4}} \sec^n x\, dx + (n-2)\int_0^{\frac{\pi}{4}} \sec^{n-2} x\, dx$$

$$\Rightarrow I_n = 2^{\left(\frac{n}{2}-1\right)} - (n-2)\cdot I_n + (n-2)\cdot I_{n-2}$$

$$\Rightarrow I_n(1 + n - 2) = 2^{\left(n/2 - 1\right)} + (n-2)\, I_{n-2}$$

$$\therefore (n-1)\cdot I_n = 2^{\left(\frac{n}{2}-1\right)} + (n-2)\cdot I_{n-2}$$

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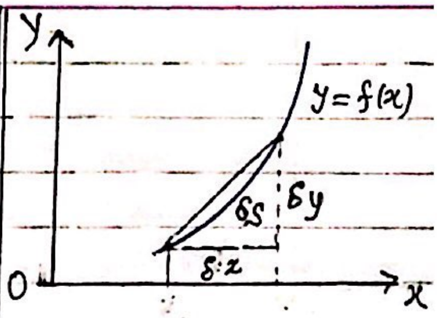
§ Arc Length:

Given the equation of a curve C:

$y = f(x)$

(i) The length of arc in Cartesian form $x = x_1$ to $x = x_2$ is given by!

$$S = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$



$$\delta s^2 = \delta x^2 + \delta y^2$$

$$\Rightarrow \left(\frac{\delta s}{\delta x}\right)^2 = 1+\left(\frac{\delta y}{\delta x}\right)^2$$

Lt $\delta x \to 0$

$$\frac{ds}{dx} = 1+\left(\frac{dy}{dx}\right)^2$$

$$S = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Example (i) Find the length of curve $y = 4x^{3/2}$ from $x=1$ to $x=2$;

Solution: $y = 4x^{3/2}$

$$\frac{dy}{dx} = 4\times\frac{3}{2}x^{1/2} = 6x^{1/2} \quad \text{---(i)}$$

$\therefore$ length of arc $s = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 \sqrt{1+36x}\,dx$

$$= \left[\frac{2}{3}\frac{(1+36x)^{3/2}}{36}\right]_1^2 = \frac{1}{54}\left[73^{3/2} - 37^{3/2}\right]$$

$$= \frac{1}{54}(398.6499) = 7.382$$

$\therefore$ Length of arc $s = 7.382$

Example (ii): The curve C has equation $y = \cosh x$ for $1 \le x \le 2$, Find the length of arc C.

Solution: $y = \cosh x$

$$\frac{dy}{dx} = \sinh x$$

Length of arc. $S = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1 \sqrt{1+\sinh^2 x}\,dx$

$$= \int_1^2 \sqrt{\cosh^2 x}\,dx = \int_1^2 \cosh x\,dx$$

$$= \left[\sinh x\right]_1^2 = \sinh 2 - \sinh 1 = 2.45$$

$\therefore S = 2.45$

§ Arc Length in Parametric form:

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Given $x = f(t)$ and $y = g(t)$.

$\delta s^2 = \delta x^2 + \delta y^2$

$\left(\frac{\delta s}{dt}\right)^2 = \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2$

Lt $\delta t \to 0$

$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$

$\Rightarrow S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Example 14: The curve C has equation:

$x = \frac{1}{2}t^2 - \ln t$ and $y = 2t + 1$ for $\frac{1}{2} \leq t \leq 2$

Find the exact length of C. ---[5]

S-20/23/Q5

Solution: C: $x = \frac{1}{2}t^2 - \ln t$ & $y = 2t+1$ for $\frac{1}{2} \leq t \leq 2$

$\frac{dx}{dt} = t - \frac{1}{t}$ & $\frac{dy}{dt} = 2$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(t - \frac{1}{t}\right)^2 + 2^2 = t^2 - 2 + \frac{1}{t^2} + 4 = t^2 + 2 + \frac{1}{t^2}$

$= (t + t^{-1})^2$ ---(1)

Length of arc. $S = \int_{1/2}^{2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{1}^{2} (t + t^{-1})\, dt$

$= \left[\frac{1}{2}t^2 + \ln t\right]_{1/2}^{2}$

$= (2 + \ln 2) - \left(\frac{1}{8} + \ln \frac{1}{2}\right)$

$= \left(2 - \frac{1}{8}\right) + (\ln 2 - \ln 2^{-1})$

$= \frac{15}{8} + (\ln 2 + \ln 2)$

$S = \frac{15}{8} + 2\ln 2.$ ✓

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Example 15: The curve C has equation $y = \ln\coth(\frac{1}{2}x)$ for $x > 0$

(i) Show that $\frac{dy}{dx} = -\operatorname{cosech} x$ ---[3]

(ii) It is given that the arc length of C from $x = a$ to $x = 2a$ is $\ln 4$, where 'a' is a positive constant.
Show that $\cosh a = 2$ and find, in logarithmic form, the exact value of a.
[W-20/21/Q8(c)(d)] --[7]

Solution (i) $y = \ln\coth x/2 \;;\; x > 0$

$$\frac{dy}{dx} = \frac{1}{\coth x/2} \times -\operatorname{cosech}^2 x/2 \times \frac{1}{2} \qquad \left[\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x\right]$$

$$= -\frac{1}{2} \times \frac{\sinh x/2}{\cosh x/2} \times \frac{1}{\sinh^2 x/2} = \frac{-1}{2\sinh x/2 \cdot \cosh x/2}$$

$$\frac{dy}{dx} = \frac{-1}{\sinh x} = -\operatorname{cosech} x \quad \text{--- (i)}$$

(ii) Arc length $C = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_a^{2a} \sqrt{1 + (-\operatorname{cosech} x)^2} \qquad [\because 1 + \operatorname{cosech}^2 x = \coth^2 x]$$

$$= \int_a^{2a} \sqrt{\coth^2 x}\, dx = \int_a^{2a} \coth x\, dx \qquad \left[\because \int \coth x\, dx = \ln|\sinh x|\right]$$

$$\therefore\; s = [\ln\sinh x]_a^{2a} = \ln\sinh 2a - \ln\sinh a$$

$$= \ln\left(\frac{\sinh 2a}{\sinh a}\right) = \ln\left(\frac{2\cancel{\sinh a}\cosh a}{\cancel{\sinh a}}\right)$$

$$s = \ln(2\cosh a) = \ln 4 \text{ (given)}$$

$$\Rightarrow 2\cosh a = 4 \Rightarrow \cosh a = 2$$

$$\Rightarrow a = \cosh^{-1} 2$$

$$= \ln(2 + \sqrt{2^2 - 1}) \qquad \left[\because \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \;;\; x \geq 1\right]$$

$$= \ln(2 + \sqrt{3})$$

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§ Lenth of Arc when curve is given in polar form:

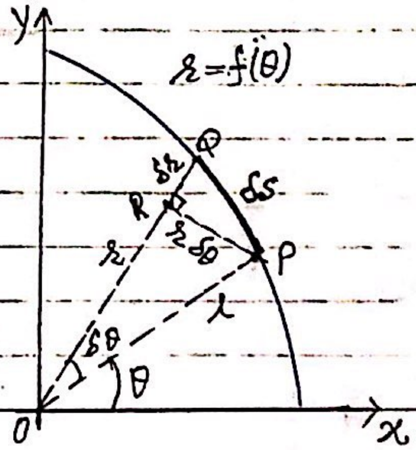
In rt Δ QRP; $\delta s^2 = \delta r^2 + r\delta\theta^2$

$$\left(\frac{\delta s}{\delta\theta}\right)^2 = \left(\frac{\delta r}{\delta\theta}\right)^2 + r^2$$

now lt $\delta\theta \to 0$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$\therefore$ Length of arc $S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$



Example 16: The polar equation of a curve C is, $r = a(1+\cos\theta)$; $0 \leq \theta \leq 2\pi$ where 'a' is a positive constant.
Find the arc length of C between the points where $\theta = 0$ and the point where $\theta = \frac{1}{3}\pi$

W-17/11/Q11

Solution: Given equation of curve C; $r = a(1+\cos\theta)$ ---(i)
diff w.r.t θ, $\frac{dr}{d\theta} = -a\sin\theta$ ---(ii)

length of arc for $\theta = 0$ to $\theta = \frac{1}{3}\pi$ is

$$S = \int_0^{\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{(a(1+\cos\theta))^2 + (-a\sin\theta)^2}\, d\theta$$

$$= a \cdot \int_0^{\frac{\pi}{3}} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= a\int_0^{\pi/3} \sqrt{2(1+\cos\theta)}\, d\theta = a\int_0^{\pi/3} \sqrt{2 \times 2\cos^2\theta/2}\, d\theta$$

$$= a\int_0^{\pi/3} 2\cos\theta/2\, d\theta = a\left[2\frac{\sin\theta/2}{1/2}\right]_0^{\pi/3}$$

$$= a\left[4\sin\theta/2\right]_0^{\pi/3}$$

$$= 4a\left(\sin\frac{\pi}{6} - 0\right) = 4a \times \frac{1}{2} = 2a$$

$\therefore S = 2a$ ✓

Example 17: The curve C has polar equation $r = e^{4\theta}$ for $0 \leq \theta \leq \alpha$, where α is measured in radians. The length of C is 2015. Find the value of α. ---[6]

S-15/13/Q.2

Solution: Given $r = e^{4\theta}$ ---(i) $\Rightarrow \frac{dr}{d\theta} = 4e^{4\theta}$ ---(ii)

$$\text{Length of arc} = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\alpha} \sqrt{(e^{4\theta})^2 + (4e^{4\theta})^2}\, d\theta = \int_0^{\alpha} \sqrt{e^{8\theta} + 16e^{8\theta}}\, d\theta$$

$$= \int_0^{\alpha} \sqrt{17e^{8\theta}}\, d\theta = \sqrt{17} \int_0^{\alpha} e^{4\theta}\, d\theta$$

$$= \sqrt{17}\left[\frac{e^{4\theta}}{4}\right]_0^{\alpha} = \frac{\sqrt{17}}{4}\left(e^{4\alpha} - 1\right)$$

$$\therefore \text{length of arc } C = \frac{\sqrt{17}}{4} e^{4\alpha} - \frac{\sqrt{17}}{4} = 2015 \text{ (Given)}$$

$$\Rightarrow e^{4\alpha} = 1955.837 \Rightarrow 4\alpha = \ln(1955.837) = 7.5785$$

$$\Rightarrow \alpha = 1.89 \text{ radians}$$

Example 18: The curve C is defined parametrically by:

$$x = e^t - t \;;\; y = 4e^{\frac{1}{2}t}$$

Find the length of arc 'C' from the point where $t=0$ to the point where $t=3$. S-18/11/Q1 ---[5]

Solution: $$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \text{ ---(i)}$$

Given $x = e^t - t$; $y = 4e^{\frac{1}{2}t}$

diff. $\frac{dx}{dt} = e^t - 1$; $\frac{dy}{dt} = 4e^{\frac{1}{2}t} \times \frac{1}{2} = 2e^{\frac{1}{2}t}$

$$\therefore \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t - 1)^2 + (2e^{\frac{1}{2}t})^2$$

$$= e^{2t} - 2e^t + 1 + 4e^t$$

$$= e^{2t} + 2e^t + 1$$

$$= (e^t + 1)^2 \text{ ---(ii)}$$

from (i) and (ii)

$$S = \int_0^3 \sqrt{(e^t+1)^2}\, dt$$

$$\text{Length of arc } S = \int_0^3 (e^t + 1)\, dt$$

$$= \left[e^t + t\right]_0^3$$

$$= (e^3 + 3) - (1 + 0)$$

$$S = (e^3 + 2)$$

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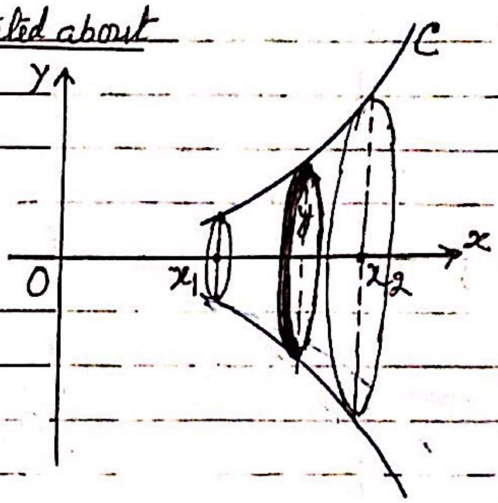
§ The Surface area when an arc is rotated about x-axis through 360° (2π radians):

$\delta s = 2\pi y\, \delta s$



(i) Given $y = f(x)$

Surface area $S = \int_{x_1}^{x_2} 2\pi y \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

(ii) For Parametric curve; $x = f(t)$, $y = g(t)$

$$S = \int_{t_1}^{t_2} 2\pi y \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 19: A curve has equation $y = \cosh x$, for $0 \le x \le \frac{1}{2}$. Find in terms of π and e, the area of the surface generated when the curve is rotated through 2π radians about the x-axis.

[W-20/22/Q2]

Solution: Given $y = \cosh x$ ---(i) ; $0 \le x \le \frac{1}{2}$

Diff. w.r.t x; $\frac{dy}{dx} = \sinh x$ ---(ii)

Surface Area $S = \int_0^{\frac{1}{2}} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$S = \int_0^{\frac{1}{2}} 2\pi \cdot \cosh x \cdot \sqrt{1+\sinh^2 x}\, dx$

$= 2\pi \int_0^{1/2} \cosh x \cdot \sqrt{\cosh^2 x}\, dx$ $\qquad \boxed{1+\sinh^2 x = \cosh^2 x}$

$S = \frac{2\pi}{2} \times \int_0^{\frac{1}{2}} 2\cosh^2 x\, dx$ $\qquad \boxed{1+\cosh 2x = 2\cosh^2 x}$

$= \pi \cdot \int_0^{\frac{1}{2}} (1+\cosh 2x)\, dx$

$= \pi \cdot \left[x + \frac{1}{2}\sinh 2x\right]_0^{\frac{1}{2}}$ $\qquad \boxed{\sinh x = \frac{e^x - e^{-x}}{2}}$

$= \pi\left[\frac{1}{2} + \frac{1}{2}\sinh 1 - 0\right] = \frac{\pi}{2}\left[1 + \frac{e - e^{-1}}{2}\right] = \frac{\pi}{4}\left[2 + e - e^{-1}\right]$ ✓

Example 20: The curve C has parametric equations:

$$x = 2\cosh t, \quad y = \frac{3}{2}t - \frac{1}{4}\sinh 2t, \text{ for } 0 \le t \le 1.$$

(a) Find $\frac{dx}{dt}$ and show that, $\frac{dy}{dt} = 1 - \sinh^2 t$. ---[3]

The area of the surface generated when C is rotated through 2π radians about the x-axis is denoted by A.

(b) (i) Show that $A = \pi \int_0^1 \left(\frac{3}{2}t - \frac{1}{4}\sinh 2t\right)(1 + \cosh 2t)\,dt$ ---[4]

(ii) Hence find A in terms of π, $\sinh 2$ and $\cosh 2$. ---[6]

S-21/21/Q8

Solution: $x = 2\cosh t$; $y = \frac{3}{2}t - \frac{1}{4}\sinh 2t$ ---(i)

(a) $\frac{dx}{dt} = 2\sinh t$ ✓

$$\frac{dy}{dt} = \frac{3}{2} - \frac{1}{4} \times \cosh 2t \times 2$$
$$= \frac{3}{2} - \frac{1}{2}\cosh 2t$$
$$= \frac{3}{2} - \frac{1}{2}(1 + 2\sinh^2 t)$$
$$= 1 - \sinh^2 t \; ✓$$

(b) (i) Consider $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$

$$= (2\sinh t)^2 + (1 - \sinh^2 t)^2$$
$$= 4\sinh^2 t + 1 - 2\sinh^2 t + \sinh^4 t$$
$$= \sinh^4 t + 2\sinh^2 t + 1$$
$$= (\sinh^2 t + 1)^2$$
$$= (\cosh^2 t)^2$$
$$= \cosh^4 t \quad \text{----(ii)}$$

Surface Area $A = \int_0^1 2\pi y \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$$A = 2\pi \cdot \int_0^1 \left(\frac{3}{2}t - \frac{1}{4}\sinh 2t\right) \cdot \sqrt{\cosh^4 t}\,dt$$

(from (i) & (ii))

$$= 2\pi \cdot \int_0^1 \left(\frac{3}{2}t - \frac{1}{4}\sinh 2t\right) \cdot \cosh^2 t\,dt$$
$$= 2\pi \int_0^1 \left(\frac{3}{2}t - \frac{1}{4}\sinh 2t\right) \cdot \left(\frac{1 + \cosh 2t}{2}\right)dt$$
$$= \pi \int_0^1 \left(\frac{3t}{2} - \frac{1}{4}\sinh 2t\right)(1 + \cosh 2t)\,dt \quad \text{....(iii)}$$

(b)(ii) from (iii)

$$A = \pi\left[\int_0^1 \frac{3}{2}t(1 + \cosh 2t)\,dt - \frac{1}{4}\int_0^1 \sinh 2t(1 + \cosh 2t)\,dt\right] \quad \text{----(iv)}$$

Now consider;

$$\frac{1}{4}\int_0^1 \sinh 2t(1 + \cosh 2t)\,dt$$
$$= \frac{1}{4}\int_0^1 \frac{1}{2}u\,du \quad \begin{cases} \text{let } 1 + \cosh 2t = u \\ 2\sinh 2t\,dt = du \end{cases}$$
$$= \frac{1}{4} \times \frac{1}{2}\frac{u^2}{2} = \frac{1}{16}\left[(1 + \cosh 2t)^2\right]_0^1$$
$$= \frac{1}{16}\left[(1 + \cosh 2)^2 - (1+1)^2\right]$$
$$= \frac{1}{16}(1 + \cosh 2)^2 - \frac{1}{4} \quad \text{---(v)}$$

Again also consider;

$$\frac{3}{2}\int_0^1 t(1 + \cosh 2t)\,dt$$
$$= \frac{3}{2}\left[\left(t + \frac{1}{2}\sinh 2t\right)\right]_0^1 - \int 1 \times \left(t + \frac{1}{2}\sinh 2t\right)dt$$
$$= \frac{3}{2}\left[\left(t + \frac{1}{2}\sinh 2t\right) - \left(\frac{t^2}{2} + \frac{\cosh 2t}{2 \times 2}\right)\right]_0^1$$
$$= \frac{3}{2}\left[\left(1 + \frac{1}{2}\sinh 2\right) - \left(\frac{1}{2} + \frac{1}{4}\cosh 2 - \frac{1}{4}\right)\right]$$
$$= \frac{3}{2}\left(\frac{1}{2}\sinh 2 - \frac{1}{4}\cosh 2 + \frac{3}{4}\right) \text{---(vi)}$$

from (v) and (vi) in (iv)

$$A = \pi\left[\frac{3}{4}\sinh 2 - \frac{3}{8}\cosh 2 + \frac{9}{8} - \frac{1}{16}(1 + \cosh 2)^2 + \frac{1}{4}\right]$$

$$A = \pi\left[\frac{3}{4}\sinh 2 - \frac{3}{8}\cosh 2 + \frac{11}{8} - \frac{1}{16}(1 + \cosh 2)^2\right] \; ✓$$

Example 21: The curve C has parametric equations:

$$x = e^{t} - 4t + 3, \quad y = 8e^{\frac{1}{2}t}, \quad \text{for } 0 \leq t \leq 2$$

(a) Find in terms of e, the length of C. ---[5]

(b) Find, in terms of π and e, the area of the surface generated when C is rotated through 2π radians about x-axis. ---[5]

[SP-20/02/Q5]

Solution: $x = e^{t} - 4t + 3$ and $y = 8e^{\frac{1}{2}t}$

(a) $\frac{dx}{dt} = e^{t} - 4$ ..(i) / $\frac{dy}{dt} = 4e^{\frac{1}{2}t}$

$$\therefore \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^{t}-4)^2 + (4e^{t/2})^2$$
$$= e^{2t} - 8e^{t} + 16 + 16e^{t}$$
$$= e^{2t} + 8e^{t} + 16$$
$$= (e^{t}+4)^2 \quad \text{---(i)}$$

Arc length $s = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_0^2 \sqrt{(e^{t}+4)^2}\, dt \quad \text{from (i)}$$
$$= \int_0^2 (e^{t}+4)\, dt$$
$$= [e^{t} + 4t]_0^2$$
$$= (e^{2}+8) - 1$$

$\therefore$ length of C $= (e^{2}+7)$ ✓

(b) Surface Area $S = \int_0^2 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$S = 2\pi \int_0^2 8e^{t/2} \cdot (e^{t}+4)\, dt \quad \text{from (i)}$$
$$= 16\pi \int_0^2 \left(e^{3/2 t} + 4e^{t/2}\right) dt$$
$$= 16\pi \left[\frac{2}{3} e^{3/2 t} + 8e^{t/2}\right]_0^2$$
$$= 16\pi \left[\frac{2}{3} e^{3} + 8e - \left(\frac{2}{3} + 8\right)\right]$$
$$= 16\pi \left[\frac{2}{3} e^{3} + 8e - \frac{26}{3}\right] ✓$$

'Note'

Instead of rotating the curve about x-axis.

If the curve is rotated about y-axis, to find the surface area, the radius will be x (instead of y)

(i) In Cartesian form:

$$S = \int_{x_1}^{x_2} 2\pi x \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

and

(ii) In parametric form:

$$S = \int_{t_1}^{t_2} 2\pi x \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

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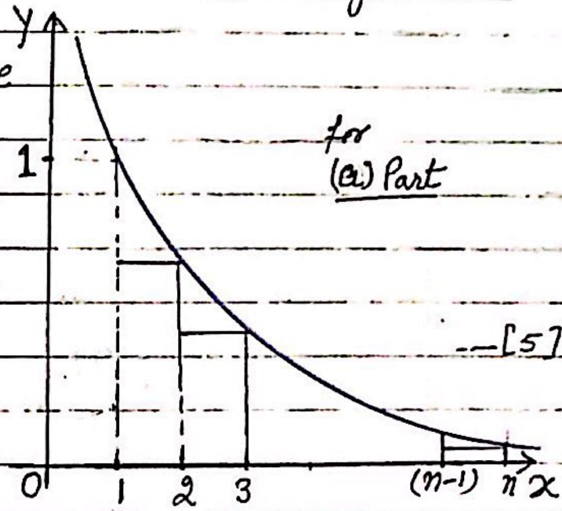
§ Rectangles and the area under a curve (Estimating an area):

Example 22: The diagram shows the curve with equation $y = \frac{1}{x^2}$ for $x > 0$, together with a set of $(n-1)$ rectangles of unit width.



(a) By considering the sum of areas of these rectangles, show that: ---[5]

$$\sum_{r=1}^{n} \frac{1}{r^2} < \frac{2n-1}{n}$$

(b) Use a similar method to find, in terms of n, a lower bound for $\sum_{r=1}^{n} \frac{1}{r^2}$. ---[3]

SP-20/02/Q4

Solution: Curve: $y = \frac{1}{x^2}$ ---(i)

(a) Sum of area of the rectangles below the curve; $S = \frac{1}{2^2} + \frac{1}{3^2} \cdots + \frac{1}{n^2}$ --(ii)

hence

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \int_1^n \frac{1}{x^2}\,dx$$
$$= \int x^{-2}\,dx$$
$$= \left[\frac{-1}{x}\right]_1^n$$

$\therefore \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \left(1 - \frac{1}{n}\right)$

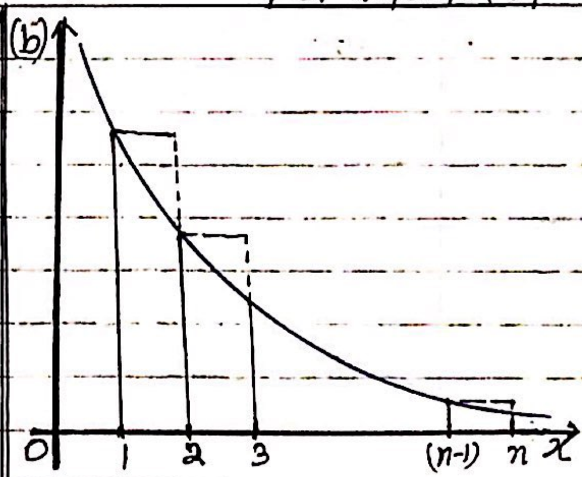
add $\frac{1}{1^2}$ (or 1) on both sides

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

$$\sum_{r=1}^{n} \frac{1}{r^2} < \frac{2n-1}{n} \text{ or } ✓$$

(Upper bound of $\sum_{r=1}^{n} \frac{1}{r^2}$)

(b)



Sum of the areas of the rectangles above the curve:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots \frac{1}{(n-1)^2} > \int_1^n \frac{1}{x^2}\,dx = \left(1 - \frac{1}{n}\right)$$

add $\frac{1}{n^2}$ on both the sides:

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} > 1 - \frac{1}{n} + \frac{1}{n^2}$$

$$\Rightarrow \sum_{r=1}^{n} \frac{1}{r^2} > \frac{n^2 - n + 1}{n} ✓$$

(Lower bound of $\sum_{r=1}^{n} \frac{1}{r^2}$)

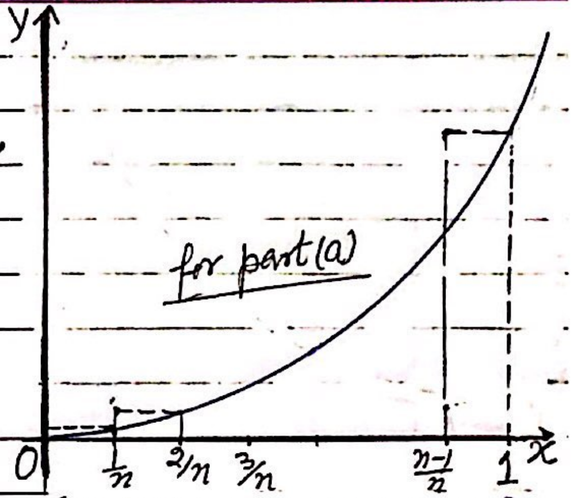
Example 23: The diagram shows the curve with equation $y=x^2$ for $0 \le x \le 1$, together with a set of rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that

$$\int_1^2 x^2\,dx < \frac{2n^2+3n+1}{6n^2}$$

(b) Use a similar method to find, in terms of n, a lower bound for:

$$\int_0^1 x^2\,dx.$$



'1' unit divided in 'n' equal Parts of each '$\frac{1}{n}$' width.

S-20/21/Q4

Solution: The sum of areas of the

(a) rectangles above the curve:

$$\frac{1}{n}\cdot\left(\frac{1}{n}\right)^2+\frac{1}{n}\left(\frac{2}{n}\right)^2+\cdots+\frac{1}{n}\left(\frac{n}{n}\right)^2 > \int_0^1 x^2\,dx$$

$$\Rightarrow \frac{1}{n^3}\left(1^2+2^2+\cdots+n^2\right) > \int_0^1 x^2\,dx$$

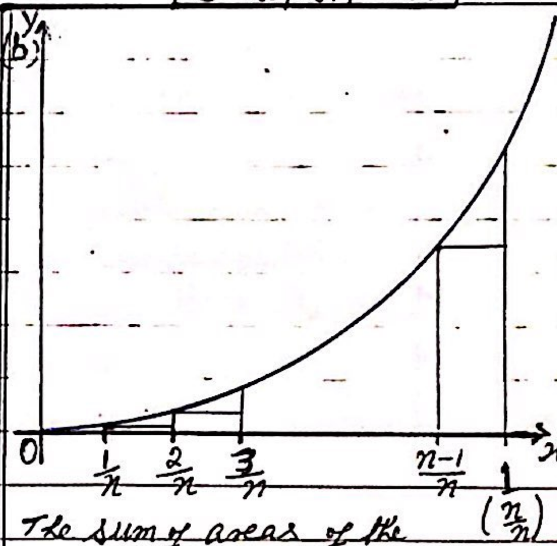
$$\Rightarrow \frac{1}{n^3}\sum_{r=1}^{n} r^2 > \int_0^1 x^2\,dx$$

$$\Rightarrow \frac{1}{n^3}\times\frac{n(n+1)(2n+1)}{6} > \int_0^1 x^2\,dx$$

$$\Rightarrow \int_0^1 x^2\,dx < \frac{(n+1)(2n+1)}{6n^2}$$

$$\Rightarrow \int_0^1 x^2\,dx < \frac{2n^2+3n+1}{6n^2} \checkmark$$

(Shows $\frac{2n^2+3n+1}{6n^2}$ is the upper bound of $\int_0^1 x^2\,dx$)



The sum of areas of the rectangles below the curve:

$$\frac{1}{n}\cdot\left(\frac{1}{n}\right)^2+\frac{1}{n}\left(\frac{2}{n}\right)^2+\cdots\frac{1}{n}\left(\frac{n-1}{n}\right)^2 \quad \int_0^1 x^2\,dx$$

$$\frac{1}{n^3}\left(1^2+2^2+\cdots(n-1)^2\right) < \int_0^1 x^2\,dx$$

or $\int_0^1 x^2\,dx > \frac{1}{n^3}\sum_{r=1}^{(n-1)} r^2$

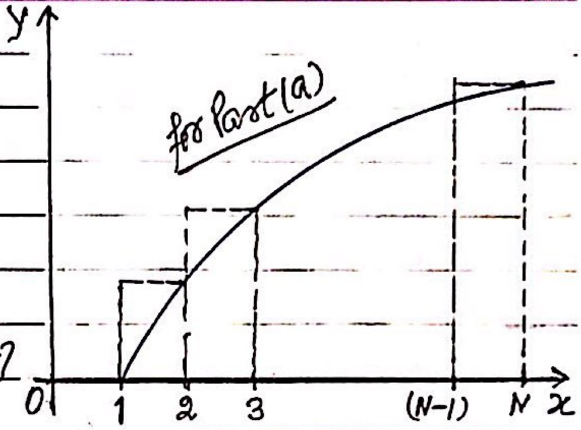
$$\Rightarrow \int_0^1 x^2\,dx > \frac{1}{n^3}\frac{(n-1)\cdot n\cdot(2(n-1)+1)}{6}$$

$$\int_0^1 x^2\,dx > \frac{(n-1)(2n-1)}{6n^2}$$

$$\Rightarrow \int_0^1 x^2\,dx > \frac{2n^2-3n+1}{6n^2} \checkmark$$

($\frac{2n^2-3n+1}{6n^2}$ is the lower bound of $\int_0^1 x^2\,dx$.)

Example 24: The diagram shows the curve with equation $y = \ln x$, for $x \geq 1$, together with a set $(N-1)$ rectangles of unit width.



(a) By considering the sum of areas of these rectangles, show that:

$\ln N! > N\ln N - N + 1$ ---[5]

(b) Use a similar method to find, in terms of N, an upper bound for $\ln N!$ ---[3]

[S-20/23/Q4]

Solution: The sum of the areas of the rectangles above the curve:

$1\times\ln 2 + 1\times\ln 3 + \cdots + 1\times\ln N > \int_1^N \ln x\,dx$

$\ln 2 + \ln 3 + \cdots + \ln N > [\ln x\cdot x - x]_1^N$ (Integrating by parts)

add $\ln 1$ on both

$\ln 1 + \ln 2 + \ln 3 + \cdots + \ln N > \ln 1 + N\ln N - N + 1$

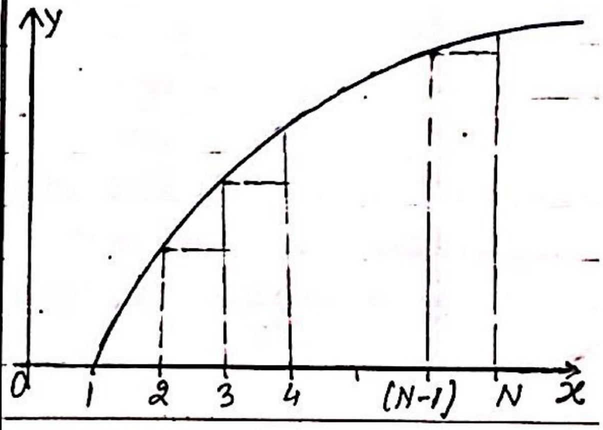
$(\because \ln 1 = 0)$

$\Rightarrow \ln(1\times 2\times\cdots N) < N\ln N - N + 1$

$\Rightarrow \ln N! > (N\ln N - N + 1)$ ✓

(Shows that $(N\ln N - N + 1)$ is the lower bound of $\ln N!$)

Part (b)



The sum of the areas of the rectangles below the curve:

$\ln 2 + \ln 3 + \cdots + \ln(N-1) < \int_1^N \ln x\,dx$

$(\ln 1 = 0)$ add (Integrating by parts)

$\ln 1 + \ln 2 + \ln 3 + \cdots + \ln(N-1) < [\ln x\cdot x - x]_1^N$

add $\ln N$ on both sides

$\Rightarrow \ln 1 + \ln 2 + \cdots + \ln(N-1) + \ln N < (N\ln N - N + 1) + \ln N$

$\Rightarrow \ln(1\cdot 2\cdot 3\cdot \cdots N) < N(\ln N + 1) - N + 1$

$\Rightarrow \ln N! \leq (N(\ln N + 1) - N + 1)$ ✓

(Shows that $(N(\ln N + 1) - N + 1)$ is the upper bound of $\ln N!$)

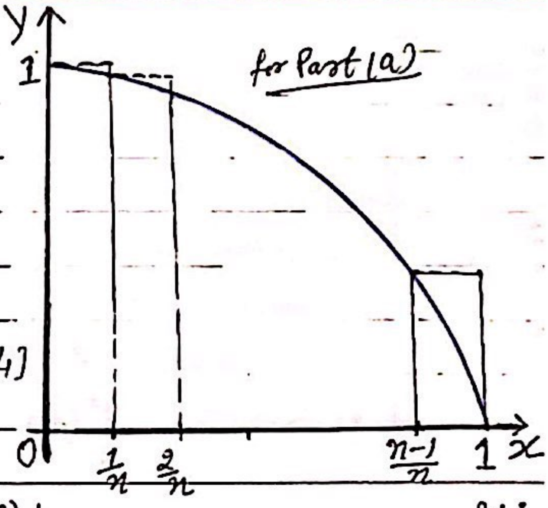
Example 25: The diagram shows the curve with equation $y = 1 - x^3$ for $0 \le x \le 1$, together with a set of n rectangles of width $1/n$.

(a) By considering the sum of the areas of the rectangles, show that:

$$\int_0^1 (1-x^3)\,dx \le \frac{3n^2+2n-1}{4n^2} \quad \text{---[4]}$$

(b) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 (1-x^3)\,dx$. --[4]

[W-20/21/Q4]



Solution: The sum of areas of the rectangles above the curve:

$$\frac{1}{n} + \frac{1}{n}\left(1 - \frac{1}{n^3}\right) + \cdots + \frac{1}{n}\left(1 - \left(\frac{n-1}{n}\right)^3\right) \ge \int_0^1 (1-x^3)\,dx$$

$$\Rightarrow \left(\frac{1}{n} + \frac{1}{n} + \cdots n \text{ term}\right) - \frac{1}{n^4}\left(1^3 + 2^3 + \cdots (n-1)^3\right) \ge \cdots$$

$$\Rightarrow 1 - \frac{1}{n^4}\sum_{r=1}^{n-1} r^3 \ge \int_0^1 (1-x^3)\,dx$$

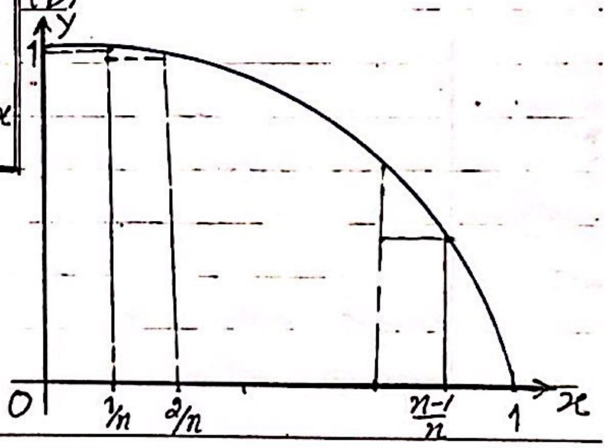
$$\Rightarrow 1 - \frac{1}{n^4} \cdot \frac{(n-1)^2 n^2}{4} \ge \cdots$$

$$\frac{4n^2 - (n-1)^2}{4n^2} \ge \cdots$$

$$\Rightarrow \frac{3n^2+2n-1}{4n^2} \ge \int_0^1 (1-x^3)\,dx \checkmark$$

(here $\left(\frac{3n^2+2n-1}{4n^2}\right)$ is the upper bound of $\int_0^1 (1-x^3)\,dx$)

(b)



The sum of areas of the rectangles below the curve:

$$\frac{1}{n}\left(1 - \frac{1}{n^3}\right) + \frac{1}{n}\left(1 - \frac{2^3}{n^3}\right) \cdots + \frac{1}{n}\left(1 - \frac{(n-1)^3}{n^3}\right) \le \int_0^1 (1-x^3)\,dx$$

$$\Rightarrow \left(\frac{1}{n} + \frac{1}{n} + \cdots (n-1) \text{ term}\right) - \frac{1}{n}\left(\frac{1}{n^3} + \frac{2^3}{n^3} + \cdots \frac{(n-1)^3}{n^3}\right) \le \int_0^1 (1-x^3)\,dx$$

$$\Rightarrow \frac{n-1}{n} - \frac{1}{n^4}\sum_{r=1}^{n-1} r^3 \le \int_0^1 (1-x^3)\,dx$$

$$\Rightarrow \frac{n-1}{n} - \frac{1}{n^4} \times \frac{(n-1)^2 n^2}{4} \le \int_0^1 (1-x^3)\,dx$$

$$\Rightarrow \frac{n-1}{n} - \frac{(n-1)^2}{4n^2} \le \cdots \Rightarrow \frac{4n(n-1) - (n-1)^2}{4n^2} \le \cdots$$

$$\Rightarrow \left(\frac{3n^2-2n-1}{4n^2}\right) \le \int_0^1 (1-x^3)\,dx \checkmark$$

(here $\left(\frac{3n^2-2n-1}{4n^2}\right)$ is the lower bound of $\int_0^1 (1-x^3)\,dx$)

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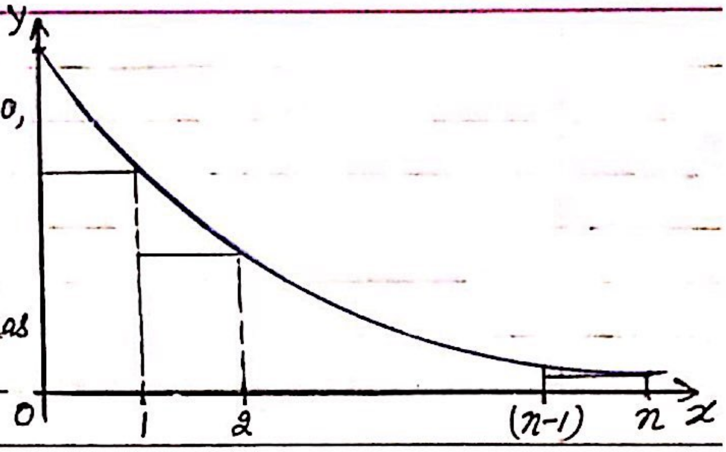


Example 26: The diagram shows the curve $y = \frac{1}{\sqrt{x^2+x+1}}$, for $x \geq 0$, together with a set of n rectangles of unit width. By considering the sum of the areas of these rectangles, show that:

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r^2+r+1}} < \ln\left(\frac{1}{3} + \frac{2}{3}n + \frac{2}{3}\sqrt{n^2+n+1}\right) \quad ----[10]$$

[W-20/22/Q8]

Solution: Sum of the areas of the rectangles below the curve:

$$\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{7}} + ---- + \frac{1}{\sqrt{(n^2+n+1)}} < \int_0^n \frac{1}{\sqrt{x^2+x+1}}\,dx$$

$$\Rightarrow \sum_{r=1}^{n} \frac{1}{\sqrt{r^2+r+1}} < \int_0^n \frac{1}{\sqrt{x^2+x+1}}\,dx \quad ------(i)$$

Consider $x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4}$ ---(ii)

Hence $\int_0^n \frac{1}{\sqrt{x^2+x+1}}\,dx = \int_0^n \frac{1}{\sqrt{(x+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}}$ from (ii)

$$= \left[\sinh^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)\right]_0^n = \left[\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right]_0^n$$

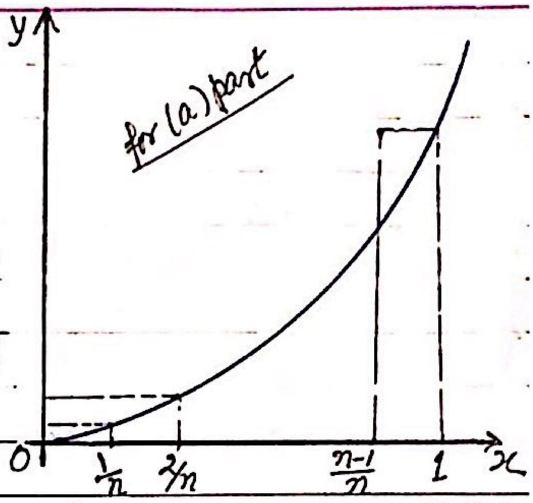
$$= \left[\ln\left(\frac{2x+1}{\sqrt{3}} + \sqrt{\frac{(2x+1)^2+3}{3}}\right)\right]_0^n \quad \left(\because \sinh^{-1}x = \ln\left(x+\sqrt{x^2+1}\right)\right)$$

$$= \ln\left[\frac{2x+1+2\sqrt{x^2+x+1}}{\sqrt{3}}\right]_0^n = \ln\left(\frac{2n+1+2\sqrt{n^2+n+1}}{\sqrt{3}}\right) - \ln\left(\frac{3}{\sqrt{3}}\right)$$

$$= \ln\left[\frac{(2n+1+2\sqrt{n^2+n+1})}{\sqrt{3}} \times \frac{1}{\sqrt{3}}\right] = \ln\left(\frac{1}{3} + \frac{2}{3}n + \frac{2}{3}\sqrt{n^2+n+1}\right) --(iii)$$

from (i) and (iii) $\Rightarrow \sum_{r=1}^{n} \frac{1}{\sqrt{r^2+r+1}} < \ln\left(\frac{1}{3} + \frac{2}{3}n + \frac{2}{3}\sqrt{n^2+n+1}\right)$ ✓

Example 27: The diagram shows the curve with equation $y = x^3$ for $0 \leq x \leq 1$, together with a set of n rectangles width $1/n$.



(a) By considering the sum of the areas of these rectangles, show that:

$$\int_0^1 x^3\,dx < U_n \;;\; \text{where } U_n = \left(\frac{n+1}{2n}\right)^2.$$

[4]

(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 x^3\,dx$. --[4]

S-21/21/Q3

(C) Find the least value of n such that:

$$U_n - L_n < 10^{-3}$$

---[2]

Solution: The sum of areas of the

(a) rectangles above the curve:

$$\frac{1}{n}\times\frac{1}{n^3} + \frac{1}{n}\times\left(\frac{2}{n}\right)^3 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^3 > \int_0^1 x^3\,dx$$

$$\Rightarrow \frac{1}{n^4}\left[1^3 + 2^3 + \cdots + n^3\right] > \int_0^1 x^3\,dx$$

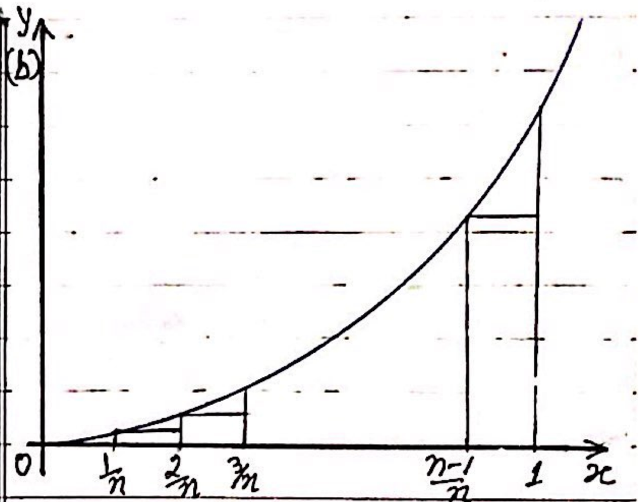
$$\Rightarrow \frac{1}{n^4}\sum_{r=1}^{n} r^3 > \int_0^1 x^3\,dx$$

$$\Rightarrow \int_0^1 x^3\,dx < \frac{1}{n^4}\times\left(\frac{n(n+1)}{2}\right)^2$$

$$\Rightarrow \int_0^1 x^3\,dx < \frac{(n+1)^2}{n^2 \times 2^2}$$

$$\Rightarrow \int_0^1 x^3\,dx < \left(\frac{n+1}{2n}\right)^2$$

$$\therefore \Rightarrow \int_0^1 x^3\,dx < U_n \;:\; U_n = \left(\frac{n+1}{2n}\right)^2 \text{---(i)}$$



(b) The sum of areas of the rectangles below the curve:

$$\int_0^1 x^3\,dx > \frac{1}{n}\left(\frac{1}{n}\right)^3 + \frac{1}{n}\left(\frac{2}{n}\right)^3 + \cdots + \frac{1}{n}\left(\frac{n-1}{n}\right)^3$$

$$\Rightarrow \int_0^1 x^3\,dx > \frac{1}{n^4}\sum_{r=1}^{n-1} r^3 = \frac{(n-1)^2 n^2}{n^4 \times 4}$$

$$\Rightarrow \int_0^1 x^3\,dx > \left(\frac{n-1}{2n}\right)^2 = L_n \;\cdots\text{(ii)}$$

(lower bound)

(C) Now $U_n - L_n < 10^{-3}$ (Given)

$$\Rightarrow \left(\frac{n+1}{2n}\right)^2 - \left(\frac{n-1}{2n}\right)^2 < \frac{1}{1000} \text{ (from (i) \& (ii))}$$

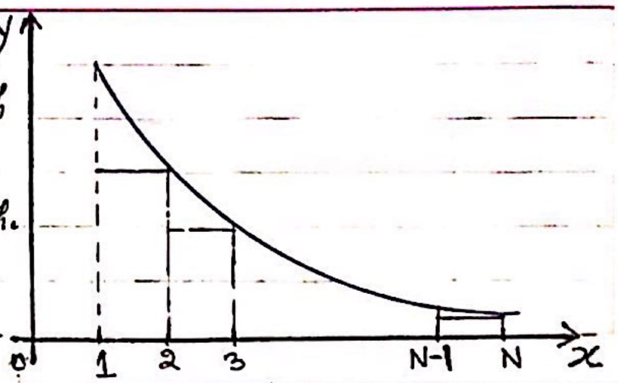
$$\Rightarrow \frac{(n^2+2n+1) - (n^2 - 2n + 1)}{4n^2} < \frac{1}{1000}$$

$$\Rightarrow \frac{1}{n} < \frac{1}{1000} \Rightarrow n > 1000$$

$\therefore$ Least Value of $n = 1001$ ✓

Note: $\sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$

Example 28: The diagram shows the curve, $y = \frac{x}{2x^2-1}$ for $x \geq 1$, together with a set of N-1 rectangles of unit width.



(a) By consider the sum of the areas of these rectangles, show that:

$$\sum_{r=1}^{N} \frac{r}{2r^2-1} < \frac{1}{4}\ln(2N^2-1)+1 \quad \text{---[7]}$$

(b) Use a similar method to find, in terms of N, a lower bound for:

$$\sum_{r=1}^{N} \frac{r}{2r^2-1} \quad \text{---[3]}$$

S-21 | 23 | Q3

Solution: The sum of the areas of the
(a) rectangles below the curve:

$$1\times\left(\frac{2}{2(2)^2-1}\right)+1\times\left(\frac{3}{2\times3^2-1}\right)+\dots+1\times\left(\frac{N}{2N^2-1}\right) < \int_1^N \frac{x}{2x^2-1}\,dx$$

$$\Rightarrow \sum_{r=2}^{N} \frac{r}{(2r^2-1)} < \int_1^N \frac{x}{2x^2-1}\,dx$$

add 1 on both the sides:

$$\Rightarrow 1+\sum_{r=2}^{N} \frac{r}{(2r^2-1)} < 1+\int_1^N \frac{x}{2x^2-1}\,dx$$

$$\Rightarrow \sum_{r=1}^{N} \frac{r}{(2r^2-1)} < 1+\int_1^N \frac{x}{(2x^2-1)}\,dx \quad \text{--(i)}$$

Consider $\int_1^N \frac{x}{(2x^2-1)}\,dx = \frac{1}{4}\int_1^N \frac{4x}{(2x^2-1)}\,dx$

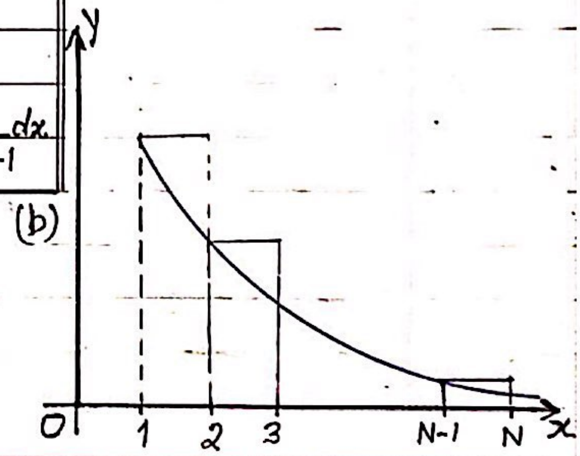
$$= \frac{1}{4}\left[\ln(2x^2-1)\right]_1^N$$

$$= \frac{1}{4}\ln(2N^2-1) \quad \text{--(ii)}$$

Put from (ii) in (i) we get:

$$\sum_{r=1}^{N} \frac{r}{(2r^2-1)} < 1+\frac{1}{4}\ln(2N^2-1) \checkmark$$

(b)



The sum of areas of the rectangles above the curve:

$$\frac{1\times1}{(2\cdot1^2-1)}+\frac{1\times2}{(2\cdot2^2-1)}+\dots+\frac{1\times N-1}{(2(N-1)^2-1)} > \int_1^N \frac{x}{2x^2-1}\,dx$$

add $\frac{N}{(2N^2-1)}$ on both the sides

$$\Rightarrow \sum_{r=1}^{N} \frac{r}{(2r^2-1)} > \int_1^N \frac{x}{2x^2-1}\,dx + \frac{N}{(2N^2-1)}$$

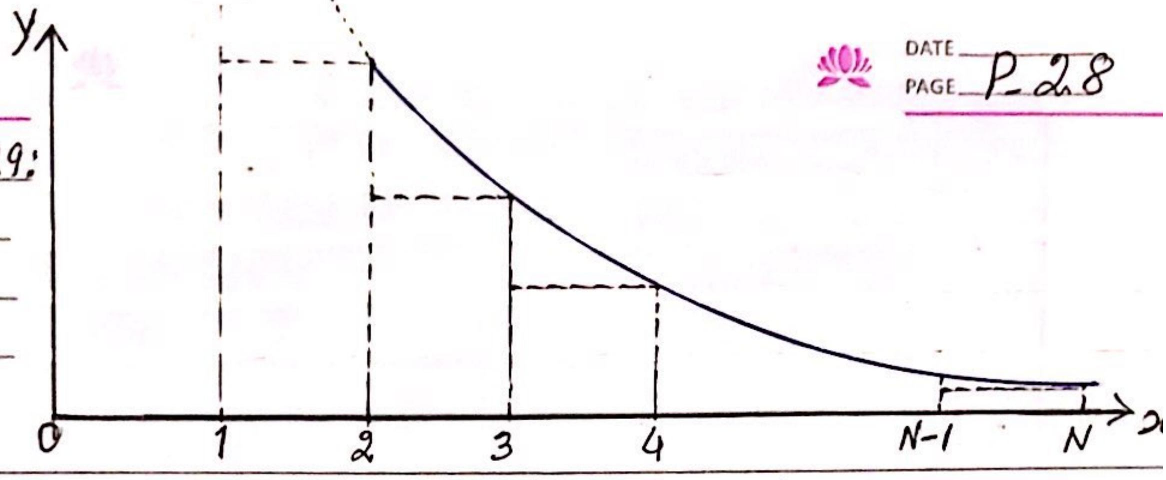
using (ii)

$$\Rightarrow \sum_{r=1}^{N} \frac{r}{(2r^2-1)} > \frac{1}{4}\ln(2N^2-1) + \frac{N}{(2N^2-1)}$$

Hence the lower bound of $\sum_{r=1}^{N} \frac{r}{(2r^2-1)}$ is

$$\frac{1}{4}\ln(2N^2-1) + \frac{N}{(2N^2-1)} \checkmark$$

Example 29:



The diagram shows the curve with equation $y = \frac{\ln x}{x^2}$ for $x \geq 2$, together with a set of $(N-2)$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that:

$$\sum_{r=1}^{N} \frac{\ln r}{r^2} < \frac{2+3\ln 2}{4} - \frac{1+\ln N}{N} \quad ---[7]$$

(b) Use a similar method to find, in terms of N, a lower bound for

$$\sum_{r=1}^{N} \frac{\ln r}{r^2} \quad ---[3]$$

[W-21/21/Q4]

Solution: $\sum_{r=1}^{N} \frac{\ln r}{r^2} = \frac{\ln 1}{1^2} + \frac{\ln 2}{2^2} + \sum_{r=3}^{N} \frac{\ln r}{r^2}$

$= 0 + \frac{\ln 2}{4} + \sum_{r=3}^{N} \frac{\ln r}{r^2}$ using by parts:

$< \frac{\ln 2}{4} + \int_2^N \frac{\ln x}{x^2}\,dx \rightarrow \int_2^N \frac{\ln x}{x^2}dx = \ln x \cdot \frac{x^{-1}}{-1} - \int \frac{1}{x} \times \frac{x^{-1}}{-1}dx$

$= -\frac{\ln x}{x} + \int x^{-2}dx$

$= \left[-\frac{\ln x}{x} + \left(\frac{x^{-1}}{-1}\right)\right]_2^N$

$= \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_2^N$ ①

$= -\left[\frac{\ln x + 1}{x}\right]_2^N = \left[-\left(\frac{\ln N+1}{N}\right) + \frac{\ln 2+1}{2}\right]$

$\sum_{r=1}^{N} \frac{\ln r}{r^2} < \frac{\ln 2}{4} + \left[-\left(\frac{\ln N+1}{N}\right) + \frac{\ln 2+1}{2}\right]$

$= \frac{2+3\ln 2}{4} - \left(\frac{1+\ln N}{N}\right)$ ✓

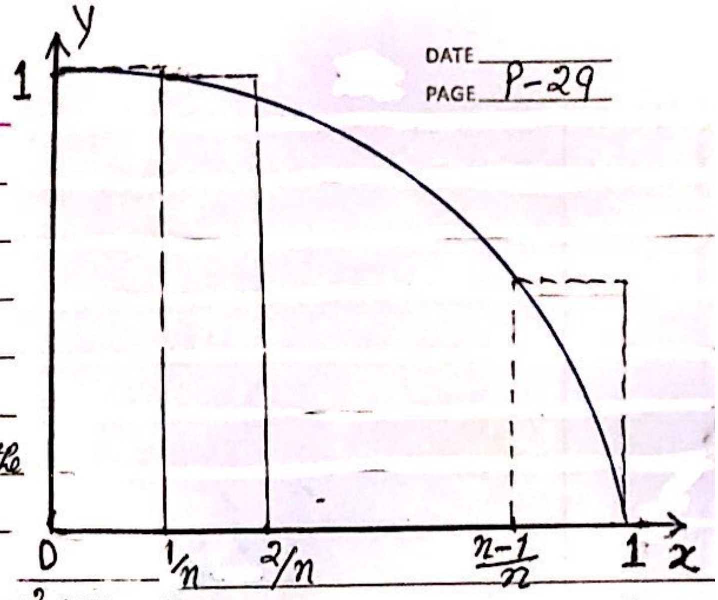
(upper bound)

Now for lower bound:

$\sum_{r=1}^{N} \frac{\ln r}{r^2} = \frac{\ln 1}{2} + \sum_{r=2}^{N-1} \frac{\ln r}{r^2} + \frac{\ln N}{N^2} > 0 + \frac{\ln N}{N^2} + \int_2^N \frac{\ln x}{x^2}dx$

$= \frac{\ln N}{N^2} - \left(\frac{\ln N+1}{N}\right) + \frac{(\ln 2+1)}{2}$

$\Rightarrow \sum_{r=1}^{N} \frac{\ln r}{r^2} = \frac{\ln 2+1}{2} - \frac{(\ln N+1)}{2} + \frac{\ln N}{N^2}$



Example 30: The diagram shows the curve with equation: $y = 1-x^2$ for $0 \leq x \leq 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of the rectangles, show that: $\int_0^1 (1-x^2)\,dx < \frac{4n^2+3n-1}{6n^2}$ ---[4]

(b) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 (1-x^2)\,dx$ |W-21|22|Q3| ---[4]

Solution:
(a) $\int_0^1 (1-x^2)\,dx <$ sum of areas of the n rectangles

$$= \frac{1}{n} + \frac{1}{n}\left(1-\frac{1}{n^2}\right) + \cdots + \frac{1}{n}\left(1-\frac{(n-1)^2}{n^2}\right)$$

$$= \left(\frac{1}{n}+\frac{1}{n}+\cdots\frac{1}{n}\right) - \frac{1}{n^3}\sum_{r=1}^{n-1} r^2$$

$$= 1 - \frac{(n-1)n.(2n-1)}{6n^3}$$

$$= \frac{6n^2-(n-1)(2n-1)}{6n^2} = \frac{4n^2+3n-1}{6n^2}$$ ✓

To find the lower bound:
(b) $\int_0^1 (1-x^2)\,dx >$ sum of areas of the $(n-1)$ rectangles (from $x = \frac{1}{n}, \frac{2}{n}, \cdots 1$)

$$= \frac{1}{n}\left(1-\frac{1}{n^2}\right) + \frac{1}{n}\left(1-\frac{2^2}{n^2}\right) + \cdots + \frac{1}{n}\left(1-\frac{(n-1)^2}{n^2}\right)$$

$$= \left(\frac{1}{n}+\frac{1}{n}+\cdots(n-1)\text{ terms}\right) - \frac{1}{n^3}\sum_{r=1}^{n-1} r^2$$

$$= \frac{n-1}{n} - \frac{1}{n^3}\frac{(n-1)n(2n-1)}{6}$$

$$= \frac{6(n-1)-(n-1)(2n-1)}{6n^2}$$

$$= \frac{4n^2-3n-1}{6n^2}$$ ✓ is the lower bound

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Example 31: The curve C has parametric equations:

$x = 3t + 2t^{-1} + at^3$, $y = 4t - \frac{3}{2}t^{-1} + bt^3$, for $1 \le t \le 2$

where a and b are constants.

(a) It is given that $a = 2/3$ and $b = -\frac{1}{2}$; Show that: --[6]

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{25}{4}(t^2 + t^{-2})^2$$ and find the exact length of C.

(b) It is given instead that $a = 0, b = 0$; find the value of $\frac{d^2y}{dx^2}$ for $t = 1$, --[4]

W-21/22/Q5

Solution (a) $x = 3t + 2t^{-1} + \frac{2}{3}t^3$ and $y = 4t - \frac{3}{2}t^{-1} - \frac{1}{2}t^3$ (for $a = \frac{2}{3}$; $b = -\frac{1}{2}$)

$$\frac{dx}{dt} = 3 - 2t^{-2} + 2t^2 \; ; \; \frac{dy}{dt} = 4 + \frac{3}{2}t^{-2} - \frac{3}{2}t^2$$

$$\therefore \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3 - 2t^{-2} + 2t^2)^2 + \left(4 + \frac{3}{2}t^{-2} - \frac{3}{2}t^2\right)^2$$

$$= \left(9 + \frac{4}{t^4} + 4t^4 - \frac{12}{t^2} - 8 + 12t^2\right) + \left(16 + \frac{9}{4t^4} + \frac{9t^4}{4} + \frac{12}{t^2} - \frac{9}{2} - 12t^2\right)$$

$$= \frac{25}{4}t^4 + \frac{25}{4}t^{-4} + \frac{25}{2}$$

$$= \frac{25}{4}\left[t^4 + t^{-4} + 2\right] = \underline{\frac{25}{4}(t^2 + t^{-2})^2}$$

Length of arc C.

$$\int_1^2 \sqrt{\frac{25}{4}(t^2 + t^{-2})^2} = \frac{5}{2}\int_1^2 (t^2 + t^{-2})\,dt$$

$$\frac{5}{2}\left[\frac{1}{3}t^3 - t^{-1}\right]_1^2 = \underline{85/12}$$

(b) for $a = b = 0$; $x = 3t + 2t^{-1}$; $y = 4t - \frac{3}{2}t^{-1}$

$$\frac{dx}{dt} = 3 - 2t^{-2} \; ; \; \frac{dy}{dt} = 4 + \frac{3}{2}t^{-2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4 + \frac{3}{2}t^{-2}}{3 - 2t^{-2}}$$

$$\because \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$\text{Now } \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{(3 - 2t^{-2})(-3t^{-3}) - (4 + \frac{3}{2}t^{-2})(4t^{-3})}{(3 - 2t^{-2})^2}$$

$$= \frac{-9t^{-3} + \cancel{6t^{-5}} - 16t^{-3} - \cancel{6t^{-5}}}{(3 - 2t^{-2})^2} = \frac{-25t^{-3}}{(3 - 2t^{-2})^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{1}{\frac{dx}{dt}} = \frac{-25t^{-3}}{(3 - 2t^{-2})^2} \times \frac{1}{(3 - 2t^{-2})}$$

$$= \frac{-25t^{-3}}{(3 - 2t^{-2})^3}$$

at $t = 1$,

$$\therefore \underline{\frac{d^2y}{dx^2} = -25}$$

Example 32 (i) Using the substitution $u = \tanh x$, find $\int \operatorname{sech}^2 x \tanh^2 x \, dx$ ---[2]

(ii) It is given that, for $n \geq 0$, $I_n = \int_0^{\ln 3} \operatorname{sech}^n x \tanh^2 x \, dx$
Show that, for $n \geq 2$
$$(n+1) I_n = \left(\frac{4}{5}\right)^3 \cdot \left(\frac{3}{5}\right)^{n-2} + (n-2) I_{n-2} \quad \text{---[5]}$$

W-21/22 | Q8(b)(c)

Solution (i) $\int \operatorname{sech}^2 x \cdot \tanh^2 x \, dx$ $\quad \left\{ \begin{array}{l} \text{Let } u = \tanh x \\ du = \operatorname{sech}^2 x \, dx \end{array} \right.$

$$= \int u^2 \, du = \frac{1}{3} u^3 = \frac{1}{3} \tanh^3 x + C$$

(ii) Given $I_n = \int_0^{\ln 3} \operatorname{sech}^n x \cdot \tanh^2 x \, dx$
$$= \int_0^{\ln 3} \operatorname{sech}^{n-2} dx \cdot \operatorname{sech}^2 x \tanh^2 x \, dx$$

using by parts:
$$= \operatorname{sech}^{n-2} x \cdot \int \operatorname{sech}^2 x \cdot \tanh^2 x \, dx - \int \left(\frac{d}{dx} \operatorname{sech}^{n-2} x \cdot \int \operatorname{sech}^2 x \tanh^2 x \, dx \right) dx$$

(from Part (i))
$$I_n = \left[\operatorname{sech}^{n-2} x \cdot \frac{1}{3} \tanh^3 x \right]_0^{\ln 3} - \int \left((n-2) \cdot \operatorname{sech}^{n-3} x \cdot (-\operatorname{sech} x \tanh x) \cdot \frac{1}{3} \tanh^3 x \right) dx$$

$$I_n = \frac{1}{3} \left(\frac{3}{5}\right)^{n-2} \cdot \left(\frac{4}{5}\right)^3 + \frac{1}{3}(n-2) \int \operatorname{sech}^{n-2} \tanh^4 x \, dx$$

$$= \frac{1}{3} \left(\frac{3}{5}\right)^{n-2} \cdot \left(\frac{4}{5}\right)^3 + \frac{1}{3}(n-2) \int \operatorname{sech}^{n-2} x \left(1 - \operatorname{sech}^2 x\right) \cdot \tanh^2 x \, dx$$

$$\left\{ \begin{array}{l} \operatorname{sech}(\ln 3) = \dfrac{2}{3 + \frac{1}{3}} = 3/5 \\ \tanh^2(\ln 3) = 1 - \operatorname{sech}^2(\ln 3) \\ \qquad = 1 - \left(\frac{3}{5}\right)^2 = \left(\frac{4}{5}\right)^2 \\ \tanh 0 = 0 \end{array} \right.$$

$$= \frac{1}{3} \left(\frac{3}{5}\right)^{n-2} \left(\frac{4}{5}\right)^3 + \frac{1}{3}(n-2) \left[\int_0^{\ln 3} \left(\operatorname{sech}^{n-2} x \tanh^2 x \, dx - \int \operatorname{sech}^n x \cdot \tanh^2 x \right) dx \right.$$

$$I_n = \text{-- -- -- --} + \frac{1}{3}(n-2)\left(I_{n-2} - I_n\right)$$

$$\Rightarrow I_n \left(1 + \frac{1}{3}(n-2)\right) = \frac{1}{3} \left(\frac{3}{5}\right)^{n-2} \left(\frac{4}{5}\right)^3 + \frac{1}{3}(n-2) I_{n-2}$$

$$\Rightarrow (n+1) I_n = \left(\frac{3}{5}\right)^{n-2} \cdot \left(\frac{4}{5}\right)^3 + (n-2) I_{n-2}$$

$$\left(\operatorname{sech} x = \frac{2}{e^x + e^{-x}} \right)$$

Example 33: The curve C has parametric equations:

$x = 2\cosh 2t + 3t$; $y = \frac{3}{2}\cosh 2t - 4t$ for $-\frac{1}{2} \le t \le \frac{1}{2}$

The area of the surface when C is rotated through 2π radians about the **y-axis** is denoted by A.

(i) Show that $A = 10\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\cosh 2t + 3t)\cosh 2t\, dt$ --- [4]

(ii) Hence find A in terms of π and e. [W-21/21/Q8(b)] - [7]

Solution: When the curve is rotated about y-axis, the surface area:

(i) $A = \int_{-\frac{1}{2}}^{\frac{1}{2}} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ --- (1) [Equation of the curve is given in parametric form.

$x = 2\cosh 2t + 3t$ and $y = \frac{3}{2}\cosh 2t - 4t$

$\frac{dx}{dt} = 4\sinh 2t + 3$; $\frac{dy}{dt} = 3\sinh 2t - 4$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (4\sinh 2t + 3)^2 + (3\sinh 2t - 4)^2 = 25(\sinh^2 2t + 1) = 25\cosh^2 2t$

from (1) $\therefore A = \int_{-\frac{1}{2}}^{\frac{1}{2}} 2\pi(2\cosh 2t + 3)\cdot\sqrt{25\cosh^2 2t}\, dt$

$= 2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\cosh 2t + 3)\cdot 5\cosh 2t\, dx$

$A = 10\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\cosh^2 2t + 3t)\cosh 2t\, dt$ --- (2)

(ii) Consider

$10\pi \int 2\cosh^2 2t\, dt$

$= 10\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} (1 + \cosh 4t)\, dt$

$= 10\pi \left[t + \frac{\sinh 4t}{4}\right]$ --- (3)

Again consider $30\pi \int t\cosh 2t\, dt$ (By part)

$= 30\pi \left[t\cdot\frac{\sinh 2t}{2} - \int 1\cdot\frac{\sinh 2t}{2}\, dt\right]$

$= 15\pi\left[t\sinh 2t - \frac{\cosh 2t}{2}\right]$ --- (4)

from (3) and (4) in (2)

$A = 10\pi\left[t + \frac{\sinh 4t}{4}\right]_{-\frac{1}{2}}^{\frac{1}{2}} + 15\pi\left[t\sinh 2t - \frac{\cosh 2t}{2}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$

$= 10\pi\left[1 + \frac{1}{4}(e^2 - e^{-2})\right] + 0$

$\therefore A = 10\pi\left[1 + \frac{e^2 - e^{-2}}{4}\right]$