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FP-2

Further Pure Maths. 2.

Matrices 2
Notes and Revision

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Matrices-2 (Notes and Revision)

Solving simultaneous linear equations: (Geometrical interpretation)

§ Two linear equation in two variables,

$$\begin{matrix} ax+by=p \\ cx+dy=q \end{matrix} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} p \\ q \end{pmatrix} \quad \text{--- (1)}$$

here $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

(a) for a unique solution $\det A \neq 0$ (or A is Non-Singular)
geometrically: two intersecting lines. (A^{-1} exists)

(b) (i) for No Solution: $\det A = 0$ (or Matrix A is singular)
$\frac{p}{q} \neq \frac{a}{c} = \frac{b}{d}$ geometrically: two parallel line, (A^{-1} does not exists)
(ii) for many solutions: $\det A = 0$ (A is singular)
$\frac{a}{c} = \frac{b}{d} = \frac{p}{q}$ geometrically - overlapping lines.

Example 1: Solve $2x - y = 5$
$3x + 4y = 2$

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \text{--- (1)}$$

$A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$; $\det A = 11$ ($\det A \neq 0$)

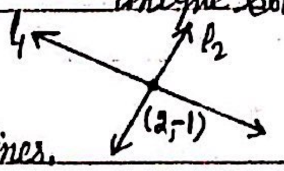
$A^{-1} = \frac{1}{|A|}\operatorname{adj} A = \frac{1}{11}\begin{pmatrix} 4 & 1 \\ -3 & 2 \end{pmatrix}$ ✓

form (i) $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 4 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$
$= \frac{1}{11}\begin{pmatrix} 22 \\ -11 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\therefore x = 2,\ y = -1$ ✓

Unique solution.

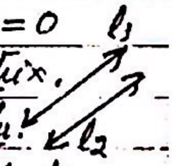
Geometrically:
A pair of intersecting lines.



Example 2(a) Solve $2x - y = 5$
$4x - 2y = 7$

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \Rightarrow \det A = -4 + 4 = 0$
A is Singular matrix,
and $\frac{2}{4} = \frac{-1}{-2} \neq \frac{5}{7}$ No Solu.!



Geometrically: a pair of parallel lines

2(b) Solve. $2x - y = 5$ --- (i)
$4x - 2y = 10$ --- (ii)

multiply (i)×2 is equate (ii)

$\operatorname{Det} A = \begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} = 0$ Singular Matrix

Many Solutions:
for $y = 2x - 5$

$\left(\frac{2}{4} = \frac{-1}{-2} = \frac{5}{10}\right)$ $\begin{cases} x = a \\ y = 2a - 5 \end{cases}$ $a \in R$

Geometrically:
A pair of coincident lines.

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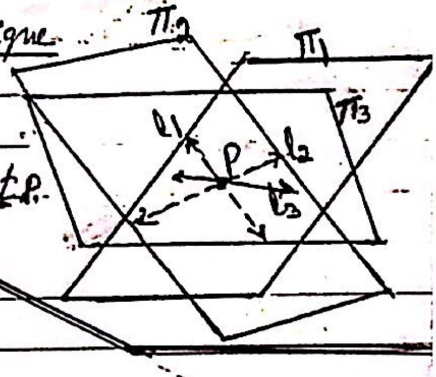
§ To Solve Three linear equation in three variables:

$$\begin{aligned} ax + dy + gz &= p \\ bx + ey + hz &= q \\ cx + fy + iz &= r \end{aligned} \Rightarrow \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdots$$

or $A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} p \\ q \\ r \end{pmatrix} \quad \text{---} (1)$$

Case I If $\det A \neq 0$ there will be a unique solution
(A is Non-Singular Matrix)
Geometrically: The three planes will intersect at only one point P.



Example 2: It is given that a is a positive constant. Show that the system of equations:

$$\begin{aligned} ax + (2a+5)y + (a+1)z &= 1, \\ -4y &= 2, \\ 3y - z &= 3, \end{aligned}$$

has a unique solution and interpret this geometrically. ---[3]

[W-20/22/9(a)]

Solution: The three linear equation may be expressed as.

$$\begin{pmatrix} a & 2a+5 & a+1 \\ 0 & -4 & 0 \\ 0 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow A^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

here

$$\det A = |A| = \begin{vmatrix} a & 2a+5 & a+1 \\ 0 & -4 & 0 \\ 0 & 3 & -1 \end{vmatrix} = 4a \neq 0 \quad \left(\text{given } a \text{ is a positive contant}\right)$$

$\therefore$ Unique solution.

Geometrically the three planes intersect at a point;

$$\begin{cases} x = \frac{11}{2}a + \frac{8}{a} \\ y = -\frac{1}{2} \\ z = -\frac{9}{2} \end{cases}$$

Example 3 (a) Given that a is an integer, show that the system of equations:

$$ax+3y+z=14$$
$$2x+y+3z=0$$
$$-x+2y-5z=17 \qquad \text{---[4]}$$

Case I

has a unique solution and interpret this situation geometrically.

(b) Find the value of a for which $x=1, y=4, z=-2$ is the solution of equations in part (a) ---[1]

[S-21/21/Q1]

Solution: (a)

$$\begin{aligned} ax+3y+z&=14\\ 2x+y+3z&=0\\ -x+2y-5z&=17 \end{aligned} \Rightarrow \begin{pmatrix} a & 3 & 1\\ 2 & 1 & 3\\ -1 & 2 & -5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}14\\0\\17\end{pmatrix} \Rightarrow A\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}14\\0\\17\end{pmatrix}$$

here $A=\begin{pmatrix} a & 3 & 1\\ 2 & 1 & 3\\ -1 & 2 & -5\end{pmatrix} \Rightarrow \det A=\begin{vmatrix} a & 3 & 1\\ 2 & 1 & 3\\ -1 & 2 & -5\end{vmatrix}=-11a+26$

here $\det A=-11a+26\neq 0$ (as $-11a+26=0$ not possible $\Rightarrow a=\frac{26}{11}$ ✗ as a is an integer)

$\therefore$ the system of equation has a unique solution. ✓

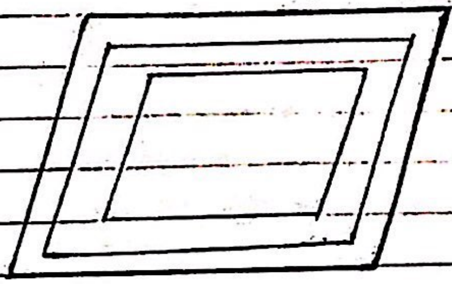
Geometric interpretation: The three planes intersect at a single point. ✓

(b) Put $x=1, y=4, z=-2$ in the first equation:

$$ax+3y+z=14$$
$$a\times 1+3\times 4+(-2)=14$$
$$a=14+2-12=4.$$

$\underline{a=4}$ ✓

Case II (a) The system of the three linear equations have No Unique Solution, (det A = 0) Inf. many solution as the equations are same; or geometrically represent three coincident planes.



Example 4 (a) Solve:

$3x+4y-z=4$ --- ①

$6x+8y-2z=8$ --- ②

$-3x-4y+z=-4$ --- ③

equation ② × $\frac{1}{2}$ = equation ①

and

equation ③ × (−1) = equation ①

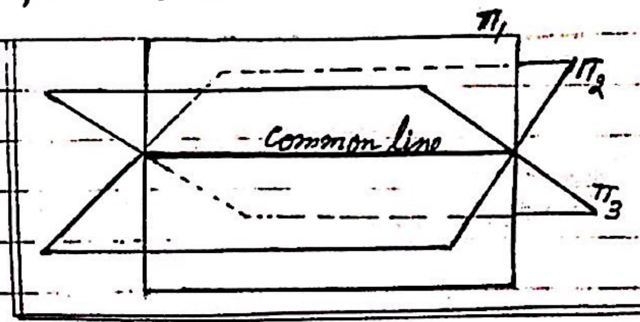
The solution of equation ① is $x=\lambda$, $y=\mu$ and $z=(3\lambda+4\mu-4)$, $\lambda, \mu \in R$.

(Many Solutions)

Case II (b) The system of three linear equations have No Unique solution (det A = 0) Infinite many solutions.

All the three equations has solutions lying on the same line.

Geometrically all the three planes intersect in one line; called a Sheaf of Planes



Example 4 (b) Solve:

π_1: $3x+3y+4z=6$ --- (i)

π_2: $x+z=1$ --- (ii)

π_3: $-6x+3y-5z=-3$ --- (iii)

$$\text{Det } A = \begin{vmatrix} 3 & 3 & 4 \\ 1 & 0 & 1 \\ -6 & 3 & -5 \end{vmatrix} = -9-3+12=0$$

∴ No Unique Solution. ✓ Det A = 0 ✓

here $9\pi_2+\pi_3=\pi_1$ hence they form a Sheaf

Now let $z=\lambda$, from (ii) $x=1-\lambda$ from (i) $y=1-\frac{1}{3}\lambda$

for $\lambda=0 \Rightarrow$ $z=0$, $x=1$, $y=1$

Eqn of line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -\frac{1}{3} \\ 1 \end{pmatrix}$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$ ✓

Case II (c) Two coincident plane and the third is non-parallel. det A = 0, Infinite many solution.

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Example 5: Find the values of k for which the set of linear equations:

(i)

$$x+3y+kz=4$$
$$4x-2y-10z=-5$$
$$x+y+2z=1$$

has no unique solution. ---[3]

(ii) For this value of k, find the set of possible solution, giving your answer in the form: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a + tb$

where a and b are vectors and t is a scalar. ---[3]

[Case II (b) det A = 0]

[S-17 | 11 | Q4]

Solution: $x+3y+kz=4$ ---①
$4x-2y-10z=-5$ ---②
$x+y+2z=1$ ---③

$$\Rightarrow \begin{pmatrix} 1 & 3 & k \\ 4 & -2 & -10 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}$$

or $A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}$ ---④

Now consider $\det A = \begin{vmatrix} 1 & 3 & k \\ 4 & -2 & -10 \\ 1 & 1 & 2 \end{vmatrix} = 0$ for No Unique Solution

$$\Rightarrow 6-54+6k=0 \Rightarrow \underline{k=8} ✓$$

for $k=8$ in ① $\Rightarrow x+3y+8z=4$ —①′ $\times 5 \Rightarrow 5x+15y+40z=20$

from ② $\Rightarrow 4x-2y-10z=-5$ ② $\times 4 \Rightarrow 16x-8y-40z=-20$

add $\Rightarrow 21x+7y=0$

$\Rightarrow y=-3x$

Put $y=-3x$ in ③

$\Rightarrow x-3x+2z=1$

$\Rightarrow z=\frac{1}{2}+x$

$\Leftarrow$ Let $x=t$
$y=-3t$
$z=\frac{1}{2}+t$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} ✓$$

(Infinite Many Solutions)

Geometrically: The three planes intersect in a line (Sheaf)

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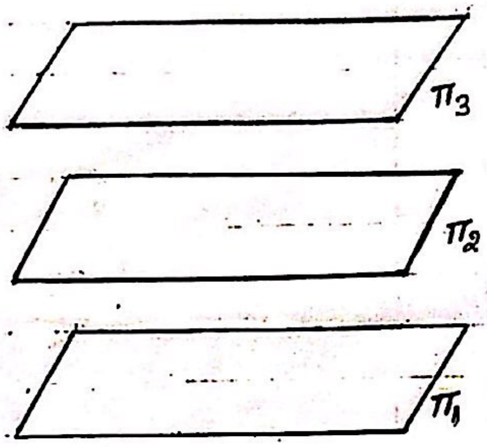
Case III $\boxed{\det A = 0 \\ \text{No. Solu}}$

§ Case III (a) All the three planes are parallel, but are not the same. There is no intersection of any of the planes. $\det A = 0$; No Solution

Example 6(a) Solve:

$$4x + 2y - 6z = 5 \quad \text{---} (1)$$
$$6x + 3y - 9z = 6 \quad \text{---} (2)$$
$$8x + 4y - 12z = 11 \quad \text{---} (3)$$

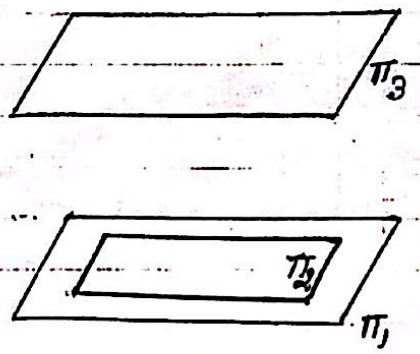


$$A = \begin{pmatrix} 4 & 2 & -6 \\ 6 & 3 & -9 \\ 8 & 4 & -12 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 4 & 2 & -6 \\ 6 & 3 & -9 \\ 8 & 4 & -12 \end{vmatrix} = 0$$

(1), (2), (3) represent three parallel plane (but not coincident) $\therefore$ No Solution.

Case III (b) Two planes are coincident and the third plane is parallel. $\det A = 0$ and No Solution.



Example 6(b) Solve:

$$\pi_1: 4x + 2y - 6z = 5 \quad \text{---} (1)$$
$$\pi_2: 8x + 4y - 12z = 10 \quad \text{---} (2)$$
$$\pi_3: 6x + 3y - 9z = 11 \quad \text{---} (3)$$

$$A = \begin{pmatrix} 4 & 2 & -6 \\ 8 & 4 & -12 \\ 6 & 3 & -9 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 4 & 2 & -6 \\ 8 & 4 & -12 \\ 6 & 3 & -9 \end{vmatrix} = 0 \quad ; \quad \text{No Solution.}$$

Two planes π_1 and π_2 are coincident and plane π_3 is parallel to π_1 and π_2.

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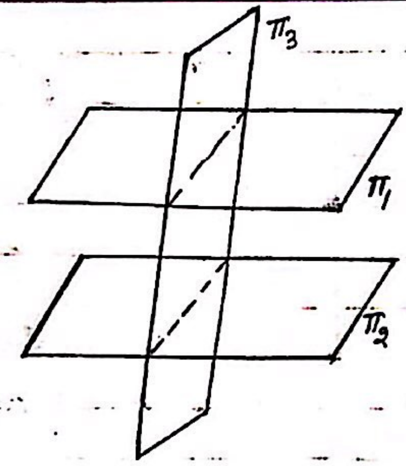
(c) § Case III (c) Two planes are parallel and the third is not:

det A = 0 & No Solution;



Example 6(c): Solve:

π_1 : $4x + 2y - 6z = 5$

π_2 : $8x + 4y - 12z = 11$

π_3 : $3x + 2y + z = 4$

$$A = \begin{pmatrix} 4 & 2 & -6 \\ 8 & 4 & -12 \\ 3 & 2 & 1 \end{pmatrix} ; \quad \det A = \begin{vmatrix} 4 & 2 & -6 \\ 8 & 4 & -12 \\ 3 & 2 & 1 \end{vmatrix} = 0$$

here planes π_1 and π_2 are parallel but π_3 is not parallel to π_1 or π_2 $\therefore$ No Solution.

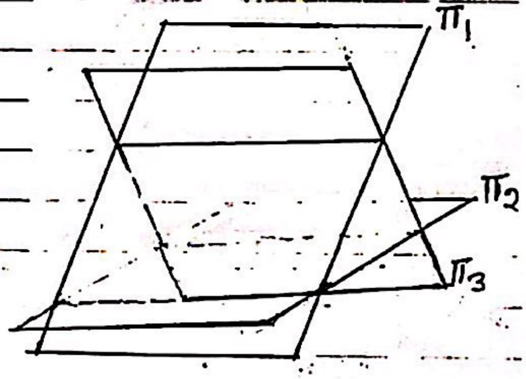
§ Case III (d) The three planes form a triangular prism:

Example 6(d): Solve:

π_1: $6x - 2y - 4z = 10$

π_2: $7x - 4y - 3z = 5$

π_3: $2x + 2y - 4z = 12$



$$A = \begin{pmatrix} 6 & -2 & -4 \\ 7 & -4 & -3 \\ 2 & 2 & -4 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 6 & -2 & -4 \\ 7 & -4 & -3 \\ 2 & 2 & -4 \end{vmatrix} = 132 - 44 - 88 = 0$$

det A = 0

here no two planes are parallel $\rightarrow$ No Solution

"Triangular prism of planes"

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Example 7(a) Find the set of values of a for which the system of equations:

$$x-2y-2z+7=0$$
$$2x+(a-9)y-10z+11=0$$
$$3x-6y+2az+29=0$$

has a unique solution. ---[4]

(b) Given that $a=-3$, show that the system of equations in part (a) has no solution. Interpret this situation geometrically. ---[3]

| SP-20 | 02 | Q8 |

Solution:

(a)
$$x-2y-2z=-7$$
$$2x+(a-9)y-10z=-11 \Rightarrow \begin{pmatrix}1&-2&-2\\2&a-9&-10\\3&-6&2a\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-7\\-11\\-29\end{pmatrix}$$
$$3x-6y+2az=-29$$

Now $\det A=\begin{vmatrix}1&-2&-2\\2&a-9&-10\\3&-6&2a\end{vmatrix}=2a(a-9)+2(4a+30)-2\{-12-3(a-9)\}$

$$=2a^2-4a-30$$

for unique solution $\det A\neq 0 \Rightarrow 2a^2-4a-30\neq 0$

$$\Rightarrow a^2-2a-15\neq 0$$
$$(a-5)(a+3)\neq 0$$
or $$a\neq 5,\ a\neq -3$$

(b) Now for $a=-3$

$x-2y-2z=-7$ --- ①

$2x-12y-10z=-11$ --- ②

$3x-6y-6z=-29$ --- ③

dividing ③ by 3 $\Rightarrow x-2y-2z=-\frac{29}{3}$ --- ③'

equations ① and ③' are inconsistent.

$\therefore$ The system of equations has no solution.

Geometrically: Equations ① and ③' represent two parallel planes and ② is not parallel to these.

② will intersect ① and ③' in two lines

(Case 3(c) - Page 7)

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Example 8: Find the value of a for which the system of equations

$$3x+y+z=0$$
$$ax+6y-z=0$$
$$ay-2z=0$$

---[3]

does not have a unique solution. [S-20/21/Q 8]

Solution:
$$\begin{pmatrix} 3 & 1 & 1 \\ a & 6 & -1 \\ 0 & a & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow A\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now
$$\det A = \begin{vmatrix} 3 & 1 & 1 \\ a & 6 & -1 \\ 0 & a & -2 \end{vmatrix} = 3(-12+a)-1(-2a)+1(a^2-0)$$
$$= a^2+5a-36$$

for No unique solution $\det A = 0 \Rightarrow a^2+5a-36=0$

$$(a+9)(a-4)=0$$

$$\therefore\ a=4 ;\ a=-9.$$ ✓

Example 9: Find the value of a for which the system of equations.

$$13x+18y-28z=0$$
$$-4x-ay+8z=0$$
$$2x+6y-5z=0$$

---[2]

does not have a unique solution. [S-21/23/ Q8(a)]

Solution: Consider det A.

$$= \begin{vmatrix} 13 & 18 & -28 \\ -4 & -a & 8 \\ 2 & 6 & -5 \end{vmatrix} = 13(5a-48)-18(20-16)-28(-24+2a)$$
$$= 65a-624-72+672-56a$$
$$= 9a-24$$ ✓

for not a unique solution $\det A = 0$

$$\Rightarrow 9a-24=0$$

$$\text{or } a = 8/3$$ ✓

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Example 10: Show that the system of equation: $x-2y-4z=1$, $x-2y+kz=1$, $-x+2y+2z=1$

(a) where k is a constant, does not have a unique solution. --- [2]

(b) Given that $k=-4$, show that the system of equations in part (a) is consistent, interpret this situation geometrically. --- [3]

(c) Given instead that $k=-2$, show that the system in part (a) is inconsistent. Interpret this situation geometrically. --- [2]

(d) For the case where $k \neq -2$ and $k \neq -4$, show that the system of equations in part (a) is inconsistent. Interpret this situation geometrically. $\boxed{W-20/21/Q4}$ --- [2]

Solution:

(a) Consider $\det A = \begin{vmatrix} 1 & -2 & -4 \\ 1 & -2 & +k \\ -1 & 2 & 2 \end{vmatrix} = -4-2k+2(2+k)-4\times 0 = 0$

$\Rightarrow \det A = 0$ $\therefore$ The system of equations does not have a unique solution.

(b) Now for $k=-4$;

$x-2y-4z=1$ --- ①

$x-2y-4z=1$ --- ②

$-x+2y+2z=1$ --- ③

Here ① and ② are coincident planes but the ③ is not parallel. $\therefore$ There will be a line of intersection. Hence consistent, will have infinite many solution.

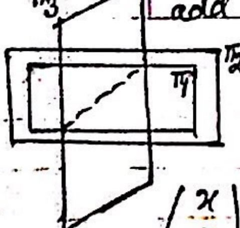
Case II (c)

add ② and ③ $\Rightarrow z=-1$

from ① $x-2y=-3$

Solutions are $\begin{cases} x=2\lambda-3 \\ y=\lambda \\ z=-1 \end{cases}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \checkmark$$



(c) for $k=-2$

$x-2y-4z=1$ --- ①

$x-2y-2z=1$ --- ②

$-x+2y+2z=1$ --- ③ $\Big\} \Rightarrow 1=-1$

Plane ② and ③ are parallel, not identical $\Rightarrow$ inconsistent. $\boxed{P-6}$ $\boxed{\text{Case III (b)}}$

(d) $x-2y-4z=1$ --- ①

$x-2y+kz=1$ --- ②

$-x+2y+2z=1$ --- ③

add ① & ③ $\Rightarrow -2z=2 \Rightarrow z=-1$

Subtract ② from ① $\Rightarrow (-4-k)z=0$ $\Big\}$

is inconsistent as $k\neq -4$; $4+k=0 \Rightarrow k=-4$

Geometrically; The three planes form a triangular prism.

(Page 7. Case III (d)) No. Solution

Example 11: Find the value of k for which the system of equations:

$$2x - 3y + 4z = 1$$
$$3x - y \quad = 2$$
$$x + 2y + kz = 1$$

(i) does not have a unique solution. ---[2]

(ii) For this value of k, solve the system of equations. ---[4]

[S-15/11/Q2]

Solution:

(i) $\det A = \begin{vmatrix} 2 & -3 & 4 \\ 3 & -1 & 0 \\ 1 & 2 & k \end{vmatrix} = 2(-k) + 3(3k) + 4(6+1)$
$= 7k + 28$

for not a unique solution $\det A = 0 \Rightarrow 7k + 28 = 0$
$\Rightarrow \underline{k = -4}$ ✓

(ii) For $k = -4$,

$$2x - 3y + 4z = 1 \text{ --- } ①$$
$$3x - y \quad = 2 \text{ --- } ②$$
$$x + 2y - 4z = 1 \text{ --- } ③$$

add ① & ③ $\Rightarrow 3x - y = 2$ (same as ②)

$\Rightarrow y = 3x - 2 \leftarrow ④$ put in ①

Case II (b)
$\Pi_1 + \Pi_3 = \Pi_2$
Page-4

from ① $2x - 3(3x - 2) + 4z = 1$
$4z = 7x - 5$ --⑤

Now put $x = t$ in ④ & ⑤

$$\begin{cases} y = 3t - 2 \\ z = \frac{7}{4}t - \frac{5}{4} \end{cases} \quad ✓ \qquad : t \in R$$

(Infinite many solutions)

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Example 12: Find two values of k (constant), the system of equations.

(i) have no unique solution:

$$kx + y + z = 2$$
$$x + ky + z = -1$$
$$x + y + kz = -1 \quad \text{---[4]}$$

(ii) Show that for one of these values of k, the equations have no solution, and solve the equations for the other value of k --[3]

$\boxed{\text{S-16/13/Q3}}$

Solution:

(i)

$$kx + y + z = 2$$
$$x + ky + z = -1$$
$$x + y + kz = -1$$

Consider:

$$\det A = \begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix} = k(k^2-1) - 1\times(k-1) + 1(1-k)$$
$$= k^3 - 3k + 2$$
$$= (k-1)^2(k+2)$$

for no unique solution: $\det A = 0$

$$\Rightarrow (k-1)^2(k+2) = 0 \Rightarrow k = 1,\ k = -2$$

(ii) for $k=1$

$$x+y+z=2 \quad \text{---}①$$
$$x+y+z=-1 \quad \text{---}②$$
$$x+y+z=-1 \quad \text{---}③$$

here equations ① and ② are inconsistent → No Solution

Geometrically: planes ② and ③ are coincident and ① is parallel to ② & ③.

$\boxed{\text{case III (b) Page-6}}$

for $k=-2$

$$-2x+y+z=2 \quad \text{---}①'$$
$$x-2y+z=-1 \quad \text{---}②'$$
$$x+y-2z=-1 \quad \text{---}③'$$

multiply ③' by 2 ⇒

$$2x+2y-4z=-2 \quad \text{---}④'$$

add ①' & ④'.

$$\Rightarrow 3y - 3z = 0$$
$$\Rightarrow y = z \quad \text{---}⑤'$$

Put $y=z$ in ③' ⇒ $x = z-1$

Hence for $z=t$: $t \in R$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t-1 \\ t \\ t \end{pmatrix}$$

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Example 13: Given a matrix A: $A = \begin{pmatrix} 2 & 5 & 3 \\ 3 & 7 & 4 \\ -1 & 5 & 5 \end{pmatrix}$

(a) Calculate adj A and A^{-1}.

(b) Hence solve the system of linear equations: $2x+5y+3z=20$, $3x+7y+4z=29$, $-x+5y+5y=7$ for x, y and z.

Solution: Given $A = \begin{pmatrix} 2 & 5 & 3 \\ 3 & 7 & 4 \\ -1 & 5 & 5 \end{pmatrix}$

(a)

$$\Rightarrow \det A = \begin{vmatrix} 2 & 5 & 3 \\ 3 & 7 & 4 \\ -1 & 5 & 5 \end{vmatrix} = 2\times 15 - 5\times 19 + 3\times 22 = 30 - 95 + 66 = 1$$

$$\text{adj } A \text{ (or } |A|) = \begin{pmatrix} \text{Cofactor} \\ \text{Matrix} \end{pmatrix}^T = \begin{pmatrix} 15 & -19 & 22 \\ -10 & 13 & -15 \\ -1 & 1 & -1 \end{pmatrix}^T = \begin{pmatrix} 15 & -10 & -1 \\ -19 & 13 & 1 \\ 22 & -15 & -1 \end{pmatrix} \quad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} 15 & -10 & -1 \\ -19 & 13 & 1 \\ 22 & -15 & -1 \end{pmatrix} = \frac{1}{1}\begin{pmatrix} 15 & -10 & -1 \\ -19 & 13 & 1 \\ 22 & -15 & -1 \end{pmatrix} \quad \text{---} (1) \quad \left[A^{-1} = \frac{1}{|A|} \text{Adj } A\right]$$

(b)

$$\begin{aligned} 2x+5y+3z&=20 \\ 3x+7y+4z&=29 \\ -x+5y+5y&=7 \end{aligned} \Rightarrow \overset{A}{\begin{pmatrix} 2 & 5 & 3 \\ 3 & 7 & 4 \\ -1 & 5 & 5 \end{pmatrix}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 20 \\ 29 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 20 \\ 29 \\ 7 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 & -10 & -1 \\ -19 & 13 & 1 \\ 22 & -15 & -1 \end{pmatrix}\begin{pmatrix} 20 \\ 29 \\ 7 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 300-290-7 \\ -380+377+7 \\ 440-435-7 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$$

$\therefore$ $x = 3$, $y = 4$, $z = -2$

System of equations - row operations:

§ Row operations:

There are three type of row operations:

(i) Row switching, $r_i \leftrightarrow r_j$

(ii) Row multiplication, $r_i \to k r_i$

(iii) Row addition, where $r_i \to r_i + k r_j$

§ To reduce a matrix in augmented form (or row echelon form)

Example: Solve the system of linear equations:

$$2x + y + z = 1$$
$$2x + 3y + 10z = 3$$
$$4x - z = 1$$

Solution: The system of equation can be expressed as:

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 10 \\ 4 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

Then for augmented matrix:
$$\left(\begin{array}{ccc:c} 2 & 1 & 1 & 1 \\ 2 & 3 & 10 & 3 \\ 4 & 0 & -1 & 1 \end{array}\right)$$

applying row operation $R_3 \to R_3 - 2R_1$ and $R_2 \to R_2 - R_1$

$$\Rightarrow \left(\begin{array}{ccc:c} 2 & 1 & 1 & 1 \\ 0 & 2 & 9 & 2 \\ 0 & -2 & -3 & -1 \end{array}\right)$$

For Unique Solution:
$$\left(\begin{array}{ccc:c} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \alpha & \beta \end{array}\right)$$
$\alpha \neq 0,\ \beta \in R$

$R_3 \to R_3 + R_2$

$$\Rightarrow \left(\begin{array}{ccc:c} 2 & 1 & 1 & 1 \\ 0 & 2 & 9 & 2 \\ 0 & 0 & 6 & 1 \end{array}\right)$$

Augmented Matrix.

$\Rightarrow 2x + y + z = 1$ --- (1)

$2y + 9z = 2$ --- (2)

$6z = 1$ --- (3)

$\Rightarrow$

from (3) $z = \frac{1}{6}$ ✓

from (2) $2y + 9 \times \frac{1}{6} = 2 \Rightarrow y = \frac{1}{4}$ ✓

from (1) $2x + \frac{1}{4} + \frac{1}{6} = 1 \Rightarrow x = \frac{7}{24}$ ✓

$\therefore x = \frac{7}{24},\ y = \frac{1}{4}$ and $z = \frac{1}{6}$

a Unique Solution

Example 15: solve the system of equations: $x+4y-2z=2$
$-2x-10y-6z=3$
$3x+14y+4z=7$

Solution: The given system of equations can be expressed as:

$$\begin{pmatrix} 1 & 4 & -2 \\ -2 & -10 & -6 \\ 3 & 14 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$$

Then for augmented matrix: $\left(\begin{array}{ccc:c} 1 & 4 & -2 & 2 \\ -2 & -10 & -6 & 3 \\ 3 & 14 & 4 & 7 \end{array}\right)$

Now applying row operations: $R_3 \to R_3 - 3R_1$ and $R_2 \to R_2 + 2R_1$

$$\Rightarrow \left(\begin{array}{ccc:c} 1 & 4 & -2 & 2 \\ 0 & -2 & -10 & 7 \\ 0 & 2 & 10 & 1 \end{array}\right)$$

$R_3 \to R_3 + R_2$

$$\Rightarrow \left(\begin{array}{ccc:c} 1 & 4 & -2 & 2 \\ 0 & -2 & -10 & 7 \\ 0 & 0 & 0 & 8 \end{array}\right)$$

System of equation has no solution: (as $0 \cdot z = 8$ false)

For No Solution

$$\left(\begin{array}{ccc:c} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \alpha & \beta \end{array}\right)$$

$\alpha = 0$ but $\beta \neq 0$

Example 16: Solve the system of equations:

$$x+5y-z=1$$
$$2x+7y-4z=0$$
$$4x+11y-10z=-2$$

Solution: The given system of equations can be expressed as:

$$\begin{pmatrix} 1 & 5 & -1 \\ 2 & 7 & -4 \\ 4 & 11 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

for augmented matrix: $\left(\begin{array}{ccc:c} 1 & 5 & -1 & 1 \\ 2 & 7 & -4 & 0 \\ 4 & 11 & -10 & -2 \end{array}\right)$

Now apply Row operations: $R_3 \to R_3 - 4R_1$ and $R_2 \to R_2 - 2R_1$

$$\Rightarrow \left(\begin{array}{ccc:c} 1 & 5 & -1 & 1 \\ 0 & -3 & -2 & -2 \\ 0 & -9 & -6 & -6 \end{array}\right)$$

apply $R_3 \to R_3 - 3R_2$

$$\Rightarrow \left(\begin{array}{ccc:c} 1 & 5 & -1 & 1 \\ 0 & -3 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

For Infinite Many Solutions

$$\left(\begin{array}{ccc:c} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \alpha & \beta \end{array}\right)$$

$$\alpha = \beta = 0$$

$$\Rightarrow \left.\begin{array}{l} x+5y-z=1 \quad \text{---}(1) \\ -3y-2z=-2 \quad \text{---}(2) \end{array}\right\}$$

Infinite Solutions

for $z=t$ in (2)

$$\Rightarrow y = \frac{2}{3} - \frac{2}{3}t$$

from (1) $x = 1-5y+z = 1-5\left(\frac{2}{3}-\frac{2}{3}t\right)+t$

$$= -\frac{7}{3} + \frac{13}{3}t$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7/3 \\ 2/3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 13/3 \\ -2/3 \\ 1 \end{pmatrix}$$

or $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7/3 \\ 2/3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 13 \\ -2 \\ 3 \end{pmatrix}$$