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S-1

Probability and Statistics-1

Discrete Random Variable

Binomial and Geometric distribution

Notes

Content

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S1

Discrete random Variable Notes

§ **Discrete random variable:**

If a random variable takes countable number of values, it is called a discrete random variable.

§ **Probability distribution:**

If a random variable X takes values $x_1, x_2, \ldots x_n$; with respective probabilities, $p_1, p_2 \ldots p_n$; Then:

X :	x_1	x_2	x_3	---	x_n
$P(X=x)$:	p_1	p_2	p_3	----	p_n

is defined as the probability distribution of X.

Note: $\sum p_i = p_1 + p_2 + \ldots + p_n = 1$

§ **Expected value of a random variable X:**

$$E(X) = \sum p_i x_i$$

§ **Variance of a random variable X:**

$$\sigma^2 = Var(X) = \sum p_i (x_i - E(x))^2$$

or

$$= \sum p_i \cdot x_i^2 - (E(X))^2$$

1. A book club sends 6-paperbacks and 2-hardback books to Mrs Hunt. She chooses "4" of these books at random to take with her on holiday. The random variable X represents the number of paperback books she chooses.

(a) Show that the probability that she chooses exactly 2 paperback books is 3/4. --- [2]

(b) Draw up the probability distribution table for X. --- [3]

(c) Find the value of E(X) and Var(X). --- [2]

SP-20/05/Q3

Solution: Paperback-books = 6 ; hardback-books = 2 ; Total No = 8.

Number of books chosen by Mrs Hunt = 4

(a) $P(\text{Exactly 2-paperback}) = \dfrac{{}^6C_2 \times {}^2C_2}{{}^8C_4} = \dfrac{15 \times 1}{70} = \dfrac{3}{14}$ ✓

(b) as number of hardback-books = 2, so the minimum number of paperback books = 2.

X	2	3	4
$P(X=x)$	$\dfrac{{}^6C_2 \times {}^2C_2}{{}^8C_4} = \dfrac{3}{14}$	$\dfrac{{}^6C_3 \times {}^2C_1}{{}^8C_4} = \dfrac{8}{14}$	$\dfrac{{}^6C_4 \times {}^2C_0}{{}^8C_4} = \dfrac{3}{14}$

(c) $E(X) = \Sigma x_i \cdot p_i = 2 \times \dfrac{3}{14} + \dfrac{3 \times 8}{14} + \dfrac{4 \times 3}{14} = \dfrac{42}{14} = 3$ ✓ ---(i)

and $\Sigma x_i^2 \times p_i = 2^2 \times \dfrac{3}{14} + 3^2 \times \dfrac{8}{14} + 4^2 \times \dfrac{3}{14} = \dfrac{132}{14} = \dfrac{66}{7}$ ✓ ---(ii)

from (i) and (ii)

Now $Var(X) = \Sigma x_i^2 p_i - (E(X))^2 = \dfrac{66}{7} - 3^2 = \dfrac{3}{7} = 0.429$ ✓

2 A fair four-sided spinner has edges numbered 1, 2, 2, 3.
A fair three-sided spinner has edges numbered −2, −1, 1.
Each spinner is spun and the number on the edge on which it comes to rest is noted. The random variable X is the sum of two numbers that have been noted.

(a) Draw up the probability distribution table for X --[3]

(b) Find Var(X). --[3]

S-20/53/Q4

Solution:

(a)

	Four-sided spinner				
		1	2	2	3
Three sided	−2	−1	0	0	1
	−1	0	1	1	2
	1	2	3	3	4

Sum

X	−1	0	1	2	3	4
fi	1	3	3	2	2	1

$n = \Sigma fi = 12$

Prob. distribution:

X	−1	0	1	2	3	4
$P(X=x)$	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$

(b) $E(X) = \Sigma p_i x_i = \frac{1}{12}\times(-1) + \frac{3}{12}\times 0 + \frac{3}{12}\times 1 + \frac{2}{12}\times 2 + \frac{2}{12}\times 3 + \frac{1}{12}\times 4 = \frac{16}{12} = \frac{4}{3}$ ✓

And $\Sigma x_i^2 \times p_i = (-1)^2\times\frac{1}{12} + 0^2\times\frac{3}{12} + 1^2\times\frac{3}{12} + 2^2\times\frac{2}{12} + 3^2\times\frac{2}{12} + 4^2\times\frac{1}{12} = \frac{46}{12}$ ✓

Now $Var(X) = \Sigma x_i^2 . p_i - (E(X))^2 = \frac{46}{12} - \left(\frac{4}{3}\right)^2 = \frac{46}{12} - \frac{16}{9}$

$= \frac{23}{6} - \frac{16}{9} = \frac{37}{18}$

$Var(X) = 2.06$ ✓

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3. Two ordinary fair dice are thrown. The resulting score is found as follows:

- If two dice show different numbers, the score is smaller of of the two numbers.
- If two dice show equal numbers, the score is 0.

(i) Draw up the probability distribution table for the score. ---[4]

(ii) Calculate the expected score. ---[2]

[S-16/63/Q3]

Solution: Sample space on one die = $\{1, 2, 3, 4, 5, 6\}$

(i) Sample space of a pair of dice = $\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), (2,3)$ - - - -
- - - - - - - - -
- - - - - - - $(6,6)\}$

Total number of outcomes:
$n(S) = 6 \times 6 = \underline{36}$ ✓

- If both the score are same $(1,1), (2,2), (3,3) --- (6,6)$, Score $X = 0$ ✓
(six outcomes)

- If different number on two dice:

$\rightarrow \{(1,2), (1,3), (1,4), (1,5), (1,6)$ or $(2,1), (3,1)$ - - - - - $(6,1)\}$ $\rightarrow$ 10 outcomes, Score $X = 1$ ✓

$\rightarrow \{(2,3), (2,4), (2,5), (2,6)$
$(3,2), (4,2), (5,2), (6,2)\}$ 8 outcomes, Score $X = 2$ ✓

$\rightarrow \{(3,4), (3,5), (3,6), (4,3), (5,3), (6,3)\}$ 6 outcomes; Score $X = 3$ ✓

$\rightarrow \{(4,5), (4,6), (5,4), (6,4)\} \rightarrow$ 4 outcomes; Score $X = 4$

$\rightarrow \{(5,6), (6,5)\} \longrightarrow$ 2 outcomes; Score $X = 5$

Prob. distribution table:

X	0	1	2	3	4	5
$P(X=x)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$2/36$

(ii) $E(X) = \sum x_i \cdot p_i = 0 \times \frac{6}{36} + \frac{1 \times 10}{36} + \frac{2 \times 8}{36} + \frac{3 \times 6}{36} + \frac{4 \times 4}{36} + 5 \times \frac{2}{36} = \frac{70}{36}$

$= \frac{70}{36} = \underline{1.94}$

4. In a game, Jim throws three darts at a board. This is called a turn. The centre of the board is called the bull's-eye.
The random variable X is the number of darts in a turn that hits bull's-eye. The probability distribution of X is given in the following table.

x	0	1	2	3
$P(X=x)$	0.6	p	q	0.05

It is given that $E(X) = 0.55$

(a) Find the value of p and q. --- [4]

(b) Find $Var(X)$. --- [2]

[W-21/53/Q6]

Solution: For a prob. distribution $\Sigma p_i = 1 \Rightarrow 0.6 + p + q + 0.05 = 1$

(a) $\Rightarrow p + q = 0.35$ ------(i)

and $E(X) = 0.55$ (Given)

$\rightarrow \Sigma p_i \cdot x_i = 0.55 \Rightarrow 0 \times 0.6 + 1 \times p + 2 \times q + 3 \times 0.05 = 0.55$

$\Rightarrow p + 2q = 0.4$ ----(ii)

Solving (i) and (ii) $\rightarrow$ $p = 0.3$

and $q = 0.05$

(b) Consider $\Sigma x_i^2 \cdot p_i = 0^2 \times 0.6 + 1^2 \times p + 2^2 \times q + 3^2 \times 0.05$

$= 0 + 1 \times 0.3 + 4 \times 0.05 + 0.45 = 0.95$

Now $Var(X) = \Sigma x_i^2 \cdot p - (E(x))^2$

$= 0.95 - (0.55)^2$ [Given $E(x) = 0.55$]

$= 0.95 - 0.3025$

$= 0.6475$

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5. The randomly variable X can only take values, −2, −1, 0, 1, 2. The probability of X is given in the following table:

x	−2	−1	0	1	2
$P(X=x)$	p	p	0.1	q	q

Given that $P(X \geq 0) = 3P(X<0)$, find the value of p and q --[4]

[S-21/53/Q2]

Solution: $\Sigma p_i = 1 \Rightarrow p+p+0.1+q+q=1 \Rightarrow 2p+2q=0.9$ ----(i)

Given $P(X\geq 0) = 3\cdot P(X<0) \Rightarrow (0.1+q+q) = 3(p+p)$

$\Rightarrow 6p-2q=0.1$ ---(ii)

add (i) & (ii) $8p=1 \Rightarrow p=\frac{1}{8}=0.125$; from (i) $q = 0.325$

6. The random variable X takes values, −1, 1, 2, 3 only. The prob. that X takes the value x is kx^2, where k is a constant.

(i) Draw up the prob. distribution table for X, in terms of k, and find the value of k. ---[3]

(ii) Find $E(X)$ and $Var(X)$. ---[3]

[M-19/62/Q4]

Solution: Given $P(x) = kx^2$

(i)

x	−1	1	2	3
$P(X=x)$	k	k	$4k$	$9k$

$\Rightarrow \Sigma p_i = 1$

$\Rightarrow k+k+4k+9k=1$

$\Rightarrow 15k=1 \rightarrow k = 1/15$

(ii) $E(X) = \Sigma x_i\cdot p_i = -1\times k + 1\times k + 2\times 4k + 3\times 9k = 35k = 35\times\frac{1}{15} = \frac{7}{3}$

Consider $\Sigma x_i^2\cdot p_i = (-1)^2\times k + 1^2\times k + 2^2\times 4k + 3^2\times 9k = 99k = 99\times\frac{1}{15} = \frac{33}{5}$

$\therefore Var(X) = \Sigma x_i^2\cdot p_i - (E(X))^2$

$= \frac{33}{5} - \left(\frac{7}{3}\right)^2 = \frac{33}{5} - \frac{49}{9} = \frac{52}{45} = 1.16$

$= 1.16$

6. At a funfair, Amy pays \$1 for two attempts to make a bell ring by shooting at it with a water pistol.

- If she makes the bell ring on her first attempt, she receives \$3 and stops playing. This means that over all she has gained \$2.
- If she makes the bell ring on her second attempt, she receives \$1.50 and stops playing. This means that overall she has gained \$0.50.
- If she does not make the bell ring in two attempts, she has lost her \$1.

The probability that Amy makes the bell ring on any attempt is 0.2 independently of others.

(i) Show that the probability that Amy loses her original \$1 is 0.64 --[2]

(ii) Complete the prob. distribution table for the amount that Amy gains --[4]

Amy's Gain (\$)			
Probability	0.64		

(iii) Calculate Amy's expected gain. --[1]

[S-19/61/Q6]

Solution: Prob. of ringing the bell in any attempt, $p = 0.2$

$q = 1 - 0.2 = 0.8$

(i) She loses \$1 if she does not make the bell ring in both the attempts

Prob $= 0.8 \times 0.8 = \underline{0.64}$

$\therefore$ this way her gain $= \$ -1$ (loses \$1)

(ii) P(She makes the bell ring in second attempt $= qp = 0.8 \times 0.2 = 0.16$

Overall gain $= 1.50 - 1 = \$0.50$

and P(She rings the bell in first attempt) $= p = 0.2$

and overall gain $= \$3 - \$1 = \$2$

$\therefore$ The Random varible X is overall gain.

Amy's Gain (\$) x	-1	0.50	2
$P(X = x)$	0.64	0.16	0.2

(iii) $E(X) = \Sigma x_i \cdot p_i = -1 \times 0.64 + 0.50 \times 0.16 + 2 \times 0.2$

$= -0.64 + 0.48$

$= \underline{-0.16}$ (or Loss of 16 cents)

7. Noor has 3 T-Shirts, 4 blouses and 5 jumpers. She chooses three items at random. The random variable X is the number of T-Shirts chosen.

(i) Show that the probability that Noor chooses exactly one T-Shirt is $27/55$. --[3]

(ii) Draw up the probability distribution table for X. --[4]

[W-16/62/Q2]

Solution: Number of T-Shirts = 3 & blouses and jumpers = 4+5 = 9

(i) Total number = 3+9 = 12

P(Exactly 1 T-Shirt) = P(1 T-Shirt and 2 any other)

$$= \frac{{}^3C_1 \times {}^9C_2}{{}^{12}C_3} = \frac{27}{55}$$

(ii) Prob. distribution table for X.

Number of Shirts x	0	1	2	3
$P(X=x)$	$\frac{{}^3C_0 \times {}^9C_3}{{}^{12}C_3} = \frac{84}{220}$	$\frac{{}^3C_1 \times {}^9C_2}{{}^{12}C_3} = \frac{108}{220}$	$\frac{{}^3C_2 \times {}^9C_1}{{}^{12}C_3} = \frac{27}{220}$	$\frac{{}^3C_3 \times {}^9C_0}{{}^{12}C_3} = \frac{1}{220}$

8. A fair coin is tossed three times. The random variable X denotes the number of heads obtained.

(i) Draw up the probability distribution table for X.

(ii) Find the prob. of getting at most 2 heads.

Solution: Sample Space. S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

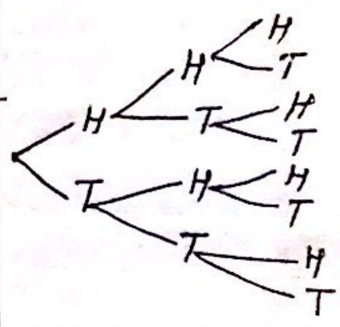
(i) Total number of out comes = 8

$X = 0$, No head → 3 Tails → 1 time

$X = 1$, One head & 2 tails → 3 times

$X = 2$, Two heads & 1 tail → 3 times

$X = 3$, Three heads 'HHH' → 1 time



Prob. distribution table for X

X	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(ii) $P(x \le 2) = P(0) + P(1) + P(2) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = 7/8$

§ **Binomial Probability Distribution:**

Let there be 'n' independent trials of an experiment, such that the probability 'p' of happening (success) of an event remain same in each trial, and let X denotes the number of successes in these trials.

Then the probability of getting exact 'r' successes is given by: $P(X=r) = {}^nC_r \cdot p^r \cdot q^{n-r}$ $[P(\text{failure}) = q = 1-p]$

Binomial prob. distribution is denoted by: $X \sim B(n, p)$

For Binomial distribution: $E(X) = np$ and $\sigma^2 = Var(X) = npq$.

9. A fair coin is tossed 3 times. The random variable X denotes the number of heads. Draw up the probability distribution table for X. Justify the application of Binomial Prob. distribution.

Solution: When a coin is tossed sample space $S = \{H, T\}$

p = Prob of getting a head $= \frac{1}{2}$

$q = P(\text{not getting head}) = 1 - \frac{1}{2} = \frac{1}{2}$

Number of trials $= 3$

The '3' trials are independent $\rightarrow P(H) = \frac{1}{2}$ in each trial.

Hence we can use Binomial Prob. distribution.

$P(X=0) = q^n = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$P(X=1) = {}^3C_1 p^1 q^2 = {}^3C_1 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^2 = 3/8$

$P(X=2) = {}^3C_2 \cdot p^2 q^1 = {}^3C_2 \times \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right) = 3/8$

$P(X=3) = {}^3C_3 p^3 q^0 = {}^3C_3 \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^0 = \frac{1}{8}$

x	0	1	2	3
$P(X=x)$	1/8	3/8	3/8	1/8

10. In a certain college, 22% of students own a car.
(a) 3 students from the college are chosen at random. Find the probability that all 3 student own a car. ---[1]
(b) 16 students from the college are chosen at random. Find the probability that the number of these student who own a car is at least 2 and atmost 4. --[3]

[S-20/53/Q2]

Solution: P(owns a car) = 0.22 = p ; q = 1 − 0.22 = 0.78.

(a) $n = 3,\ p = 0.22,\ q = 0.78$ $\qquad P(X=r) = {}^nC_r\, p^r \cdot q^{n-r}$

Using Binomial Prob dis.

$$P(r=3) = {}^3C_3 \cdot (0.22)^3 \cdot (0.78)^0 = \underline{0.0106}$$

(b) Now $n = 16,\ p = 0.22;\ q = 0.78$

$$P(2 \leq r \leq 4) = P(2,3,4)$$
$$= {}^{16}C_2\,(0.22)^2 \cdot (0.78)^{14} + {}^{16}C_3\,(0.22)^3(0.78)^{13} + {}^{16}C_4\,(0.22)^4(0.78)^{12}$$
$$= 0.179205 + 0.235877 + 0.216221 = 0.631$$

11. An urn contains 5 white, 7 red and 8 black balls. If four balls are drawn one by one with replacement.
What is the probability that:
(i) Only 3 are white
(ii) atleast three are white.

Note: When balls are drawn with replacement. Prob. remains same in each successive draw.

Solution: White = 5, red and black = 7+8 = 15, Total balls = 20

$$p = P(\text{White}) = \frac{5}{20} = \frac{1}{4};\ q = 1 - \frac{1}{4} = \frac{3}{4}$$

4 balls are drawn one by one with replacement ⇒ p remains same.

(i) $P(3 \text{ white}) = {}^4C_3 \cdot \left(\frac{1}{4}\right)^3 \cdot \left(\frac{3}{4}\right)^1 = \underline{\frac{3}{64}}$

(ii) $P(\text{atleast } 3 \text{ white}) = P(r \geq 3) = P(3,4)$

$$= {}^4C_3\left(\frac{1}{4}\right)^3 \cdot \left(\frac{3}{4}\right) + {}^4C_4\left(\frac{1}{4}\right)^4 = \underline{\frac{13}{256}}$$

12. An ordinary fair die is thrown '3' times. The random variable X is the number of times that a 1 or 6 is obtained.

(a) Draw up the probability distribution table for X. --[3]

(b) Find $E(X)$. --[2]

[M-20|52|Q2]

Solution: $n=3$ ✓, When a die is thrown once $S=\{1,2,3,4,5,6\}$

$p = P(\text{getting 1 or 6}) = \frac{2}{6} = \frac{1}{3}$ ✓ and $q = 1-p = 1-\frac{1}{3} = \frac{2}{3}$ ✓

p remains same in each throw, use Binomial Prob dis.

(a) Prob. distribution: $\{P(X=r) = {}^nC_r\, p^r q^{n-r}$

$X=r$	0	1	2	3
$P(X=r)$	${}^3C_0 (\frac{1}{3})^0 (\frac{2}{3})^3 = 8/27$	${}^3C_1 (\frac{1}{3})^1 (\frac{2}{3})^2 = \frac{12}{27}$	${}^3C_2 (\frac{1}{3})^2 (\frac{2}{3}) = \frac{6}{27}$	${}^3C_3 (\frac{1}{3})^3 (\frac{2}{3})^0 = \frac{1}{27}$

(b) $E(X) = \Sigma x_i p_i = 0\times\frac{8}{27} + 1\times\frac{12}{27} + 2\times\frac{6}{27} + 3\times\frac{1}{27} = \frac{27}{27} = 1$ ✓

Alternatively: $E(X) = np = 3\times\frac{1}{3} = 1$ ✓

13. The results of a survey by a large supermarket show that 35% of its customers shop online.

(i) Six customers are chosen at random. Find the prob. that more than three of them shop online. --[3]

(ii) For a random sample of 'n' customers, the prob. that at least one of them shop online is greater than 0.95. Find the least possible value of n.

[M-19|62|Q6] --[3]

Solution: $p = 0.35$, $q = 0.65$, $n = 6$

(a) $P(X>3) = P(4,5,6)$

$= {}^6C_4 (0.35)^4 (0.65)^2 + {}^6C_5 (.35)^5 (.65)$

$+ {}^6C_6 (0.35)^6$

$= 0.117$ ✓

(b) $P(\text{at least one}) = 1 - P(0)$

$= 1 - q^n > 0.95$

Given $1-(0.65)^n > 0.95$

$\Rightarrow 0.65^n < 0.05$

$\Rightarrow n \ln 0.65 < \ln 0.05$ (*)

$\Rightarrow n > \frac{\ln 0.05}{\ln 0.65} = 6.95$

$\therefore n = 7$ ✓

(*) Note: $\ln 0.65 = -0.4307 < 0$

($\because$ when we divide by $\ln 0.65$; the inequality will reverse)

14. Every day Richard takes a flight between Asten and Begin. On any day, the prob. that the flight arrives early is 0.15, the prob. that it arrives on time is 0.55 and the prob. that it arrives late is 0.3.

(a) Find the probability on each of 3 randomly chosen days, Richard's flight does not arrive late. ---[1]

(b) Find the probability that for 9 randomly chosen days, Richard's flight arrives early at least 3 times. ---[3]

[S-21/52/Q5]

Solution: P(arrives early) = 0.15; P(on time) = 0.55; P(late) = 0.3

(a) $p = P(\text{late}) = 0.3$;

$q = P(\text{does not arrives late any day}) = 1 - 0.3 = 0.7$

Now for $n = 3$

$P(\text{does not arrives late for 3 days}) = {}^3C_3\, p^0 q^3 = (0.7)^3$

$= \underline{0.343}$ ✓

(b) Now $n = 9$, $p = P(\text{arrives early}) = 0.15$

$q = 1 - 0.15 = 0.85$

At least 3 day arrives early:

$P(r \geq 3) = 1 - P(0, 1, 2)$

$= 1 - \{0.85^9 + {}^9C_1(0.15)^1 \cdot (0.85)^8 + {}^9C_2(0.15)^2 \cdot (0.85)^7$

$= 1 - \{0.231617 + 0.367862 + 0.259667)$

$= \underline{0.141}$ ✓

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15. Visitors to a wildlife park in Africa, have independent prob. of seeing zebra 0.85. 50 people independently visit the wildlife park. Find the mean and variance of number of these people who see zebra. [W/15/61/Q3(iii)]--[2]

Solution: $n = 50$, $p = 0.85$, $q = 1 - 0.85 = 0.15$; $X \sim B(50, 0.85)$

Mean $E(X) = np = 50 \times 0.85 = \underline{42.5}$ ✓ and Variance $\sigma^2 = npq = 50 \times 0.85 \times 0.15$

$= \underline{6.375}$ ✓

16. Robert uses his calculator to generate 5 random integers between 1 and 9 inclusive.

(i) Find the prob. that at least 2 of the 5 integers are less than or equal to 4. --[3]

Robert now generates n random integers between 1 and 9 inclusive. The random variable X is the number of these n integers which are less than or equal to a certain integer k between 1 and 9 inclusive. It is given that mean of X is 96 and the variance of X is 32.

(ii) Find the value of n and k. --[4]

[S-13/62/Q4]

Solution (i) Let Y represent the number of integers that are 4 or less.

$P(Y \geq 2) = 1 - P(Y = 0 \text{ or } 1)$; $n = 5$; $p = 4/9$; $q = \frac{5}{9}$

$= 1 - \left\{ \left(\frac{5}{9}\right)^5 + {}^5C_1 \left(\frac{4}{9}\right)^1 \cdot \left(\frac{5}{9}\right)^4 \right\} = 1 - 0.26461 = \underline{0.735}$ ✓

(ii) $E(X) = np = 96$ ---(i)
$\sigma^2 = Var(X) = npq = 32$ --(ii)
$\Rightarrow$ (ii) ÷ (i) → $\frac{npq}{np} = \frac{32}{96} \Rightarrow q = \frac{1}{3}$

$\Rightarrow p = 1 - \frac{1}{3} = \frac{2}{3}$ ✓

Now $p = 2/3$, from (i) $np = 96 \Rightarrow n \times \frac{2}{3} = 96 \Rightarrow n = \underline{144}$ ✓

P(an integer is k or less out of 9) $= \frac{k}{9}$

$\therefore p = \frac{2}{3} = \frac{k}{9} \Rightarrow$ $\boxed{k = 6}$ ✓

17. Screws are sold in packets of 15. Faulty screws occur randomly. A large number of packets are test for faulty screws and the mean number of faulty screws per packet is found to 1.2. (i) Show that the variance of the number of faulty screws in a packet is 1.104. [W-14/61/Q5] --[2]

(ii) Damion buys 8 packets of screws at random. Find the prob. that there are exactly 7 packet in which there is at least 1 faulty screw. [4]

Solution: (i) $n = 15$, Mean $np = 1.2$
$\Rightarrow 15p = 1.2 \Rightarrow p = 0.08, q = 0.92$
$\therefore$ Variance $= npq = 15 \times 0.08 \times 0.92$
$= \underline{1.104}$

(ii) $n = 15$, $p = 0.08$, $q = 0.92$
P(atleast one faulty) $= 1 - P(0) = 1 - (0.92)^{15}$
$= 0.7137$ ✓
Now p, $q = 0.2863$
Now 7 out of 8
$P(X = 7) = {}^8C_7 (0.7137)^7 \cdot (0.2863)$
$= \underline{0.216}$ ✓

§ The Geometric Distribution: $X \sim Geo(p)$

- Consider: A pair of fair coins is thrown repeatedly until a pair of tails is obtained. The random variable X denotes the number of throws required to obtain a pair of tails.

When a pair of coins is thrown;

The sample space $S = \{HH, HT, TH, TT\}$

Prob. of success 'p' is getting both tails $= \frac{1}{4}$

In any single throw it remains the same

$$p = P(\text{both tails}) = \frac{1}{4}$$

$$q = P(\text{Not getting both tails}) = 1 - \frac{1}{4} = \frac{3}{4}$$

Now → Prob. of getting both tail in first throw $= p = \frac{1}{4}$

" " " " " second " $= qp = \frac{3}{4} \cdot \frac{1}{4}$

" " " " " third " $= q^2p = \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4}$

and so on.

The repeated trials are independent,

Can be infinite.

Here $p, q, q^2p, q^3p, ---- \infty$ which is an infinite G.P.

$$\Sigma p_i = \frac{p}{1-q} = \frac{p}{1-(1-p)} = \frac{p}{p} = 1 \checkmark$$

§ A discrete random variable, X, is said to have a **geometric distribution**, and its parameter is 'p'. denoted by:

$$X \sim Geo(p).$$

For,

$$P(X = r) = q^{r-1} \cdot p$$

is the prob. of getting success first time in r_{th} trial.

The Geometric Prob. distribution:

X	1	2	3	4	---	r	----
$P(X=x)$	p	qp	q^2p	q^3p	---	$q^{r-1} \cdot p$	----

§ Note (i) $P(X \leq r) = 1 - q^r$ (iii) $E(X) = p$

(ii) $P(X > r) = q^r$

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§ Expectation of Geometric distribution: $X \sim Geo(p)$

$$E(X) = \sum x_i \cdot P(x_i)$$

$$= 1\cdot p + 2\cdot qp + 3\cdot q^2 p + 4\cdot q^3 p + \dots$$

$$E(X) = p[1 + 2q + 3q^2 + 4q^3 + \dots] \quad \text{---(i)}$$

$$\Rightarrow q \cdot E(X) = p[q + 2q^2 + 3q^3 + 4q^4 + \dots] \quad \text{---(ii)}$$

Now (i) − (ii)

$$\Rightarrow E(X)(1-q) = p[1 + q + q^2 + q^3 + \dots] \qquad (1-q = p)$$

$$\Rightarrow E(X)\cdot \cancel{p} = \cancel{p}\cdot[1 + q + q^2 + q^3 + \dots]$$

$$\Rightarrow E(X) = 1 + q + q^2 + q^3 \qquad \{\text{infinite G.P. } r = q$$

$$E(X) = \frac{1}{1-q} = \frac{1}{p}$$

$$\therefore \underline{E(X) = \frac{1}{p}}$$

---

§ $P(X \leq r) = 1 - q^r$

(i) Proof:

$$P(X \leq r) = P(X=1) + P(X=2) + P(X=3) + \dots P(X=r)$$

$$= p + qp + q^2 p + \dots + q^{r-1} p$$

$$= \frac{p[1 - q^r]}{1-q}$$

$$= \frac{\cancel{p}[1 - q^r]}{\cancel{p}} \qquad [1 - q = p]$$

{G.P.
{first term = p
{common ratio = q
{No of terms = r

$$\{S_n = \frac{a[1 - r^n]}{(1-r)}$$

$$\therefore \underline{P(X \leq r) = 1 - q^r}$$

§ $\underline{P(X > r) = q^r}$

§ $\underline{P(X < r) = P(X \leq r-1) = 1 - q^{r-1}}$

18. A fair six-sided die, with faces marked 1,2,3,4,5,6 is thrown 7 times repeatedly until a 3 is obtained. ---[2]
Find the probability that obtaining a 3 requires fewer than 7 throws.
[SP-20/05/Q5(C)]

Solution: $P(\text{getting } 3) = p = \frac{1}{6}$; $q = \frac{5}{6}$, $n = 7$ $[P(X \leq r) = 1 - q^r]$
$X \sim Geo(\frac{1}{6})$
$P(\text{getting 3 in less than 7 throws}) = P(X < 7) = P(X \leq 6) = 1 - q^6 = 1 - (\frac{5}{6})^6$
$= 0.665$

19. Two fair coins are thrown at the same time. The random variable X is the number of throws of the two coins required to obtain two tails at the same time.
(a) Find the prob. that two tails are obtained for the first time on the 7th throw. ---[2]
(b) Find the prob. that it takes more than 9 throws to obtain two tails for the first tails. ---[2]
[W-21/51/Q1]

Solution: When two coins are tossed, sample space $S = \{HH, HT, TH, TT\}$
$p = P(\text{Two tails}) = \frac{1}{4}$, $q = 1 - \frac{1}{4} = \frac{3}{4}$
(a) $X \sim Geo(p) \rightarrow X \sim Geo(\frac{1}{4})$
$P(X = 7) = q^6 p = (\frac{3}{4})^6 \cdot \frac{1}{4} = 0.0445$ $\{P(X = r) = q^{r-1} \cdot p$

(b) $P(X > 9) = (\frac{3}{4})^9 = 0.0751$ $\{P(X > r) = q^r$

20. In a certain region, the prob. that any given day in october is wet is 0.16, independently of other days.

(a) Find the prob. that, in a 10-day period in october, fewer than 3 days will be wet. --[3]

(b) Find the prob. that the first wet day in October is 8 october. --[2]

(c) For 4 randomly chosen years, find the prob. that in exactly 1 of these years the first wet day in October is 8 october. --[2]

[W-21 | 52 | Q5]

Solution (a) $p = P(\text{wet day}) = 0.16$, $q = 0.84$, $n = 10$, $X \sim B(n, p)$

$P(X<3) = P(0,1,2) = {}^{10}C_0 (.16)^0 (.84)^{10} + {}^{10}C_1 (.16)^1 (.84)^9 + {}^{10}C_2 (.16)^2 (.84)^8$ $\{P(X=r) = {}^nC_r\, p^r q^{n-r}$

$= 0.17490 + 0.333145 + 0.28555 = 0.794$

(b) Now Geo(p) or $X \sim Geo(0.16)$, $p = 0.16$, $q = 0.84$

$P(X=8) = q^7 \cdot p = (0.84)^7 \cdot (0.16) = \underline{0.0472}$ ✓

(c) Now $p = P(\text{wet day on } 8_{th}) = 0.0472$, $q = 1 - 0.0472 = 0.9528$, $n = 4$

$P(r=1) = {}^4C_1 \cdot (0.0472)^1 (0.9528)^3$ $\{X \sim B(n,p) = B(4, 0.0472)$

$= \underline{0.163}$ ✓

{(X) Geo & Binomial both (b) & (c)

21. In a game, Tim throws three darts at a board. This is called a 'turn'. The centre of the board is called the bull's-eye.

The random variable X is the number of darts that hit the bull's eye. The Prob distribution of X is given by:

x	0	1	2	3
$P(X=x)$	0.6	0.3	0.05	0.05

[W-21 | 53 | Q6(d)]

Jim is practising for a competition and he repeatedly throws three darts at the board. Find the prob. that Jim first succeeds in hitting the bullseye with all three darts on his 9_{th} turn. --[1]

Solution: $P = P(\text{all three darts at the board}) = 0.05$, $q = 0.95$, $X \sim Geo(0.05)$

$P(r=9) = q^8 \cdot p = (0.95)^8 \cdot (0.05) = \underline{0.0332}$ ✓

22. A fair spinner with 5 sides numbered 1,2,3,4,5 is spun repeatedly. The score on each spin is the number on the side on which the spinner lands.

(a) Find the prob. that a score of 3 is obtained for first time on 8_{th} spin. --[1]

(b) Find the prob. that fewer than 6 spins are required to obtain a score of 3 for the first time. [M-21/52/Q1] --[2]

Solution: $S = \{1,2,3,4,5\}$, $p = P(\text{score } 3) = \frac{1}{5}$, $q = \frac{4}{5}$ $\{X \sim Geo(p)\}$

(a) $P(r=8) = q^7 p = \left(\frac{4}{5}\right)^7 \cdot \left(\frac{1}{5}\right) = \underline{0.0419}$ ✓

(b) $P(X<6) = P(X \leq 5) = 1 - \left(\frac{4}{5}\right)^5 = \underline{0.672}$ ✓ $\{P(X \leq r) = 1 - q^r$

23. An ordinary fair die is thrown repeatedly until a '5' is obtained. The number of throws is denoted by the random variable X.

(a) Write down the mean of X. ---[1]

(b) Find the probability that a '5' is first obtained after the third throw but before 8_{th} throw. ---[2]

(C) Find the prob. that a '5' is first obtained in fewer than 10 throws. [S-21/52/Q1] --[2]

Solution: For an ordinary throw $S = \{1,2,3,4,5,6\}$

$p = P(\text{getting '5'}) = \frac{1}{6}$, $q = 5/6$

(a) $\text{Mean} = E(X) = \frac{1}{p} = \frac{1}{\frac{1}{6}} = \underline{6}$ ✓

(b) $P(3 < X < 8) = P(X = 4,5,6,7) = \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^5 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^6 \cdot \frac{1}{6}$

$= \underline{0.300}$ ✓

(c) $P(X<10) = P(X \leq 9) = 1 - \left(\frac{5}{6}\right)^9$ $[P(X \leq r) = 1 - q^r$

$= \underline{0.806}$ ✓

24. Three fair six-sided dice, each with faces marked 1,2,3,4,5,6, are thrown at the same time, repeatedly. For a single throw of the three dice, the score is the sum of the numbers on the top faces.

(a) Find the probability that the score is 4 on a single throw of the three dice. ---[3]

(b) Find the prob. that score of 18 is obtained for first time on the 5_{th} throw of the three dice. ---[3]

[S-21/53/Q4]

Solution: Three dice are thrown: $S=\{(1,1,1),(1,1,2),---(1,1,6)$
$(1,2,1),(1,2,2)$ - - - -
- - - - - $(6,6,6)\}$

$n(S)=6\times6\times6=216$ ✓

(a) Score $4 \rightarrow (1,1,2),(1,2,1),(2,1,1)$

$P(\text{Total score }4)=\frac{3}{216}=\frac{1}{72}=0.0138$ ✓

(b) Total score $18=(6,6,6)$

$p=P(\text{Total }18)=\frac{1}{216}$, $q=\frac{215}{216}$; $X\sim Geo(p)$ $\{P(X=r)=q^{r-1}\cdot p$

$P(X=5)=q^4\cdot p=\left(\frac{215}{216}\right)^4\cdot\frac{1}{216}=0.00454$ ✓

---

25. An ordinary fair dice is thrown until a '6' is obtained.

(a) Find the prob. that obtaining a '6' takes more than 8 throws. --[2]

Two ordinary fair dice are thrown together until a pair of 6s is obtained. The number of throws taken is denoted by the random variable X.

(b) Find the expected value of X. ---[1]

(c) Find the prob. that obtaining a pair of 6s takes either 10 or 11 throws. --[2]

[W-20/53/Q2]

Solution (a) $p=P(\text{getting }6)=\frac{1}{6}$; $q=\frac{5}{6}$; $Geo(p)$

$P(X>8)=q^8=\left(\frac{5}{6}\right)^8=0.233$ ✓

(b) $P(\text{getting a pair of }6)=\frac{1}{36}=p$, $q=\frac{35}{36}$

$\text{Expected}(X)=\frac{1}{p}=\frac{1}{\frac{1}{36}}=36$ ✓

(c) $P(X=10\text{ or }11)=q^9p+q^{10}p$

$=\left(\frac{35}{36}\right)^9\cdot\frac{1}{36}+\left(\frac{35}{36}\right)^{10}\cdot\frac{1}{36}$

$=0.0425$ ✓