Date 22.02.2022

S.1

Probability and Statistics 1

Probability Notes

Content. Pages.

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(S1) # Probability (Chance) - Notes

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§ When we perform an experiment, and want to find out, what is the probability (or chance) of happening of an outcome (or event), we must know all the outcomes.

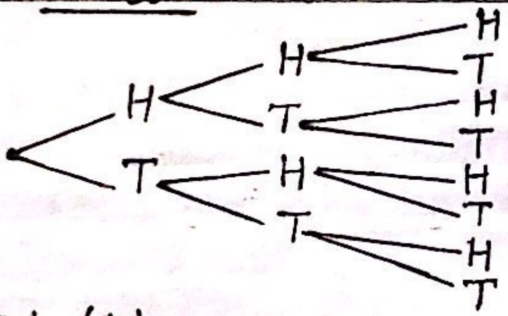
§ Sample space: Set of all posible outcomes, when we perform an experiment, is called its sample space, denoted by S, and the number of outcomes is denoted by $n(S)$.

Examples:

1. When we toss a coin (i) $S = \{H, T\}$ $n(S) = 2$

(ii) A coin is tossed twice $S = \{HH, HT, TH, TT\}$
(or a pair of coins are tossed) $n(S) = 2^2 = 4$ ✓

(iii) A coin is tossed three times $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
(or three coin are tossed at a time)



$n(S) = 2^3 = 8$ ✓

2. Die / Dice

(i) We roll a die (with sixface) $S = \{1, 2, 3, 4, 5, 6\}$; $n(S) = 6$

(ii) A pair of dice is thrown $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), \ldots$
$\ldots (6,6)\}$

$n(S) = 6^2 = 36$ ✓

(iii) Three dice are thrown; $S = \{(1,1,1), (1,1,2), (1,1,3), (1,1,4) \ldots (1,1,6)$
$(1,2,1), (1,2,2) \ldots (1,2,6)$
$\ldots$
$\ldots (6,6,6)\}$

$n(S) = 6^3 = 216$ ✓

3. A tetrahedral dice: $S = \{1, 2, 3, 4\}$; $n(S) = 4$

4 A bag contain 5 white and 4 black balls. Two balls are drawn at random, without replacement, Total number of balls = 9

$n(S) = {}^9C_2$

§ Equally Likely outcomes: When the chances of happening of each out are equal. and we say:
(i) Fair die (unbiased die) is rolled. (ii) A fair coin is tossed
(iii) A ball is drawn from a bag with replacement.

§ Event : Event $\subseteq$ Sample space
Example: A pair of dice is thrown $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$n(S) = 36$ $(2,1), (2,2)$ - - - - -
- - - - - $(6,6)\}$

Event E_1 = Same number of both the dice
, doublet, $= \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$ $n(E_1) = 6$

Event E_2 = Sum of numbers on the two dice is 8.
$= \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

1 2 3 4 5 6
6 5 4 3 2

$n(E_2) = 5$

§ Probability (Chance):
When an experiment is performed such that, all the outcomes are equally likely,
Then the probability of happening of an event 'E' denoted by $P(E)$:

$$P(E) = \frac{\text{Number of favourable (to E) outcomes}}{\text{Total number of outcomes.}} \quad \frac{n(E)}{n(S)}$$

Note:
(i) $0 \leq P(E) \leq 1$
(ii) $P(E) + P(\text{not } E) = 1$; $P(\text{not } E) = 1 - P(E)$
(or $P(E) + P(\bar{E}) = 1$)

Example: A pair of fair dice is thrown.
(i) Find the prob. of getting a doublet (E_1)
(ii) Find the prob. of getting sum of outcomes be 8. (E_2)

Solutions: (i) $P(\text{doublet}) = \frac{6}{36} = \frac{1}{6}$ ✓
(ii) $P(\text{Sum of number is } 8) = \frac{5}{36}$ ✓

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1. A bag contains 10 pink balloons, 9 yellow, 12 green and 9 white balloons. 7 balloons are selected at random without replacement. Find the probability that exactly 3 of them are green. --[3]

[M-17/62/Q2]

Solution: Total number of balls $= 10+9+12+9 = 40$ ✓

ball drawn $= 7$ (of which 3-green & 4-non green)

Number of green balls $= 12$ ✓ & non-green $= 40-12 = 28$ ✓

$$P(3\text{ Green}) = \frac{{}^{12}C_3 \times {}^{28}C_4}{{}^{40}C_7} = \boxed{0.242}$$ ✓

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2. The 8 letters in the word RESERVED are arranged in a random order. Find the probability that the arrangement has 'V' as the first letter and 'E' as the last letter. ---[3]

[W-20/53/Q6]

Solution: Total number of arrangement any way

$$= \frac{8!}{3!\,2!} = 3360$$ ✓

{RESERVED { R-2, E-3, V-1, S-1, D-1, Total No = 8

Now Given V. 'RESERD' E

Number of ways (with V the first and E in last) $= \frac{6!}{2! \times 2!} = 180$ ✓

$\therefore$ Required Prob $= \frac{180}{3360} = \boxed{0.0536}$ ✓

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3. Three six-sided fair dice, each with faces marked 1, 2, 3, 4, 5, 6. are thrown at the same time. For a single throw of the three dice, the score is the sum of numbers on the top of the faces. Find the prob. that the score is 4 on a single throw of three dice. ---[3]

[S-21/53/Q4(a)]

Solution: Sample Space $S = \{(1,1,1), (1,1,2) ---- (1,1,6)$
$(1,2,1)\ (1,2,2)\ \ - \ - - (6,6,6)\}$ $n(S) = 6\times6\times6 = 216$ ✓

Event 'E' Sum is 4; $E = \{(1,1,2), (1,2,1), (2,1,1)\}$ $n(E) = 3$

$$P(\text{Total } 4) = \frac{n(E)}{n(S)} = \frac{3}{216} = \frac{1}{72}$$ ✓

§ Probability when the outcomes are not equally likely (or, are not equiprobable):

Let $S = \{ W_1, W_2, W_3, --- W_n \}$

and $P(W_1) = p_1$, $P(W_2) = p_2$ --- $P(W_n) = p_n$

Now let event $E = \{ W_2, W_3, W_5 \}$

$\Rightarrow P(E) = (p_2 + p_3 + p_5)$ ✓

Note:
(i) $0 \leq p_i \leq 1$
(ii) $\Sigma p_i = 1$

Example:

4. The faces of a biased die are numbered 1, 2, 3, 4, 5, 6. The random variable X is the score when the die is thrown. The following is the prob. distribution table for X:

x	1	2	3	4	5	6
$P(X=x)$	p	p	p	p	0.2	0.3

(i) Find the value of p.
(ii) Find the prob. of getting 2 or 5 ; $E = \{2, 5\}$

Solution: (i) $\Sigma p_i = 1 \Rightarrow 4p + 0.2 + 0.3 = 1 \Rightarrow 4p = 0.5 \Rightarrow p = 0.125$ ✓

(ii) $P(2 \text{ or } 5) = P(2) + P(5)$

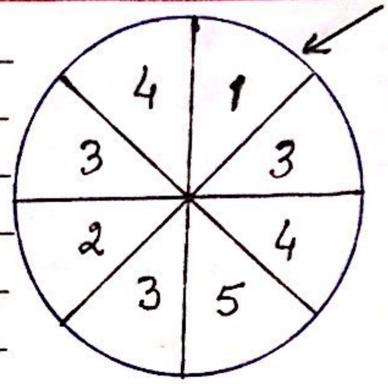
$= p + 0.2 = 0.125 + 0.2 = 0.325$ ✓

§ Trials and expectation:

When we repeat an experiment, that is trial. If the experiment is repeated n times, and $P(A)$ is the prob. of happening of an event 'A', when the experiment is performed once, then expectation is the number of times the event A is likely to happen, in 'n' trials of the experiment, denoted by: $E(A) = n \cdot P(A)$

or $E(A) = n \cdot p$ $\{ P(A) = p \}$

Example 5. A numbered wheel is divided into eight sectors of equal size, as shown. The wheel is spun until it stops with the arrow pointing at one of the numbers.



Axel decides to spin the wheel 400 times.

(a) Find the number of times the arrow is <u>not</u> expected to point at a 4.

(b) How many more times must Axel spin the wheel so that the expected number of times that the arrow points at a 4 is at least 160?

Solution: (a) $P(\text{not } 4) = \frac{6}{8} = \frac{3}{4} = p$ (let) $\Rightarrow E(\text{not } 4) = np = 400 \times \frac{3}{4} = \underline{300}$ ✓

Number of trials $= 400$;

(b) $P(4) = \frac{2}{8} = \frac{1}{4}$

Let x more times the wheel is spun $\rightarrow$ Total number $N = (400 + x)$

Expected number of $4 = (400 + x) \times \frac{1}{4}$

Given $\frac{1}{4} \times (400 + x) \geq 160 \Rightarrow 400 + x \geq 640$

$\Rightarrow x \geq 240$ ✓

$\therefore$ Axel must spin the wheel at <u>240</u> more times.

Example 6: Savita randomly picks one of the 10 cards shown.

A | C | B | C | B | A | C | C | B | C

If she repeats this 40 times, how many times is Savita expected to pick a card that is not blue and does not a letter B on it.

Solution: Number of red cards with A or C = 3 out of 10 cards.

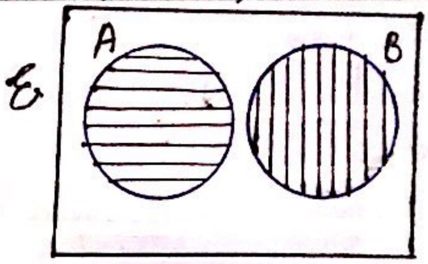
$p = P(\text{red with A or C}) = \frac{3}{10}$

Expectation $= np = 40 \times \frac{3}{10} = 12$ ✓ $(n = 40)$

§ Mutuall Exclusive events:

Two events A and B are mutually exclusive if there is no common outcome between A and B.

$$A \cap B = \phi \quad \text{or} \quad P(A \cap B) = 0$$



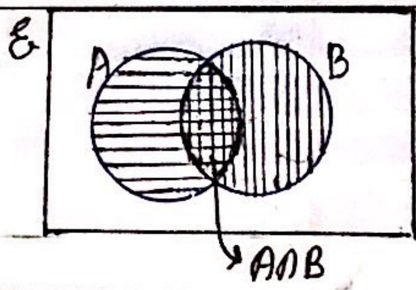
§ Addition Law of Probability:

(i) Given two (or more) mutually exclusive events A, B, ..

$$P(A \text{ or } B) = P(A) + P(B) \quad \{\text{or } P(A \cup B) = P(A) + P(B)$$

otherwise

(ii) If the events A and B are <u>not</u> mutually exclusive, then



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\left(\text{or } P(\text{or } B) = P(A) + P(B) - P(A \text{ and } B)\right)$$

Note: $A \cup B$ (or (A or B)) denotes the outcomes in A or B or both A and B.

Example 7: A pair of fair dice are rolled, find the prob. of a getting a total of 9 or a doublet (same number on both the dice).

Solution: Sample space $S = \{(1,1), (1,2), (1,3) \text{ --- } (1,6)$
$(2,1)\ (2,2) \text{ --- --- --- } (6,6)\}$

$n(S) = 6 \times 6 = 36$

Getting a total of 9 ; event $A = \{(3,6), (4,5), (5,4), (6,3)\}$ $\quad \left\{\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ & & 6 & 5 & 4 & 3 \end{matrix}\right\}$

$n(A) = 4 \quad ; \quad P(A) = 4/36$

Getting a doublet, event $B = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$, $n(B) = 6$

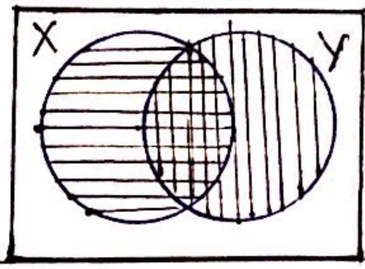
$A \cap B = \phi \quad \therefore$ A and B are <u>mutually exclusive event</u>.

$$P(A \text{ or } B) = P(A) + P(B) = \frac{4}{36} + \frac{6}{36} = \frac{10}{36} = \frac{5}{18} \checkmark$$

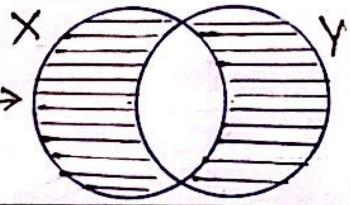
Example 8. Events X and Y are such that, $P(X)=0.5$, $P(Y)=0.6$ and $P(X\cap Y)=0.2$
(a) State, giving a reason, whether X and Y are mutually exclusive.
(b) Using a Venn diagram, or otherwise, Find $P(X\cup Y)$
(c) Find the prob. that X or Y, but not both, occur.

Solution (a) X and Y are not mutually exclusive, because $P(X\cap Y)=0.2\neq 0$ ✓

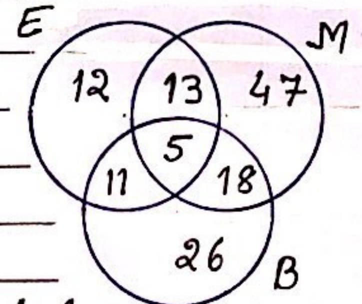
(b) $P(X\cup Y)=P(X)+P(Y)-P(X\cap Y)$
$=0.5+0.6-0.2=\underline{0.9}$ ✓



(c) $P(X$ or Y but not Both$)$
$=(0.5-0.2)+(0.6-0.2)=0.3+0.4$
$=\underline{0.7}$ ✓



Example 9: A garage repaired 132 vehicles last month. The number of vehicles that required electrical (E), mechanical (M) and bodywork (B) repairs are given in the diagram opposite.



Find the prob. that a randomly selected vehicle required.
(a) mechanical or bodywork repairs.
(b) no body work repairs.
(c) exactly two types of repairs.

Solution (a) $P(M \text{ or } B)=\dfrac{13+47+5+18+11+26}{132}=\underline{\dfrac{10}{11}}$ ✓ $\Big\}$ or $\dfrac{132-12}{132}=\dfrac{10}{11}$

(b) $\dfrac{12+13+47}{132}=\underline{\dfrac{6}{11}}$ ✓

(c) $\dfrac{13+11+18}{132}=\underline{\dfrac{7}{22}}$ ✓

§ Conditional Probability:

If A and B are two events, then the conditional prob. of happening of B, given that the event A has happened, is:

$$P(B/A) = \frac{P(A \text{ and } B)}{P(A)} \quad \text{---- (i)}$$

Multiplication Law of Probability:

$$P(A \text{ and } B) = P(A) \cdot P(B/A) \quad (\text{from (i)})$$

Example 10: On a library shelf there are seven novels, three dictionaries and two atlases. Two books are randomly selected without replacement. Find the prob. that the selected books are both novels.

Solution: Novels = 7, dictionaries = 3, atlases = 2 → Total = 12

$$P(\text{both novels}) = P(N_1) \times P(N_2/N_1) = \frac{7}{12} \times \frac{6}{11} = \frac{7}{22}$$

Example 11: A number between 10 and 100 inclusive is selected at random. Find the prob. that the number is a multiple of 5, given that none of its digit is a 5.

Solutions: Numbers between 10 and 100 (inclusive) = 91 (55 is common)

Numbers with one/both digit 5 → (15, 25 --- 95; 50, 51, --- 59) = 18

∴ Numbers none has digit 5 = 91 − 18 = 73

Let A is the event – None of the digits is 5 ⇒ $P(A) = 73/91$ --- (i)

Let B is the event – Divisible by 5 → (18 + 9) = 27

$A \cap B = \{10, 20, 30, 40, 60, 70, 80, 90, 100\}$, $n(A \cap B) = 9$

$$\therefore P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{9/91}{73/91} = \frac{9}{73} = 0.123$$

§ The Law of Total Probability:

Let S be the sample space, and let $E_1, E_2, \dots E_n$ be 'n' mutually exclusive and exhaustive events associated with a random experiment. If A is an event occurs with E_1 or E_2 or -- E_n, then:

$$P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + \dots + P(E_n) \cdot P(A/E_n)$$

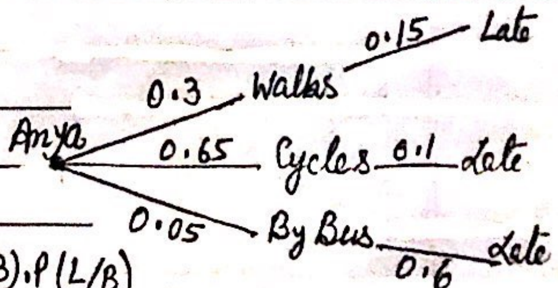
Baye's Theorem: (using the same condition as above)

$$P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{P(A)}$$

Example 12: When Anya goes to school, the prob. that she walks is 0.3, and the prob. that she cycles is 0.65. If she does not walk or cycle, she takes the bus. When Anya walks, the prob that she is late is 0.15. When she cycles the prob. that she is late is 0.1 and when she takes the bus the prob. that she is late is 0.6. Given that Anya is late, find the prob that she cycles.

[W-16/61/Q5] ---[5]

Solution: $P(C/L) = \dfrac{P(C \cap L)}{P(L)}$ ---(i)



$P(\text{By Bus}) = 1 - (0.3 + 0.65) = 0.05$ ✓

$$P(L) = P(W) \cdot P(L/W) + P(C) \cdot P(L/C) + P(B) \cdot P(L/B)$$
$$= 0.3 \times 0.15 + 0.65 \times 0.1 + 0.05 \times 0.6$$
$$= 0.14 \quad \text{----} \; (ii)$$

$$P(C \cap L) = P(C) \cdot P(L/C) = 0.65 \times 0.1 = 0.065 \quad \text{---} (iii)$$

from (i)

$$\therefore P(C/L) = \frac{P(C \cap L)}{P(L)} = \frac{0.065}{0.14} = \underline{0.464} ✓$$

$[\, P(C \cap L) = P(C) \cdot P(L/C) \,]$

Example 13: For her bedtime drink, Suki has either chocolate, tea or milk with probabilities 0.45, 0.35 and 0.2 respectively. When she has chocolate, the prob. that she has a biscuit is 0.3. When she has tea, the prob that she has a biscuit is 0.6. When she has milk, she never has a biscuit.
Find the prob. that Suki has tea given that she does not have a biscuit.

[W-21/51/Q3]

Solution:
$$P(T/B') = \frac{P(T \cap B')}{P(B')} \quad \text{---(i)}$$

$$P(B') = P(C) \cdot P(B'/C) + P(T) \cdot P(B'/T) + P(M) \cdot P(B'/M)$$
$$= 0.45 \times 0.7 + 0.35 \times 0.4 + 0.2 \times 1 = 0.655 \checkmark \text{---(ii)}$$
$$P(T \cap B') = P(T) \cdot P(B'/T) = 0.35 \times 0.4 = 0.14 \checkmark \text{----(iii)}$$

from (ii) and (iii) in (i) $P(T/B') = \frac{0.14}{0.655} = \underline{0.214} \checkmark$

Example 14: Each of the 180 students at a college plays exactly one of the piano, the guitar and the drum. The numbers of male and female students who play the piano, the guitar and the drum are given in the table:

	Piano	Guitar	Drum
Male	25	44	11
Female	42	38	0

(a) Find the prob. the student plays guita. --[1]
(b) Find the prob. that the student is male given that student plays the drums. --[2

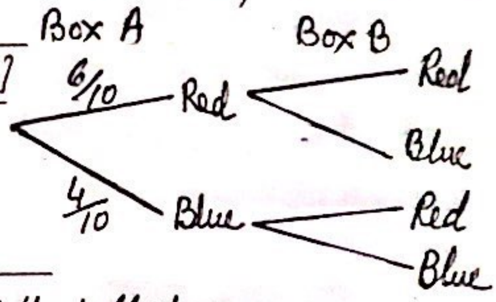
[W-21/52/Q1]

Solution (a) $P(\text{Student plays Guitar}) = \frac{\text{No. of Students playing Guita}}{\text{Total no. of students}} = \frac{82}{180} = \underline{0.456} \checkmark$

(b) $P(M/D) = \frac{P(M \cap D)}{P(D)} = \frac{11/180}{\frac{20}{180} + \frac{11}{180}} = \frac{11}{31} = \underline{0.355} \checkmark$

Example 15: Box A contains 6 red balls and 4 blue balls. Box B contains x red balls and 9 blue balls. A ball is chosen at random from box A and placed in box B. A ball is then chosen at random from box B.

(a) Complete the tree diagram, giving the remaining four prob. in terms of x. ---[3]

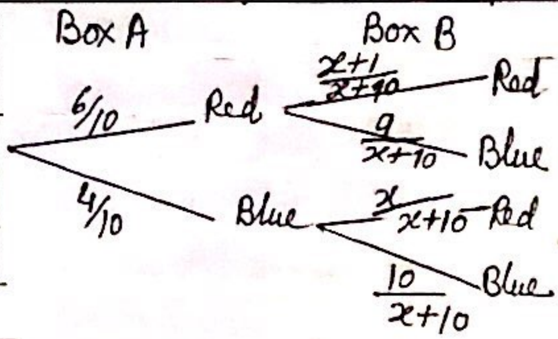


(b) Show that the prob. that both balls chosen are blue is $\frac{4}{x+10}$. ---[2]

(c) It is given that the prob. that both balls chosen are blue is $1/6$. Find the prob. that the ball chosen from Bag A is red given that the ball chosen from box B is red. ---[5]

[W-21/53/Q7]

Solution: (a) tree diagram ⟶



(b) $P(\text{both blue}) = \frac{4}{10} \times \frac{10}{x+10}$

$= \frac{4}{x+10}$ ✓

(C) Given $P(\text{both blue}) = \frac{4}{x+10} = \frac{1}{6}$ [Part (b)]

$\Rightarrow x+10 = 24 \Rightarrow x = 14$ ✓

Now $P(A\text{red}/B\text{red}) = \frac{P(A\text{red} \cap B\text{red})}{P(B\text{red})}$ ----(i)

$P(B\text{red}) = P(A\text{red})(B\text{red}/A\text{red}) + P(A\text{blue}/B\text{red})$

(for $x=14$) $\Rightarrow = \frac{6}{10} \times \frac{x+1}{x+10} + \frac{4}{10} \times \frac{x}{x+10} = \frac{6}{10} \times \frac{15}{24} + \frac{4}{10} \times \frac{14}{24}$

$= 73/120$ ✓ --(ii)

$P(A\text{red} \cap B\text{red}) = \frac{6}{10} \times \frac{x+1}{x+10} = \frac{6}{10} \times \frac{15}{24} = \frac{45}{120}$ ---(iii)

from (ii) and (iii) in (i)

$P(A\text{red}/B\text{red}) = \frac{45}{120} \Big/ \frac{73}{120} = \frac{45}{73} = \underline{0.616}$ ✓

Example 16: To gain a place at a science college, students first have to pass a written test and then a practical test. Each student is allowed a maximum of two attempts at the written test. A student is only allowed a second attempt if they fail the first attempt. No student is allowed more than one attempt at the practical test. If a student fails both attempts at the written test, then they cannot attempt the practical test.

The prob. that a student will pass the written test at the first attempt is 0.8. If the student fails the first attempt at the written test, the prob. they will pass at the second attempt is 0.6. The prob. that a student will pass the practical test is alway 0.3.

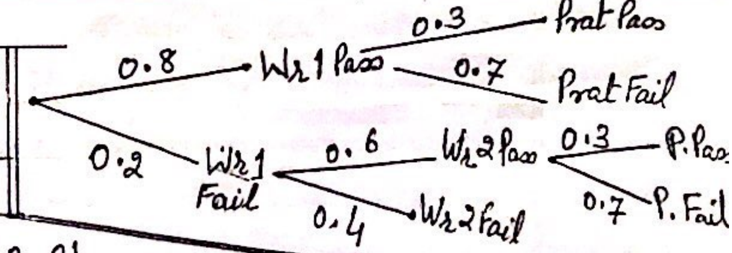
(a) Draw a tree diagram to represent this information, showing the prob. on the branches. --[3]

(b) Find the prob. that a randomly chosen student will succeed in gaining a place at the college. --[2]

(c) Find the prob. that a randomly chosen student passes the written test at the first attempt given that the student succeeds in gaining a place at the college. [S-21/51/Q4] --[2]

Solution (a)

Tree-diagram ⟶



(b) $P(\text{Gaining in college}) = P(W_1 P)\times P(Pr\ P) + P(W_1 F)\times P(W_2 P)\times P(P.P)$

$= 0.8\times 0.3 + 0.2\times 0.6\times 0.3 = \underline{0.276}$ ✓

(c) $P(W_1 P / \text{Gain college}) = \dfrac{P(W_1 \cap Pr\ Pass)}{P(\text{Gain College})} = \dfrac{0.8\times 0.3}{0.276} = \dfrac{0.24}{0.276}$

$= \underline{0.87}$ ✓

Example 17: On each day that Alexa goes to work, the prob. that she travels by bus, by train or by car are 0.4, 0.35 and 0.25 respectively. When she travels by bus, the prob. that she arrives late is 0.55. When she travels by train, the prob. that she arrives late is 0.7. When she travels by car, the prob. she arrives late is x.
On a randomly chosen day when Alexa goes to work, the prob. that she does not arrive late is 0.48.

(a) Find the value of x. ---[3]

(b) Find the prob. that Alexa travels to work by train given that she arrives late. [S-21/52/Q3] --[3]

Solution: Prob (Alexa is late to work) $= 1 - 0.48 = 0.52$ ---(i)

(a) $P(\text{Alexa is late to Work}) = P(B)\cdot P(L/B) + P(T)\cdot P(L/T) + P(C)\cdot P(L/C)$

$$= 0.4 \times 0.55 + 0.35 \times 0.7 + 0.25 \times x$$

$$= 0.22 + 0.245 + 0.25x = 0.52 \text{ from (i)}$$

$$\Rightarrow 0.25x = 0.055$$

$$\Rightarrow x = 0.22 \checkmark \quad \text{----(ii)}$$

(b) $$P(T/L) = \frac{P(T \cap L)}{P(L)} = \frac{0.35 \times 0.7}{0.52} = \frac{0.245}{0.52} = 0.471 \checkmark$$

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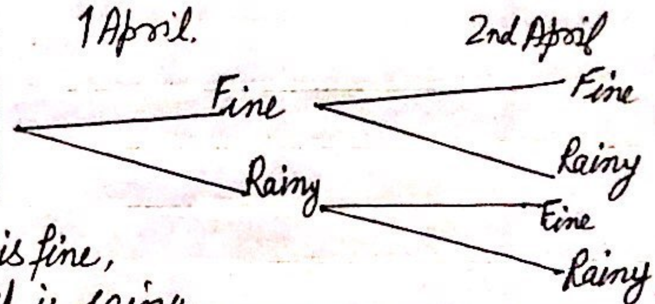
Example 18: In a certain country, weather each day is classified as fine or rainy. The Prob. that a fine day is followed by a fine day is 0.75 and that the prob. that a rainy day is followed by a fine day is 0.4. The prob. that is fine on April 1 is 0.8. The tree diagram below shows the possibilities for the weather on 1 April and 2 April.

(a) Complete the tree diagram to show the probabilities; --[1]

(b) Find the prob. that 2 April is fine. --[2]

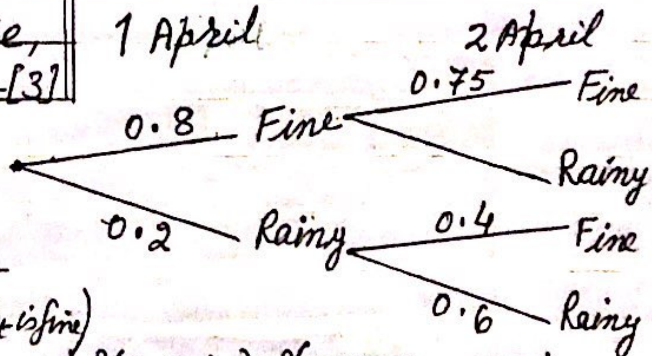


Let X be the event that 1 April is fine, and Y be the event that 3 April is rainy.

(c) Find value of $P(X \cap Y)$ --- --[3] | W-20/52/Q4 |

(d) Find the prob. that 1 April is fine, given that 3 April is rainy. ---[3]

Solution (a) Tree diagram →



(b) $P(2\text{Ap. is fine}) = P(1\text{Ap fine}) \cdot P(2\text{nd fine}/1\text{st is fine}) + P(1\text{st rainy}) \times P(2\text{nd fine}/1\text{st rainy})$

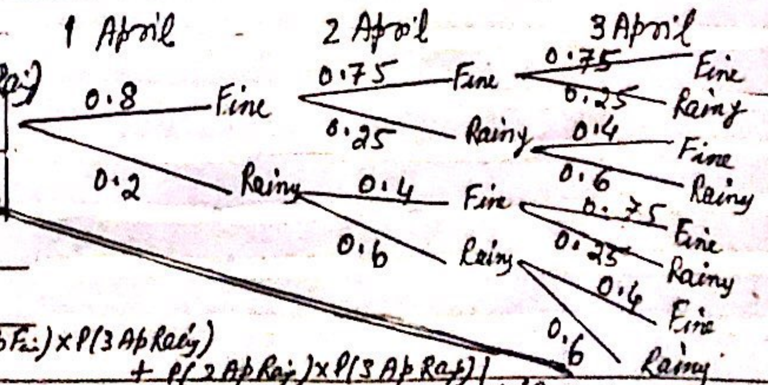
$= 0.8 \times 0.75 + 0.2 \times 0.4 = 0.6 + 0.08 = \underline{0.68}$ ✓

(c) $P(X \cap Y) = P(X) \cdot P(Y/X)$

$= P(1\text{Ap Fine})\{P(2\text{Ap Fine}) \times P(3\text{Ap Rainy}) + P(2\text{Ap Rainy}) \times P(3\text{Ap Rainy})\}$

$= 0.8\{0.75 \times 0.25 + 0.25 \times 0.6\}$

$= \underline{0.27}$ ✓ --- (i)



(d) $P(3\text{Ap. Rainy}) = P(1\text{Ap Fine})[P(2\text{Ap Fine}) \times P(3\text{Ap Rainy}) + P(2\text{Ap Rainy}) \times P(3\text{Ap Rainy})]$ → (Part C)

$+ P(1\text{Ap Rainy})[P(2\text{Ap F}) \times P(3\text{Ap R}) + P(2\text{Ap R}) \times P(3\text{Ap R})]$

$\Rightarrow P(Y) = 0.27 + 0.2\,[0.4 \times 0.25 + 0.6 \times 0.6]$

$= 0.27 + 0.092 = 0.362$ ✓ --- (ii)

$P(X/Y) = \dfrac{P(X \cap Y)}{P(Y)} = \dfrac{0.27}{0.362} = \underline{0.746}$ ✓ (from (i) and (ii))

§ Independent events and multiplication law:

Two events are said to independent if either can occur without being affected by the occurrence of the other.

Example (i) Making selections with replacement.
(ii) Rolling two dice.

Multiplication law for independent events:

$$P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B)$$

Example 19: Ellie throws two fair tetrahedral dice. Each with faces numbered 1,2,3,4. She notes the numbers on the faces that the dice land on. Event S is "the sum of two numbers is 4," and Event T is "the product of the two numbers is an odd number." Determine whether events S and T are independent. ---[5]

W-15	63	Q3

Solution: $n(\text{Sample space}) = 4 \times 4 = 16$; Tetrahedral dice $= \{1,2,3,4\}$

S = Sum of numbers is $4 = \{(1,3),(3,1),(2,2)\}$; $\Rightarrow n(S) = 3$

T = Product of two numbers is odd $= \{(1,1),(1,3),(3,1),(3,3)\}$

$\Rightarrow n(T) = 4$

$S \cap T = \{(1,3),(3,1)\} \Rightarrow n(S \cap T) = 2$

$P(S \cap T) = \frac{2}{16}$ ---(i)

$P(S) = 3/16$; $P(T) = \frac{4}{16} \Rightarrow P(S) \times P(T) = \frac{3}{16} \times \frac{4}{16} = \frac{3}{64}$ --(ii)

$\frac{2}{16} \neq \frac{3}{64}$ for (i) & (ii)

or $P(S \cap T) \neq P(S) \cdot P(T)$

$\therefore$ The events S and T are not independent.

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Example 20. For a group of 250 cars the numbers, classified by colours and country of manufacture, are shown in the table:

	Germany	Japan	Korea
Silver	40	26	34
White	32	22	26
Red	28	12	30

X is the event that the selected car is white. Y is the event that the selected car is manufactured in Germany.
Determine whether events X and Y are independent. ---[5]
[W-16/63/Q4]

Solution: $n(\text{White cars}) = 80 \Rightarrow P(X) = \frac{80}{250} = \frac{8}{25}$ ---(i)

$n(\text{Germany-Car}) = 100 \Rightarrow P(Y) = \frac{100}{250} = \frac{2}{5}$ ----(ii)

$n(\text{Silver in Germany}) = 32 \Rightarrow P(X \cap Y) = \frac{32}{250} = \frac{16}{125}$ --(iii)

Now $P(X) \times P(Y) = \frac{8}{25} \times \frac{2}{5} = \frac{16}{125}$ ----(iv)

from (iii) and (iv) $P(X) \times P(Y) = P(X \cap Y) = \frac{16}{25}$

$\therefore$ The events X and Y are independent.

Example 21: Two ordinary fair dice, one red and the other blue, are thrown.
Event A is "the score on red die is divisible by 3"
Event B is "the sum of the two scores is at least 9".
(a) Find $P(A \cap B)$. ---[2]
(b) Determine whether or not the events A and B are independent. ---[2]
[W-20/51/Q1]

Solution (a) $A = \{(3,1),(3,2) \text{ --- } (3,6),(6,1)(6,2) \text{ -- } (6,6)\}$

$B = \{(3,6),(6,3),(4,5),(5,4)\ (5,5),(5,6)(6,5),(6,6)\}$ $(6,4),(4,6)\}$

$A \cap B = \{(3,6),(6,3),(6,5),(6,6),(6,4)\}$

$\therefore P(A \cap B) = \frac{5}{36}$, $P(A) = \frac{12}{36} = \frac{1}{3}$; $P(B) = \frac{10}{36}$

$P(A) \cdot P(B) = \frac{1}{3} \times \frac{10}{36} = \frac{5}{54} \neq 5/36$

$\therefore P(A) \cdot P(B) \neq P(A \cap B)$

Hence events A and B are not independent.

Blue \ Red	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Example 22: Two ordinary fair dice are thrown and the numbers obtained are noted. Event 'S' is "the sum of the numbers is even". The event 'T' is "The sum of the numbers is either less than 6 or a multiple of 4 or both". Determine whether the events 'S' and 'T' are independent. --[4]
[S-19|62|Q1]

Solution: $S = \{\text{Sum is even}\} = \{\text{both odd or both even}\} = n(S) = \underbrace{3\times3}_{odd} + \underbrace{3\times3}_{even} = 18$

$P(S) = \frac{18}{36} = \frac{1}{2}$ ---(i)

$T = \{\text{Sum is less than 6}\} \cup \{\text{sum is a multiple of 4}\}$
$= \{\text{Sum is } 2,3,4,5 \text{ or sum is } 4, 8, 12\}$

$n(T) = 16$

$P(T) = \frac{16}{36} = \frac{4}{9}$ ---(ii)

$S \cap T = \{\text{even less than 6}\} \text{ or } \{\text{multiple of 4}\}$

$n(S \cap T) = 10$

$P(S \cap T) = \frac{10}{36} = \frac{5}{18}$ ---(iii)

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(S) \times P(T) = \frac{1}{2} \times \frac{4}{9} = \frac{2}{9} \neq \frac{5}{18}$ from (i), (ii), (iii)
$\neq P(S \cap T)$

$\therefore$ The events S and T are <u>not</u> independent.

Example 23: There are 300 students at a music college. All the students play exactly one of the guitar, the piano or the flute. The numbers of male and female students that play each of the instruments are given in the table.

	Guitar	Piano	Flute
Female Students	62	35	43
Male Students	78	40	42

Determine whether the events "a student is male" and "student does not play piano" are independent. [W-19|63|Q1] --[2]

Total no. of students = 300

Solution: S: Student is male $\rightarrow n(S) = 160$; $\Rightarrow P(S) = 160/300 = 8/15$ ---(i)

P: Students play piano $\rightarrow n(P) = 75 \rightarrow n(\text{not } P) = 225$, $P(\text{not } P) = \frac{225}{300} = \frac{9}{12} = \frac{3}{4}$ --(ii)

$n(S \cap \text{not } P) = 120 \rightarrow P(S \cap \text{not } P) = \frac{120}{300} = \frac{2}{5}$ ---(iii)

Now $P(S) \times P(\text{not } P) = \frac{8}{15} \times \frac{3}{4} = \frac{2}{5} = P(S \cap \text{not } P)$ from (i), (ii) & (iii)

Hence the events "student is male" and "does not play piano" are **Independent**

Example 24: A fair six-sided die is thrown twice and the scores are noted. Event X is defined as 'the total of the two scores is 4'. Event Y is defined as 'The first score is 2 or 5'. Are events X and Y independent? ---[4]

S-19 | 61 | Q3

Solution: X = total score is 4 $= \{(1,1,2), (1,2,1), (2,1,1)\}; n(S) = 3$

$\therefore P(X) = 3/36 = 1/12$ ---(i)

Y = first score is 2 or 5 $= \{(2,1), (2,2), \ldots (2,6); (5,1), (5,2) \ldots (5,6)\}$

$\therefore n(Y) = 12 \Rightarrow P(Y) = \frac{12}{36} = \frac{1}{3}$ ---(ii)

Now $X \cap Y = \{(2,1,1)\} \Rightarrow P(X \cap Y) = \frac{1}{36}$ ---(iii)

Now $P(X) \cdot P(Y) = \frac{1}{12} \times \frac{1}{3} = \frac{1}{36} = P(X \cap Y)$ from (i), (ii), (iii)

$\therefore$ X and Y are independent.

Example 25: A fair eight sided die has faces marked 1, 2, 3, 4, 5, 6, 7, 8. The die is thrown twice. Event R is "one of the score is exactly 3 greater than the other score". Event S is "the product of the scores is more than 19"

(i) Find the prob. of R. ---[2]

(ii) Find the prob of S. ---[2]

(iii) Determine whether events R and S are independent. ---[3]

M-16 | 62 | Q3

Solution: $R = \{(1,4), (2,5), (3,6), (4,7), (5,8); (4,1), (5,2), (6,3), (7,4), (8,5)\}$ $n(R) = 10$

$S = \{(3,8), (3,7), (4,8), (4,7), (4,6), (4,5), (5,8), (5,7), (5,6), (6,8), (6,7), (7,8)\} \times 2$
$+ \{(5,5), (6,6), (7,7), (8,8)\}$

$n(S) = 28$; $R \cap S = \{(4,7), (7,4), (5,8), (8,5)\}$; $n(R \cap S) = 4$

(i) $P(R) = \frac{10}{64} = \frac{5}{32}$ ---(i)

(ii) $P(S) = \frac{28}{64} = \frac{7}{16}$ ---(ii)

(iii) $P(R \cap S) = \frac{4}{64} = \frac{1}{16}$ ---(iii)

Now $P(R) \cdot P(S) = \frac{5}{32} \times \frac{7}{16} = \frac{35}{512}$ ---(iv)

$\frac{35}{512} \neq \frac{1}{16} \Rightarrow P(R) \cdot P(S) \neq P(R \cap S)$

$\therefore$ Events R and S are not independents.