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FP-2

Further Pure Maths 2

Matrices 1
Notes and Revision

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FP.2

Matrices 1 - Notes and Revision

Matrix 'A', eigenvector 'e', eigenvalue 'λ' and characteristic equation:

Example: Consider a matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ and a Vector $e = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$Ae = \lambda e$

$\det(A - \lambda I) = 0$ To find eigen values

Now $Ae = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1\times1 + 2\times2 \\ 4\times1 + 3\times2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (or $Ae = \lambda e$)

we observe that when matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ is multiplied with vector $e = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ we get a parallel vector '5e'

here 5 is the eigenvalue 'λ' ($\lambda = 5$)

$e = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the eigenvector.

Now $Ae = \lambda e \Rightarrow Ae = \lambda I \cdot e$

$\Rightarrow (A - \lambda I)\cdot e = 0 \Rightarrow (A - \lambda I)^{-1}$ does not exists (as $e \neq 0$)

$\Rightarrow \det(A - \lambda I) = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$

Characteristic equation:

$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$ ✓

$(\lambda - 5)(\lambda + 1) = 0$

$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}$

Eigen values: $\lambda_1 = 5,\ \lambda_2 = -1$.

we know that for $e_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ the eigenvalue is $\lambda_1 = 5$ ✓

Now to find eigen vector e_2 for $\lambda_2 = -1$

$Ae_2 = \lambda_2 e_2 \Rightarrow \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$

Let $e_2 = \begin{pmatrix} x \\ y \end{pmatrix} \Longrightarrow$

$\Rightarrow \left.\begin{array}{l} x + 2y = -x \\ 4x + 3y = -y \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2y = -2x \\ y = -4x \end{array}\right\}$

$\Rightarrow y = -x$

$\Rightarrow x = 1,\ y = -1$

for eigenvalue $\lambda_2 = -1$, eigenvector $\rightarrow e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ✓

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§ To find the eigenvalues and eigenvector given a matrix. $\boxed{P\text{-}2}$

Example 1: Find the eigenvalues and the corresponding eigenvectors of the matrix A,

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 3 & 2 & -3 \\ 1 & 1 & 2 \end{pmatrix}$$

---[6]

[W-17/11/Q11(ii)]

Solution: $A = \begin{pmatrix} 0 & 1 & 3 \\ 3 & 2 & -3 \\ 1 & 1 & 2 \end{pmatrix} \Rightarrow A - \lambda I = \begin{pmatrix} 0 & 1 & 3 \\ 3 & 2 & -3 \\ 1 & 1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$

$$= \begin{pmatrix} -\lambda & 1 & 3 \\ 3 & 2-\lambda & -3 \\ 1 & 1 & 2-\lambda \end{pmatrix}$$

For eigenvalues:

$$\text{Det}(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 3 \\ 3 & 2-\lambda & -3 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$= -\lambda[(2-\lambda)^2 + 3] - 1[3(2-\lambda)+3] + 3[3-(2-\lambda)] = 0$$

$\therefore$ Characteristic equation:

$$\Rightarrow \lambda^3 - 4\lambda^2 + \lambda + 6 = 0 \Rightarrow$$

$$\Rightarrow (\lambda+1)(\lambda^2 - 5\lambda + 6) = 0$$

$$\Rightarrow (\lambda+1)(\lambda-2)(\lambda-3) = 0$$

$$\lambda = -1, 2, 3$$

$\therefore$ Eigenvalues are $-1, 2, 3$, ✓

Let $e = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$A \cdot e = \lambda e$$

$$\begin{pmatrix} 0 & 1 & 3 \\ 3 & 2 & -3 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} y + 3z &= \lambda x \\ 3x + 2y - 3z &= \lambda y \\ x + y + 2z &= \lambda z \end{aligned}\right\} \text{--- } (1)$$

Now for $\lambda = -1 \Rightarrow$ in (1)
$$\left\{\begin{aligned} y + 3z &= -x \\ 3x + 2y - 3z &= -y \\ x + y + 2z &= -z \end{aligned}\right. \Rightarrow$$

$$x + y + 3z = 0 \text{ --- } (2)$$
$$3x + 3y - 3z = 0 \text{ --- } (3)$$
$$x + y + 3z = 0 \text{ --- } (4)$$

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$$\begin{vmatrix} i & j & k \\ 1 & 1 & 3 \\ 3 & 3 & -3 \end{vmatrix} = \begin{pmatrix} -12 \\ 12 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\therefore$ for $\lambda = -1 \Rightarrow$ eigenvector $\rightarrow e = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ ✓

Similarly for $\lambda = 2 \Rightarrow e = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ ✓ (from (1))

And for $\lambda = 3$, $e = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ✓ (Put in (1))

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§ Given eigenvector for A, to find the eigenvalue and vice-versa;

Example 2: The matrix A is given by:
$$A = \begin{pmatrix} 6 & -8 & 7 \\ 7 & -9 & 7 \\ 6 & -6 & 5 \end{pmatrix}$$

(i) Given that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of A. find the corresponding value of eigenvalue. ---[2]

(ii) Given that -1 is an eigenvalue of A, find the corresponding eigenvector. [S-17/13/Q10] ---[2]

Solution (i) Given $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of A.

$$\Rightarrow \quad Ae = \begin{pmatrix} 6 & -8 & 7 \\ 7 & -9 & 7 \\ 6 & -6 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6-8+0 \\ 7-9+0 \\ 6-6+0 \end{pmatrix}$$

$$\Rightarrow \quad Ae = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \lambda e$$
$$\Rightarrow \lambda = -2 \checkmark$$

(ii) To find eigenvector of e, given eigenvalue $\lambda = -1$

let $e = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Now $Ae = \lambda e \Rightarrow \begin{pmatrix} 6 & -8 & 7 \\ 7 & -9 & 7 \\ 6 & -6 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\left.\begin{aligned} 6x - 8y + 7z &= -x \\ 7x - 9y + 7z &= -y \\ 6x - 6y + 5z &= -z \end{aligned}\right\} \Rightarrow \begin{aligned} 7x - 8y + 7z &= 0 \quad (1) \\ 7x - 8y + 7z &= 0 \quad (2) \\ 6x - 6y + 6z &= 0 \quad (3) \end{aligned}$$

Using cross multiplication $\boxed{\text{⊗ Page 8}}$

$$\begin{vmatrix} i & j & k \\ 7 & -8 & 7 \\ 6 & -6 & 6 \end{vmatrix} = \begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

∴ Now $e = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \checkmark$ for $\lambda = -1$

Alternate method:

Given eigenvalue of A is λ:

$\Rightarrow Ae = \lambda e \Rightarrow Ae = \lambda I e$

$\Rightarrow \boxed{(A - \lambda I)e = 0}$

for $\lambda = -1$

$$\left(\begin{pmatrix} 6 & -8 & 7 \\ 7 & -9 & 7 \\ 6 & -6 & 5 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right) e = 0$$

$$\Rightarrow \begin{pmatrix} 7 & -8 & 7 \\ 7 & -8 & 7 \\ 6 & -6 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} 7x - 8y + 7z &= 0 \quad (1) \\ 7x - 8y + 7z &= 0 \quad (2) \\ 6x - 6y + 6z &= 0 \quad (3) \end{aligned}\right\}⊗$$

Proceed like →

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§ (i) Given a matrix A with eigenvector e and eigenvalue λ.
And a matrix $B=(A+KI)$ Then (i) B has the same eigenvector e
(ii) B has eigen value $\lambda+K$

Proof: $Be=(A+KI)e = Ae+KIe = \lambda e+ke = (\lambda+k)e$

§ (ii) A has eigenvector e and eigenvalue λ and
B has eigenvector e and eigenvalue μ, Then:

$$(A+B)e = (\lambda+\mu)e.$$

Proof:

$$Ae=\lambda e \quad \text{and} \quad Be=\mu e$$

$$\Rightarrow Ae+Be = \lambda e+\mu e$$

$$(A+B)e = (\lambda+\mu)e \checkmark$$

§ (iii) AB has eigenvalue $\lambda\mu$ and corresponding eigenvector e.

§ (iv) A^n has eigenvalue λ^n and eigenvector of A^n is e (same as for A)

§ (v) $C=B+B^2+B^3 \Rightarrow Ce=(B+B^2+B^3)e = (\lambda+\lambda^2+\lambda^3)e$

VI $A^{-1}e = \frac{1}{\lambda}e$

§ To find A^{-1} using Cayley-Hamilton theorem:

Characteristic equation is $P_A(\lambda) = \det(A-\lambda I)$
Then $P_A(A)=0$

Example: Let the characteristic equation for matrix A is:

$$a\lambda^3+b\lambda^2+c\lambda+d=0$$

The $aA^3+bA^2+cA+dI=0$

To find A^{-1}:

$$aA^3+bA^2+cA = -dI$$

$$A(aA^2+bA+cI) = -dI$$

$$aA^2+bA+cI = -dIA^{-1}$$

$$\Rightarrow A^{-1} = \frac{aA^2+bA+cI}{-d} \checkmark$$

Note: $(A^2)^{-1} = (A^{-1})^2$

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To find A^{-1} - Using characteristic equation

Example 3: The matrix A is given by:

$$A=\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

(i) Find the eigenvalues of A. ---[4]

(ii) Use the characteristic equation of A to find A^{-1} ---[4]

SP-20/02/Q8

Solution:

$$A=\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

for find eigenvalues $\det(A-\lambda I)=0$

expanding with C_1

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix}=0 \Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-1\times 2] - 0 - 1[2-2(2-\lambda)]=0$$

$$\Rightarrow (1-\lambda)(\lambda^2-5\lambda+6-2) - 1(2-4+2\lambda)=0$$

$$\Rightarrow (1-\lambda)(\lambda^2-5\lambda+4)-1\times 2(\lambda-1)=0$$

$$\Rightarrow (1-\lambda)(\lambda^2-5\lambda+4+2]=0$$

$$\Rightarrow (1-\lambda)(\lambda^2-5\lambda+6]=0$$

$$\Rightarrow (1-\lambda)(\lambda-2)(\lambda-3)=0 \quad ---(i)$$

$\therefore$ Eigen values of A are $\lambda=1,2,3$ ✓

Now from (i) $-\lambda^3+6\lambda^2-11\lambda+6=0$

$\therefore$ Characteristic equⁿ: $\lambda^3-6\lambda^2+11\lambda-6=0$ ---- (ii)

Using Cayley-Hamilton theorem $\Rightarrow A^3-6A^2+11A-6I=0$

$$\Rightarrow 6I = A^3-6A^2+11A$$

multiplying by A^{-1} $\Rightarrow 6A^{-1}I = A^3\cdot A^{-1} - 6A^2\cdot A^{-1} + 11AA^{-1}$

$$\Rightarrow 6A^{-1} = A^2-6A+11I$$

$$6A^{-1}=\begin{pmatrix} -1 & 5 & 10 \\ -2 & 6 & 10 \\ -4 & 4 & 9 \end{pmatrix} - 6\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$$

$$A^{-1}=\frac{1}{6}\begin{pmatrix} 4 & -1 & -2 \\ -2 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix} ✓$$

$$A^2=\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}=\begin{pmatrix} -1 & 5 & 10 \\ -2 & 6 & 10 \\ -4 & 4 & 9 \end{pmatrix}$$

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Example 4: The matrix A is given by: $A = \begin{pmatrix} 5 & -1 & 7 \\ 0 & 6 & 0 \\ 7 & 7 & 5 \end{pmatrix}$

(i) Find the eigenvalues of A. ---[4]

(ii) Use the characteristic equation of A to find A^{-1}! --[4]

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Solution: To find the eigenvalues of $A \to \det(A - \lambda I) = 0$

$$\begin{vmatrix} 5-\lambda & -1 & 7 \\ 0 & 6-\lambda & 0 \\ 7 & 7 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)\big((6-\lambda)(5-\lambda)-0\big) - 0 + 7\big(0 - 7(6-\lambda)\big) = 0$$

(Expanding with C_1)

$$\Rightarrow (5-\lambda)(6-\lambda)(5-\lambda) - 49(6-\lambda) = 0$$

$$\Rightarrow (6-\lambda)\left[(5-\lambda)^2 - 49\right] = 0$$

$$\Rightarrow (6-\lambda)(\lambda^2 - 10\lambda + 25 - 49) = 0$$

$$\Rightarrow (6-\lambda)(\lambda^2 - 10\lambda - 24) = 0 \Rightarrow (6-\lambda)(\lambda+2)(\lambda-12) = 0$$

$\therefore$ Eigenvalues of A are: $\lambda = -2, 6, 12$ ✓ ---(1)

from (1) $-\lambda^3 + 16\lambda^2 - 36\lambda - 144 = 0$

Now using Cayley-Hamilton theorem: $-A^3 + 16A^2 - 36A - 144I = 0$

$$\Rightarrow 144I = -A^3 + 16A^2 - 36A \Rightarrow 144A^{-1}I = -A^3 A^{-1} + 16A^2 A^{-1} - 36AA^{-1}$$

$$\Rightarrow 144A^{-1} = -A^2 + 16A - 36I \quad \text{--(2)}$$

Now $A^2 = \begin{pmatrix} 5 & -1 & 7 \\ 0 & 6 & 0 \\ 7 & 7 & 5 \end{pmatrix}\begin{pmatrix} 5 & -1 & 7 \\ 0 & 6 & 0 \\ 7 & 7 & 5 \end{pmatrix} = \begin{pmatrix} 74 & 38 & 70 \\ 0 & 36 & 0 \\ 70 & 70 & 74 \end{pmatrix}$ --(3)

from (3) in (2).

$$144A^{-1} = -\begin{pmatrix} 74 & 38 & 70 \\ 0 & 36 & 0 \\ 70 & 70 & 74 \end{pmatrix} + 16\begin{pmatrix} 5 & -1 & 7 \\ 0 & 6 & 0 \\ 7 & 7 & 5 \end{pmatrix} - 36\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{144}\begin{pmatrix} -30 & -54 & 42 \\ 0 & 24 & 0 \\ 42 & 42 & -30 \end{pmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{24}\begin{pmatrix} -5 & -9 & 7 \\ 0 & 4 & 0 \\ 7 & 7 & -5 \end{pmatrix} \checkmark$$

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Matrix is in row echelon form $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$ or $\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$

Then the eigenvalues are the diagonal elements

Example 5: The matrix P is given by: $P = \begin{pmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

(a) State the eigenvalues of P. ---[1]

(b) Use the characteristic equation of P to find P^{-1} --[4]

[W-20/21/Q7(a)(b)]

Solution: Given $P = \begin{pmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

(a)

The matrix P is given in row echelon form:

$\therefore$ eigenvalues are the diagonal element: $\lambda = 1, -1, 2$.

(b) Characteristic equation is: $(\lambda-1)(\lambda+1)(\lambda-2) = 0$

$\Rightarrow \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

Using Cayley Hamilton theorem $\Rightarrow P^3 - 2P^2 - P + 2I = 0$

$\Rightarrow 2I = 2P^2 + P - P^3$

$\Rightarrow 2IP^{-1} = 2P^2 \cdot P^{-1} + P \cdot P^{-1} - P^3 \cdot P^{-1}$

$2P^{-1} = 2P + I - P^2$ --- ①

$$P^2 = \begin{pmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 10 \\ 0 & +1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \text{ --- ②}$$

from ① & ②

$$2P^{-1} = 2\begin{pmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 8 & -6 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

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Diagonelisation

§ Given a matrix A, with eigenvalues $\lambda_1, \lambda_2, \lambda_3 \cdots$ and the corresponding eigenvectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ and $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$.

Now consider a matrix P:

$$P = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

and a matrix D:

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

are such that $AP = PD$

Hence $A = PDP^{-1}$ ✓

Note 1.

$$D^m = \begin{pmatrix} \lambda_1^m & 0 & 0 \\ 0 & \lambda_2^m & 0 \\ 0 & 0 & \lambda_3^m \end{pmatrix}$$ ✓

Note 2: $A^n = PD^n \cdot P^{-1}$ ✓

Vector Product: $\left.\begin{array}{l} a = a_1 i + b_1 j + c_1 k \\ b = b_1 i + b_2 j + c_2 k \end{array}\right\} \Rightarrow a \times b = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

$$\Rightarrow a \times b = (b_1 c_2 - b_2 c_1) i - j(a_1 c_2 - a_2 c_1) + k(a_1 b_2 - b_2 a_1)$$

Solving two equations in three variables:

$$\left.\begin{array}{l} a_1 x + b_1 y + c_1 z = 0 \\ a_2 x + b_2 y + c_2 z = 0 \end{array}\right\}$$

$$\Rightarrow \frac{x}{(b_1 c_2 - b_2 c_1)} = \frac{y}{-(a_1 c_2 - a_2 c_1)} = \frac{z}{(a_1 b_2 - a_2 b_1)} = k$$

Example 6: Given a matrix $A = \begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix}$

P is a matrix formed by the eigenvectors of A
D is a digonal matrix formed by eigenvalues of in the same order.

Express $A = PDP^{-1}$

Solution: $A = \begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix} \Rightarrow |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 5 \\ 3 & 1-\lambda \end{vmatrix} = 0$

$\therefore$ Characteristic equation is $(3-\lambda)(1-\lambda) - 15 = 0$

or $\lambda^2 - 4\lambda - 12 = 0 \Rightarrow (\lambda+2)(\lambda-6) = 0$

$\therefore$ eigenvalues are $\lambda = -2, 6$

$\therefore D = \begin{pmatrix} -2 & 0 \\ 0 & 6 \end{pmatrix}$ ✓ --- (i)

Now To find eigenvectors of A.

for $\lambda = -2$

$\begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -2 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} 3x + 5y = -2x \\ 3x + y = -2y \end{cases}$

$\Rightarrow \begin{matrix} 5x + 5y = 0 \\ 3x + 3y = 0 \end{matrix} \Rightarrow y = -x \quad , x = -1, y = 1$

for $\lambda = -2$, Eigenvector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ✓ —— (ii)

Again for $\lambda = 6$

$\begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 6 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 3x + 5y = 6x \\ 3x + y = 6y \end{matrix} \Rightarrow \begin{matrix} -3x + 5y = 0 \\ 3x - 5y = 0 \end{matrix}$

$\Rightarrow x = 5, y = 3$

$\therefore$ for $\lambda = 6$, eigenvector $= \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ ✓ (iii)

$\therefore P = \begin{pmatrix} -1 & 5 \\ 1 & 3 \end{pmatrix}$ —— (iv) (from (ii) & iii)

$\therefore P^{-1} = \frac{-1}{8} \begin{pmatrix} 3 & -5 \\ -1 & -1 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} -3 & 5 \\ 1 & 1 \end{pmatrix}$ —— (v)

$\therefore A = PDP^{-1} = \frac{1}{8} \begin{pmatrix} -1 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} -3 & 5 \\ 1 & 1 \end{pmatrix}$ } from (i), (iv), (v)

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Example 7: Given $A = \begin{pmatrix} 4 & 1 & 6 \\ 2 & -1 & 0 \\ 1 & 3 & 7 \end{pmatrix}$

(a) Find the eigenvalues and eigen vectors of A.
(b) Write down matrices P and D such that $A = PDP^{-1}$
(c) Find P^{-1}
(d) Verify your answers to parts (b) and (c) using matrix multiplication
(e) Calculate A^{10}.

Solution (a) $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 & 6 \\ 2 & -1-\lambda & 0 \\ 1 & 3 & 7 \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 10\lambda^2 + 9\lambda = 0 \Rightarrow$ eigenvalues are $0, 1, 9$

for $\lambda = 0$, eigenvector $= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

for $\lambda = 1$ eigenvector $= \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}$

and for $\lambda = 9$ eigenvector $= \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}$

(b) $P = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 3 & 1 \\ -1 & -2 & 4 \end{pmatrix}$

and $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}$

(c) $P^{-1} = \frac{1}{18}\begin{pmatrix} -14 & 22 & 12 \\ 9 & -9 & -9 \\ 1 & 1 & 3 \end{pmatrix}$

(d) $A = PDP^{-1}$

$= \frac{1}{18}\begin{pmatrix} 1 & 3 & 5 \\ 2 & 3 & 1 \\ -1 & -2 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}\begin{pmatrix} -14 & 22 & 12 \\ 9 & -9 & 9 \\ 1 & 1 & 3 \end{pmatrix}$

$= \begin{pmatrix} 4 & 1 & 6 \\ 2 & -1 & 0 \\ 1 & 3 & 7 \end{pmatrix}$ ✓

(e) $A^{10} = P \cdot D^{10} \cdot P^{-1}$

$= \frac{1}{18}\begin{pmatrix} 1 & 3 & 5 \\ 2 & 3 & 1 \\ -1 & -2 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}^{10}\begin{pmatrix} -14 & 22 & 12 \\ 9 & -9 & 9 \\ 1 & 1 & 3 \end{pmatrix}$

$A^{10} = \begin{pmatrix} 968551224, & 968551221, & 2905653566 \\ 193710246, & 193710243, & 581130732 \\ 77480977, & 774840979, & 2324522935 \end{pmatrix}$ ✓

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Example 8: The matrix A is given by: $A=\begin{pmatrix} 3 & 1 & 1 \\ 0 & 6 & -1 \\ 0 & 0 & -2 \end{pmatrix}$

(i) Use the characteristic equation of A to find the inverse of A^2; ... [4]
(ii) Find a matrix P and a diagonal matrix D such that $A^5 = PDP^{-1}$ ... [7]

[S-20/21/Q8]

Solution: $A=\begin{pmatrix} 3 & 1 & 1 \\ 0 & 6 & -1 \\ 0 & 0 & -2 \end{pmatrix}$

(i)

A is in the row echolon form.
$\therefore$ Eigenvalues are $\lambda = 3, 6, -2$
$\therefore$ Characteristic equation of A is:

$(\lambda-3)(\lambda-6)(\lambda+2)=0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$

$\therefore$ Using Cayley Hamilton theorem:

$A^3 - 7A^2 + 36I = 0$

$\Rightarrow 36I = 7A^2 - A^3$

$36(A^{-1})^2 . I = 7A^2 . (A^{-1})^2 - A^3 . (A^{-1})^2$

$36(A^2)^{-1} = 7I - A$

$$36(A^2)^{-1} = 7\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 1 & 1 \\ 0 & 6 & -1 \\ 0 & 0 & -2 \end{pmatrix}$$

$$(A^2)^{-1} = \frac{1}{36}\begin{pmatrix} 4 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 9 \end{pmatrix}$$

(ii) Eigen values of A are $3, 6, -2$

Eigenvector for $\lambda = 3$

$$\begin{pmatrix} 3 & 1 & 1 \\ 0 & 6 & -1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\left.\begin{aligned} 3x+y+z &= 3x \\ 6y - z &= 3y \\ z &= 3z \end{aligned}\right\} \Rightarrow \begin{aligned} y+z &= 0 \\ 3y - z &= 0 \\ -2z &= 0 \end{aligned}$$

$$\begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 0 & 3 & -1 \end{vmatrix} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ for $\lambda = 3$ eigenVector $= e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Now for $\lambda = 6$

$$\begin{pmatrix} 3 & 1 & 1 \\ 0 & 6 & -1 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} 3x+y+z &= 6x \\ 6y - z &= 6y \\ -2z &= 6z \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} -3x+y+z &= 0 \\ -z &= 0 \\ -8z &= 0 \end{aligned}\right.$$

$$\begin{vmatrix} i & j & k \\ -3 & 1 & 1 \\ 0 & 0 & -1 \end{vmatrix} = \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

for $\lambda = 6$, $e_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

Similarly for $\lambda = -2$, $e_3 = \begin{pmatrix} -9 \\ 5 \\ 40 \end{pmatrix}$

$$\therefore P = \begin{pmatrix} 1 & 1 & -9 \\ 0 & 3 & 5 \\ 0 & 0 & 40 \end{pmatrix}$$

for A^5; $D = \begin{pmatrix} 3^5 & 0 & 0 \\ 0 & 6^5 & 0 \\ 0 & 0 & (-2)^3 \end{pmatrix}$

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Example 9: The 3×3 matrix A has distict eigenvalues $b, -1, 1$, with Corresponding eigenvectors: $\begin{pmatrix}1\\0\\0\end{pmatrix}$; $\begin{pmatrix}4\\-1\\0\end{pmatrix}$; $\begin{pmatrix}2\\1\\2\end{pmatrix}$ respectively.

Find A in terms of b. ---[4]

[W-20/21/7(C)]

Solution: $A = PDP^{-1}$ ---(i)

$$D = \begin{pmatrix} b & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{---(ii)} \quad \text{and} \quad P = \begin{pmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \text{---(iii)}$$

<u>Now to find P^{-1}</u> ; eigenvalues of P are $\lambda = 1, -1, 2$

$\therefore$ Characteristic equ^n of P: $(\lambda-1)(\lambda+1)(\lambda-2) = 0$

$$\Rightarrow \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

Using Cayley Hamilton theorem $\Rightarrow P^3 - 2P^2 - P + 2I = 0$

$$\Rightarrow 2I = 2P^2 + P - P^3$$

$$\Rightarrow 2 \cdot P^{-1} = 2P + I - P^2$$

$$\Rightarrow 2P^{-1} = 2\begin{pmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} +1 & 0 & 10 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 8 & -6 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & -3 \\ 0 & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \checkmark \quad \text{---(iv)}$$

from (i)

$$A = PDP^{-1} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} b & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & -3 \\ 0 & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\therefore A = \begin{pmatrix} b & 4b+4 & -3b-1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

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Example 10: The matrix is given by:

$$A = \begin{pmatrix} a & 2a+5 & a+1 \\ 0 & -4 & 0 \\ 0 & 3 & -1 \end{pmatrix}$$

(i) Show that the eigenvalues of A are $a, -1, -4$ ---[2]

(ii) Find a matrix P such that:

$$A = P \begin{pmatrix} a & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix} P^{-1}$$ ---[5]

(iii) Use characteristic equation of A to find A^{-1}. --[6]

[W-20/22/Q9]

Solution (i) $\det(A-\lambda I) = 0$

$$\Rightarrow \begin{vmatrix} a-\lambda & 2a+5 & a+1 \\ 0 & -4-\lambda & 0 \\ 0 & 3 & -1-\lambda \end{vmatrix} = 0$$

Expand with C_1.

$(a-\lambda)(-4-\lambda)(-1-\lambda) = 0$ --- (1)

Eigenvalues: $\lambda = a, -1, -4$ ✓

(ii) for $\lambda = a$, eigenvector

$$\begin{pmatrix} a & 2a+5 & a+1 \\ 0 & -4 & 0 \\ 0 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{cases} ax + (2a+5)y + (a+1)z = ax \\ -4y = ay \\ 3y - z = az \end{cases}$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ 0 & -4-a & 0 \\ 0 & 3 & -1-a \end{vmatrix} = \begin{pmatrix} (4+a)(1+a) \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_1$$

Now for $\lambda = -1$

$$\begin{pmatrix} a & 2a+5 & a+1 \\ 0 & -4 & 0 \\ 0 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{cases} ax + (2a+5)y + (a+1)z = -x \\ -4y = -y \\ 3y - z = -z \end{cases}$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ a+1 & 2a+5 & a+1 \\ 0 & -3 & 0 \end{vmatrix} = \begin{vmatrix} 3(a+1) \\ 0 \\ -3(a+1) \end{vmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad e_2$$

using cross multiplication

Similarly for $\lambda = -4$

$$\begin{pmatrix} a & 2a+5 & a+1 \\ 0 & -4 & 0 \\ 0 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{cases} (a+4)x + (2a+5)y + (a+1)\cdot z = 0 \checkmark \\ -4y = -4y \\ 3y + 3z = 0 \checkmark \end{cases}$$

$$\begin{vmatrix} i & j & k \\ a+4 & 2a+5 & a+1 \\ 0 & 3 & 3 \end{vmatrix} = \begin{pmatrix} 3a+12 \\ -(3a+12) \\ 3a+12 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad e_3 \checkmark$$

$$\therefore P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \checkmark$$

(iii) see next page →

Continued;

Example 10(iii) from part (i) Characteristic equation of A.

$$(\lambda-a)(\lambda+1)(\lambda+4)=0 \Rightarrow \lambda^3+(5-a)\lambda^2+(4-5a)\lambda-4a=0$$

∴ Using Cayley Hamilton theorem:

$$4aI = A^3+(5-a)A^2+(4-5a)A$$

$$\Rightarrow 4aA^{-1} = A^2+(5-a)A+(4-5a)I$$

$$4aA^{-1} = \begin{pmatrix} a^2 & 2a^2-17 & a^2-1 \\ 0 & 16 & 0 \\ 0 & -15 & 1 \end{pmatrix} + (5-a)\begin{pmatrix} a & 2a+5 & a+1 \\ 0 & -4 & 1 \\ 0 & 3 & -1 \end{pmatrix} + (4-5a)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{4a}\begin{pmatrix} 4 & 5a+8 & 4a+4 \\ 0 & -a & 0 \\ 0 & -3a & -4a \end{pmatrix} \checkmark$$

Example 11: The matrix A is given by: $A=\begin{pmatrix} 5 & -22/3 & 8 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(a) Find a matrix P and a diagonal matrix D such that:

$A^2 = PDP^{-1}$ ---[7]

(b) Use characteristic equation of A to find A^3. ---[4]

[S-21/21/Q6]

Solution: The matrix A is given in Row echelon form;

∴ Eigenvalues of A are the diagonal elements, $\lambda = 5, -6$ and 1

Now to find Eigenvector:

✓ for $\lambda = 5$,

$$\begin{pmatrix} 5 & -22/3 & 8 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \left.\begin{matrix} -\frac{22}{3}y+8z=0 \\ -11y=0 \\ -4z=0 \end{matrix}\right\}$$

$$\begin{vmatrix} i & j & k \\ 0 & -\frac{22}{3} & 8 \\ 0 & -11 & 0 \end{vmatrix} = \begin{pmatrix} 88 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \checkmark = e_1$$

✓ Again for $\lambda = -6$

$$\begin{pmatrix} +5 & -\frac{22}{3} & 8 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -6\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \left.\begin{matrix} 11x-\frac{22}{3}y+8z=0 \\ -6y=-6y \\ 7z=0 \end{matrix}\right\} \Rightarrow \begin{vmatrix} i & j & k \\ 11 & -\frac{22}{3} & 8 \\ 0 & 0 & 7 \end{vmatrix} = \begin{pmatrix} -\frac{154}{3} \\ -77 \\ 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \checkmark = e_2$$

(continued →)

(Continued)

Example 11(a): for $\lambda=1 \Rightarrow \begin{pmatrix} 5 & -\frac{22}{3} & 8 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow$ $4x-\frac{22}{3}y+8z=0$

Solution: $-7y=0$

$z=z$

$$\Rightarrow \begin{vmatrix} i & j & k \\ 4 & -\frac{22}{3} & 8 \\ 0 & -7 & 0 \end{vmatrix} = \begin{pmatrix} 56 \\ 0 \\ -28 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = e_3$$

$$\therefore P = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \checkmark$$

and for $\underline{A^2 = PDP^{-1}}$

$$D = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}$$

$$\therefore D = \begin{pmatrix} 5^2 & 0 & 0 \\ 0 & (-6)^2 & 0 \\ 0 & 0 & 1^2 \end{pmatrix} = \begin{pmatrix} 25 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

(b) Characteristic equation is;

$(\lambda-5)(\lambda+6)(\lambda-1)=0 \Rightarrow \lambda^3-31\lambda+30=0$

$\therefore$ Using Cayley Hamilton theorem $A^3-31A+30I=0$

$$\Rightarrow A^3 = 31A - 30I$$

$$\Rightarrow A^3 = 31\times\begin{pmatrix} 5 & -\frac{22}{3} & 8 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 30\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 125 & -\frac{682}{3} & 248 \\ 0 & -216 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

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Example 12: The matrix A is given by: $A = \begin{pmatrix} 13 & 18 & -28 \\ -4 & -1 & 8 \\ 2 & 6 & -5 \end{pmatrix}$

(i) Find the eigenvalue of A corresponding to the eigenvector $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ --[1]

(ii) Find a matrix P and a diagonal matrix D, Such that: $A = PDP^{-1}$ --[8]

(iii) Use characteristic equation of A to find A^{-1} in terms of A ---[2]

$\boxed{S\text{-}21/23/Q8}$

Solution: $Ae = \lambda e$

(i) given $e_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 13 & 18 & -28 \\ -4 & -1 & 8 \\ 2 & 6 & -5 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \lambda = -1$ ✓

(ii) For eigenvalues, $\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 13-\lambda & 18 & -28 \\ -4 & -1-\lambda & 8 \\ 2 & 6 & -5-\lambda \end{vmatrix} = 0.$

$\Rightarrow -\lambda^3 + 7\lambda^2 - 7\lambda - 15 = 0$ --- (1)

$\Rightarrow (\lambda+1)(\lambda-3)(\lambda-5) = 0$

$\Rightarrow \lambda = -1,\ \lambda = 3$ and $\lambda = 5$ ✓

Now eigenvector for $\lambda = 3$

$$\begin{pmatrix} 13 & 18 & -28 \\ -4 & -1 & 8 \\ 2 & 6 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \left\{\begin{aligned} 10x + 18y - 28z &= 0 \\ -4x - 4y + 8z &= 0 \\ 2x + 6y - 8z &= 0 \end{aligned}\right\}$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ -4 & -4 & 8 \\ 2 & 6 & -8 \end{vmatrix} = \begin{pmatrix} -16 \\ -16 \\ -16 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e_2$$

Again for $\lambda = 5$

$$\begin{pmatrix} 13 & 18 & -28 \\ -4 & -1 & 8 \\ 2 & 6 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \left\{\begin{aligned} 8x + 18y - 28z &= 0 \\ -4x - 6y + 8z &= 0 \\ 2x + 6y - 10z &= 0 \end{aligned}\right\} \Rightarrow \text{solve by cross multiplication}$$

$$= \begin{vmatrix} i & j & k \\ -4 & -6 & 8 \\ 2 & 6 & -10 \end{vmatrix} = \begin{pmatrix} 12 \\ -24 \\ -12 \end{pmatrix} \sim \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} e_3$$ ✓

$$\therefore P = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix};\ D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

(iii) from (1) characteristic equation of A

$-\lambda^3 + 7\lambda^2 - 7\lambda - 15 = 0$

Using Cayley Hamilton theorem:

$-A^3 + \lambda A^2 - 7A + 15I = 0$

$\Rightarrow A^{-1} = \frac{1}{15}(-A^2 + 7A - 7I) = 0$

$$= \frac{1}{15}\left[\begin{pmatrix} -41 & -48 & 80 \\ 32 & 23 & -64 \\ 8 & 0 & -17 \end{pmatrix} + 7\begin{pmatrix} 13 & 18 & -28 \\ -4 & -1 & 8 \\ 2 & 6 & -5 \end{pmatrix} - 7\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{15}\begin{pmatrix} 43 & 78 & -116 \\ 4 & 9 & -8 \\ 22 & 42 & -59 \end{pmatrix}$$ ✓

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Example 13: It is given that e is an eigenvector of matrix A, with corresponding eigenvalue λ.

(i) Show that e is an eigenvector of A^2, with corresponding eigenvalue λ^2. ---[2]

(ii) The matrices A and B are given by:

$$A=\begin{pmatrix} n & 1 & 3 \\ 0 & 2n & 0 \\ 0 & 0 & 3n \end{pmatrix} \quad \text{and} \quad B=(A+nI)^2$$

where I is the 3×3 identity matrix and n is a non-zero integer.

(ii) Find in terms of n, a non-singular matrix P and a diagonal matrix D such that $B=PDP^{-1}$ ---[8]

[S-19/11/Q9]

Solution (i) $A^2e=A(Ae)=A(\lambda e)=\lambda(Ae)=\lambda(\lambda e)=\lambda^2 e$

$\therefore$ eigenvector of A^2 is e; and eigenvalue of A^2 is λ^2.

(ii) $A=\begin{pmatrix} n & 1 & 3 \\ 0 & 2n & 0 \\ 0 & 0 & 3n \end{pmatrix}$

The matrix is given in row-echelon form; hence the eigen values of A are: $n, 2n; 3n$

Now for $\lambda_1=n$, eigenvector e_1

$$\begin{pmatrix} n & 1 & 3 \\ 0 & 2n & 0 \\ 0 & 0 & 3n \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=n\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \quad y+3z=0$$
$$ny=0$$
$$2nz=0$$

$$\begin{vmatrix} i & j & k \\ 0 & 1 & 3 \\ 0 & n & 0 \end{vmatrix}=\begin{pmatrix} -3n \\ 0 \\ 0 \end{pmatrix}\sim\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}=e_1$$

Similarly $\lambda_2=2n$; $e_2=\begin{pmatrix} 1 \\ n \\ 0 \end{pmatrix}$

and for $\lambda_3=3n$; $e_3=\begin{pmatrix} 3 \\ 0 \\ 2n \end{pmatrix}$

Hence:

$$P=\begin{pmatrix} 1 & 1 & 3 \\ 0 & n & 0 \\ 0 & 0 & 2n \end{pmatrix}$$

Now eigenvalues of $B=A+nI$ are $(\lambda_1+n), (\lambda_2+n), (\lambda_3+n)$ or $2n, 3n$ and $4n$

hence diagonal Matrix D is

$$D=\begin{pmatrix} (2n)^2 & 0 & 0 \\ 0 & (3n)^2 & 0 \\ 0 & 0 & (4n)^2 \end{pmatrix}$$

$$D=\begin{pmatrix} 4n^2 & 0 & 0 \\ 0 & 9n^2 & 0 \\ 0 & 0 & 16n^2 \end{pmatrix}$$

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Example 14: The matrix M is defined by: $M = \begin{pmatrix} 2 & m & 1 \\ 0 & m & 7 \\ 0 & 0 & 1 \end{pmatrix}$

where $m \neq 0, 1, 2,$

(i) Find a matrix P and a diagonal matrix D such that $M = PDP^{-1}$ ---[7]

(ii) Find M^7P ---[3]

[W-19/11/Q8]

Solution: $M = \begin{pmatrix} 2 & m & 1 \\ 0 & m & 7 \\ 0 & 0 & 1 \end{pmatrix}$

M is upper diagonal matrix.

(i) Eigenvalues of M are $\lambda = 2, m, 1$

for $\lambda = 2$ (To find eigen vector e)

$$\begin{pmatrix} 2 & m & 1 \\ 0 & m & 7 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\Rightarrow 0 + my + z = 0$

$0 + (m-2)y + 7z = 0$

$0 + 0 - z = 0$

$$e_1 = \begin{vmatrix} i & j & k \\ 0 & m-2 & 7 \\ 0 & 0 & -1 \end{vmatrix} = \begin{pmatrix} (2-m) \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_1$$

for $\lambda = m$

$$\begin{pmatrix} 2 & m & 1 \\ 0 & m & 7 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = m\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{cases} (2-m)x + my + z = 0 \\ 0 + 0 + 7z = 0 \\ 0 + 0 + (1-m)z = 0 \end{cases}$$

$$e_2 = \begin{vmatrix} i & j & k \\ (2-m) & m & 1 \\ 0 & 0 & 7 \end{vmatrix} = \begin{pmatrix} 7m \\ 7(m-2) \\ 0 \end{pmatrix} \sim \begin{pmatrix} m \\ m-2 \\ 0 \end{pmatrix} \quad e_2$$

for $\lambda = 1$; e_3

$$\begin{pmatrix} 2 & m & 1 \\ 0 & m & 7 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\Rightarrow x + my + z = 0$

$0 + (m-1)y + 7z = 0$

$0 + 0 + \cancel{z} = 0$

$$\begin{vmatrix} i & j & k \\ 1 & m & 1 \\ 0 & (m-1) & 7 \end{vmatrix} = \begin{pmatrix} 6m+1 \\ -7 \\ m-1 \end{pmatrix} \sim \begin{pmatrix} 6m+1 \\ -7 \\ m-1 \end{pmatrix} \quad e_3$$

$$\therefore P = \begin{pmatrix} 1 & m & 6m+1 \\ 0 & m-2 & -7 \\ 0 & 0 & m-1 \end{pmatrix}; \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) $M^7 = PD^7P^{-1}$

$\Rightarrow M^7P = PD^7P^{-1}P = PD^7$

$$PD^7 = \begin{pmatrix} 1 & m & 6m+1 \\ 0 & m-2 & -7 \\ 0 & 0 & m-1 \end{pmatrix}\begin{pmatrix} 2^7 & 0 & 0 \\ 0 & m^7 & 0 \\ 0 & 0 & 1^7 \end{pmatrix}$$

$$\therefore M^7P = \begin{pmatrix} 2^7 & m^8 & 6m+1 \\ 0 & m^8 - 2m^7 & -7 \\ 0 & 0 & m-1 \end{pmatrix}$$

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Example 15: The matrix A is given by:

$$A = \begin{pmatrix} -1 & 2 & 12 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Use the characteristic equation of A to show that: $A^4 = pA^2 + qI$ ---[6]

where p and q are integers to be determined.

W-21|21| Q2

Solution:

$$A = \begin{pmatrix} -1 & 2 & 12 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

A is upper diagnal Matrix (row echelon form)

∴ eigenvalues of A are -1, 1 and 3.

∴ Characteristic equation of A is $(\lambda+1)(\lambda-1)(\lambda-3)=0$

or $\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$

∴ Using Cayley Hamilton theorem: $A^3 - 3A^2 - A + 3I = 0$ ---(i)

$\Rightarrow A^4 - 3A^3 - A^2 + 3A = 0$

$\Rightarrow A^4 = 3A^3 + A^2 - 3A$

$= 3[3A^2 + A - 3I] + A^2 - 3A$ [from (i) $A^3 = 3A^2 + A - 3I$]

$= 9A^2 + 3A - 9I + A^2 - 3A = \underline{10A^2 - 9I}$ ✓

Example 16: The matrix P is given by,

$$P=\begin{pmatrix}1 & 6 & 6\\ 0 & 2 & 6\\ 0 & 0 & -3\end{pmatrix}$$

(a) Use the characteristic equation of P to find P^{-1}. ---[5]

(b) Find the matrix A such that:

$$P^{-1}AP=\begin{pmatrix}4 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 6\end{pmatrix}$$ ---[4]

(c) State the eigenvalues and corresponding eigenvectors of A^3. ---[2]

W-21	22	Q6

Solution:

(a) $$P=\begin{pmatrix}1 & 6 & 6\\ 0 & 2 & 6\\ 0 & 0 & -3\end{pmatrix}$$

Eigenvalues of P are: $\lambda = 4, 5, 6$

Characteristic equation of P: $(\lambda-1)(\lambda-2)(\lambda+3)=0$

or $\lambda^3 - 7\lambda + 6 = 0$

$\therefore$ Using Cayley Hamilton theorem: $P^3 - 7P + 6I = 0$

$\Rightarrow 6I = 7P - P^3 \Rightarrow 6P^{-1} = 7I - P^2$

$$\Rightarrow 6P^{-1} = 7\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} - \begin{pmatrix}1 & 18 & 24\\ 0 & 4 & -6\\ 0 & 0 & 9\end{pmatrix} = \begin{pmatrix}6 & -18 & -24\\ 0 & 3 & 6\\ 0 & 0 & -2\end{pmatrix}$$

$$\Rightarrow P^{-1} = \begin{pmatrix}1 & -3 & -4\\ 0 & \frac{1}{2} & 1\\ 0 & 0 & -\frac{1}{3}\end{pmatrix}$$ ✓

$$\left[P^2 = \begin{pmatrix}1 & 18 & 24\\ 0 & 4 & -6\\ 0 & 0 & 9\end{pmatrix}\right.$$

(b) Given $$P^{-1}AP = \begin{pmatrix}4 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 6\end{pmatrix}$$

$$\Rightarrow A = P.\begin{pmatrix}4 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 6\end{pmatrix}.P^{-1} \Rightarrow A = \begin{pmatrix}1 & 6 & 6\\ 0 & 2 & 6\\ 0 & 0 & -3\end{pmatrix}\begin{pmatrix}4 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 6\end{pmatrix}\begin{pmatrix}1 & -3 & -4\\ 0 & \frac{1}{2} & 1\\ 0 & 0 & -\frac{1}{3}\end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix}4 & 30 & 36\\ 0 & 10 & 36\\ 0 & 0 & -18\end{pmatrix}\begin{pmatrix}1 & -3 & -4\\ 0 & \frac{1}{2} & 1\\ 0 & 0 & -\frac{1}{3}\end{pmatrix} = \begin{pmatrix}4 & 3 & 2\\ 0 & 5 & -2\\ 0 & 0 & 6\end{pmatrix}$$ ✓

(Continued →)

(Continued)

16(C) From Part(b)

$$A=\begin{pmatrix} 4 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 6 \end{pmatrix}$$ Matrix A is in row echelon form

$\therefore$ Eigenvalues of A are $\lambda=4, 5$ and 6. ✓ $(\lambda_1, \lambda_2, \lambda_3)$

Eigenvector for $\lambda=4$.

$$\begin{pmatrix} 4 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \left.\begin{aligned} 3y+2z&=0 \\ y-6z&=0 \\ 6y-2z&=0 \end{aligned}\right\}$$

$\therefore$ Eigenvector for $\lambda=4$ is $e_1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ✓ $\Rightarrow y=0, z=0, x=k$

for $\lambda=5$, eigenvector

$$\begin{pmatrix} 4 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \left.\begin{aligned} -x+3y+2z&=0 \\ -7z&=0 \\ z&=0 \end{aligned}\right\}$$

$$\Rightarrow x=3y \Rightarrow e_2=\begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}$$ ✓

for $\lambda=6$,

$$\begin{pmatrix} 4 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \left.\begin{aligned} -2x+3y+2z&=0 \\ -y-2z&=0 \\ 6z&=6z \end{aligned}\right\}$$

$$\Rightarrow \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 6 \\ 6 \\ -3 \end{pmatrix}$$ ✓

$\therefore$ Eigenvectors of A^3 are same as that of A

$\therefore$ Eigenvector of $A^3 \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \\ -3 \end{pmatrix}$ ✓

Eigenvalues of $A^3 \rightarrow \lambda_1^3, \lambda_2^3, \lambda_3^3 \rightarrow 4^3, 5^3, 6^3$

$= \underline{64, 125, 216}$ ✓