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S-1

Probability and Statistics-1

The Normal Distribution Notes.

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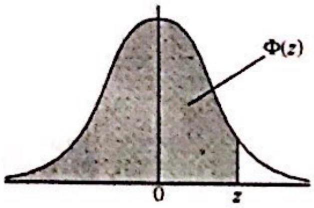
THE NORMAL DISTRIBUTION FUNCTION

If Z has a normal distribution with mean 0 and variance 1 then, for each value of z, the table gives the value of $\Phi(z)$, where

$$\Phi(z) = P(Z \leq z).$$

For negative values of z use $\Phi(-z) = 1 - \Phi(z)$.



z	0	1	2	3	4	5	6	7	8	9	ADD								
											1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	4	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0

Critical values for the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that

$$P(Z \leq z) = p.$$

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

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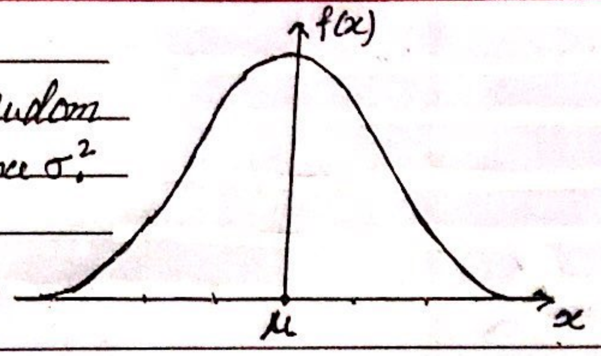
The Normal distribution- Notes

§ The normal distribution:

Given X is a continuous random variable with mean μ and variance σ^2:

$$X \sim N(\mu, \sigma^2)$$



The normal curve is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$ for all real values of x.

§ The standardard normal distribution:

The value of X is standardised to the value z, by

$$Z = \frac{X-\mu}{\sigma}$$

Now $X \sim N(\mu, \sigma^2) \Rightarrow Z \sim N(0,1)$

$Z \sim N(0,1)$ is called standard normal distribution.

Here Z is a normal varible with mean 0 and variance 1.

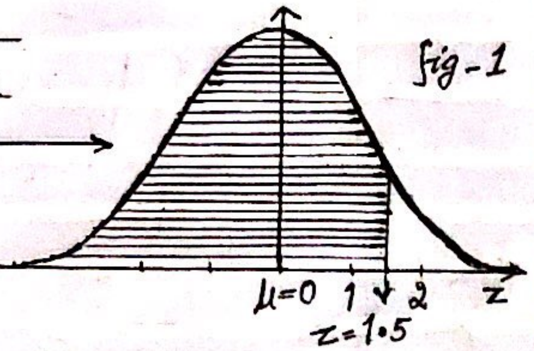
Then for each value of z, the table (on the front page) gives you the value of $\phi(z)$. (Fig-1)

where (i) $P(Z \leq z) = \phi(z)$

(Fig-2)

(ii) $P(Z > z) = 1 - \phi(z)$

(iii) $P(Z \leq -z) = P(Z > z) = 1 - \phi(z)$

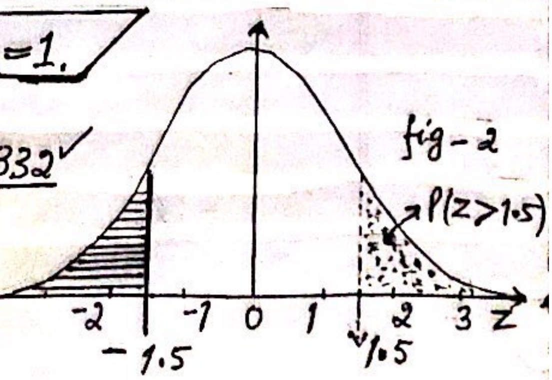


Note: Total area under the normal curve = 1. (fig-1)

Example 1 (i) $P(z \leq 1.5) = \phi(1.5) = 0.9332$ ✓

(ii) $P(Z > 1.5) = 1 - \phi(1.5)$

$= 1 - 0.9332 = 0.0668$ ✓



(iii) $P(Z \leq -1.5) = P(Z > 1.5) = 1 - \phi(1.5)$

$= 1 - 0.9332 = 0.0668$ ✓

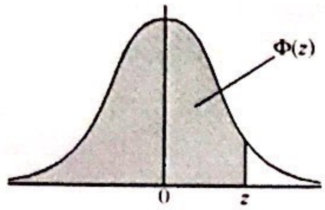
The Normal Distribution

THE NORMAL DISTRIBUTION FUNCTION

If Z has a normal distribution with mean 0 and variance 1 then, for each value of z, the table gives the value of $\Phi(z)$, where

$$\Phi(z) = P(Z \le z).$$

For negative values of z use $\Phi(-z) = 1 - \Phi(z)$.



z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
											ADD								
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
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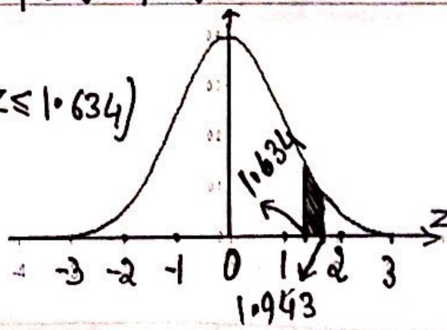
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5

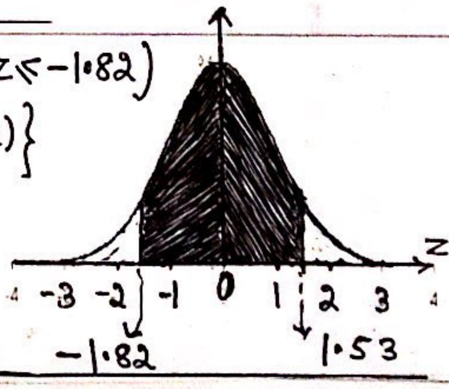
§ (iv) $P(a \le Z \le b) = P(Z \le b) - P(Z \le a) = \phi(b) - \phi(a)$

Example

$$P(1.634 \le Z \le 1.943) = P(Z \le 1.943) - P(Z \le 1.634)$$
$$= \phi(1.943) - \phi(1.634)$$
$$= \begin{pmatrix} 0.9738 \\ +0.0002 \end{pmatrix} - \begin{pmatrix} 0.9484 + \\ 0.0004 \end{pmatrix}$$
$$= 0.9740 - 0.9488$$
$$= 0.0252 \checkmark$$



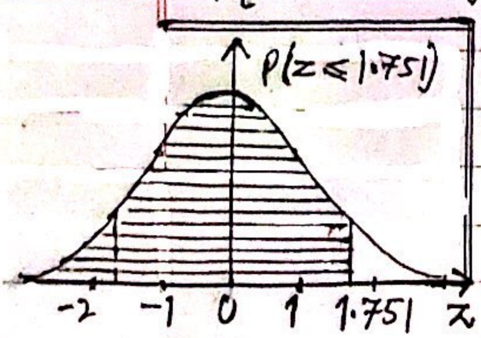
(v) $P(-1.82 \le Z \le 1.53) = P(Z \le 1.53) - P(Z \le -1.82)$
$$= \phi(1.53) - \{1 - \phi(1.82)\}$$
$$= \phi(1.53) + \phi(1.82) - 1$$
$$= 0.9370 + 0.9656 - 1$$
$$= 0.9026$$

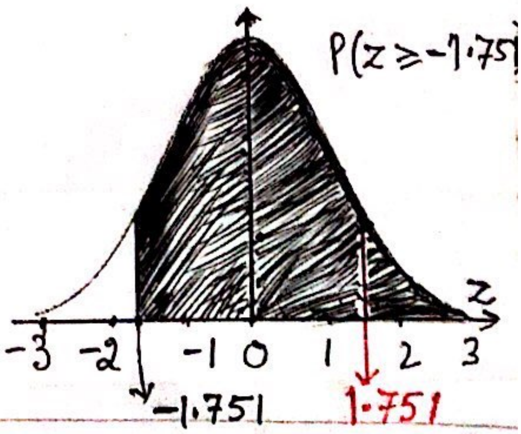


§ (VI) $\boxed{P(Z \ge -z) = P(Z \le z) = \phi(z)}$

Example:

$$P(Z \ge -1.751) = P(Z \le 1.751)$$
$$= \phi(1.751)$$
$$= (0.9599 + .0001)$$
$$= 0.9600 \checkmark$$





Case I: (i) $P(z \le a) > 0.5$ (ii) $P(z > a) < 0.5$

§ To find the value of 'a' given (i) $P(z \le a) = \phi(a) > 0.5$

Then $a > 0$ ✓

Example: $\phi(1.5) = 0.9332$

(ii) $P(z > a) < 0.5$

Then also $a > 0$ ✓

as $P(z > a) = 1 - \phi(a) < 0.5 \Rightarrow \phi(a) > 0.5$

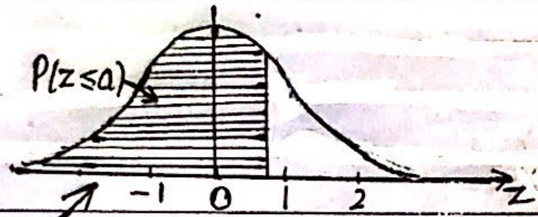
Examples: $P(z > a) = 0.0668$

$\Rightarrow 1 - \phi(a) = 0.0668$

$\Rightarrow \phi(a) = 1 - 0.0668$

$= 0.9332 > 0$

Example 2(i)



Find the value a : $P(z \le a) = 0.7360 > 0.5$

$\therefore a > 0$; $\Rightarrow a = \phi^{-1}(0.7360) = 0.631$ ✓

z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
															ADD				
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23

(ii) Find the value a :

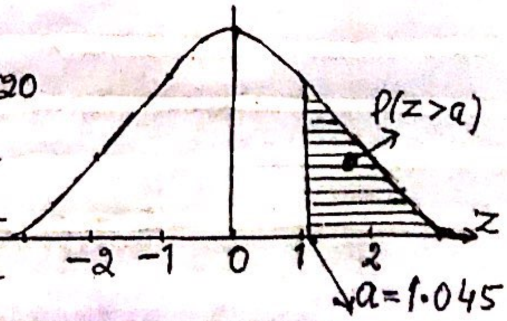
$P(z > a) = 0.1480 < 0.5 \Rightarrow a > 0$

$\Rightarrow 1 - \phi(a) = 0.1480$

$\Rightarrow \phi(a) = 1 - 0.148 = 0.8520$

$\Rightarrow a = \phi^{-1}(0.8520)$

$a = 1.045$ ✓



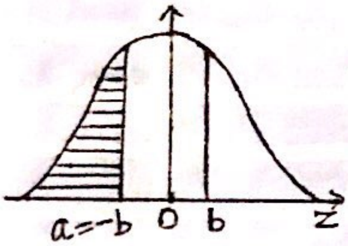
z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
															ADD				
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13

§ Case II : To find the value of a:

$$\left.\begin{array}{l}(i)\ P(z \le a) < 0.5 \\ (ii)\ P(z > a) > 0.5\end{array}\right\} \Rightarrow a < 0 \Rightarrow a = -b$$

Example 3(i) Find the value of 'a': $P(z \le a) = 0.317 < 0.5$

$\Rightarrow a = -b$ ✓



$\therefore P(z \le -b) = 0.317$

$\phi(-b) = 0.317 \Rightarrow 1 - \phi(b) = 0.317 \Rightarrow \phi(b) = 1 - 0.317$

$\Rightarrow \phi(b) = 0.683$

$\Rightarrow b = \phi^{-1}(0.683) = 0.476$

$\therefore a = -b = -0.476$ ✓ $\Rightarrow \underline{a = -0.476}$

z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
															ADD				
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
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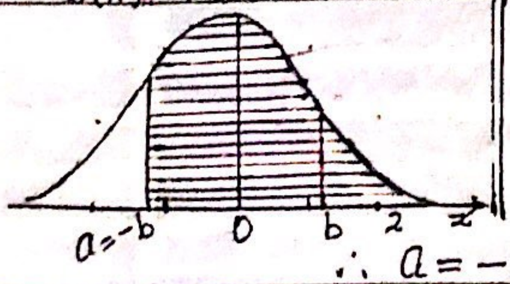
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13

3(ii) Find the value of a: $P(z > a) = 0.8950 > 0.5 \Rightarrow a = -b$



$\Rightarrow P(z > -b) = 0.8950$

$\Rightarrow P(z < b) = 0.8950$

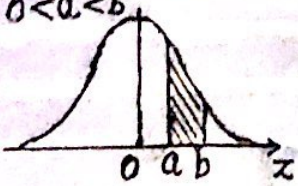
$\Rightarrow \phi(b) = 0.8950$

$\Rightarrow b = \phi^{-1}(0.8950) = 1.253$

$\therefore a = -b = -1.253 \Rightarrow \underline{a = -1.253}$ ✓

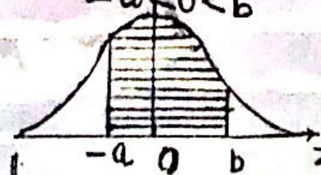
§ Note :

$0 < a < b$



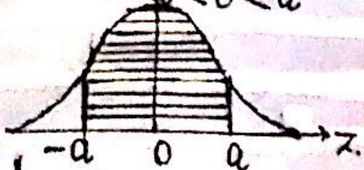
$P(a < z < b) = \phi(b) - \phi(a)$

$-a < 0 < b$



$P(-a < z < b) = \phi(b) + \phi(a) - 1$

$-a < 0 < a$



$P(-a < z < a) = 2\phi(a) - 1$

z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
											ADD								
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	4	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0

Example 4: Given, $Z \sim N(0,1)$, find the following probabilities correct to 3 s.f.

(a) $P(Z<1.96)$ (b) $P(Z>2.576)$ (c) $P(Z<-1.75)$

(d) $P(Z>-1.53)$ (e) $P(1.645<Z<2.326)$ (f) $P(-1.667<Z<2.667)$

Solution (a) $P(Z<1.96) = \phi(1.96) = 0.975$ ✓

(b) $P(Z>2.576) = 1-\phi(2.576) = 1-0.995 = 0.005$ ✓

(c) $P(Z<-1.75) = 1-\phi(1.75) = 1-0.9599 = 0.0401$ ✓

(d) $P(Z>-1.53) = P(Z<1.53) = \phi(1.53) = 0.937$

(e) $P(1.645<Z<2.326) = P(Z<2.326) - P(Z<1.645)$

$= \phi(2.326) - \phi(1.645)$

$= 0.99 - 0.95 = 0.0400$ ✓

(f) $P(-1.667<Z<2.667) = P(Z<2.667) - P(Z<-1.667)$

$= \phi(2.667) - [1-\phi(1.667)]$

$= 0.9962 - 1 + 0.9522$

$= 0.9484$

$= 0.948$ ✓ (3 s.f.)

z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
											ADD								
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21

Example 5: Given that $Z \sim N(0,1)$, find the value of k, such that:

(a) $P(Z<k)=0.5442$ (b) $P(Z>k)=0.2743$

(c) $P(Z<k)=0.25$ (d) $P(Z>k)=0.648$

(e) $P(-k<Z<k)=0.6994$.

Solution: (a) $P(Z<k)=0.5442>0.5 \therefore k>0$

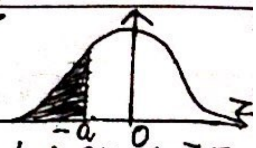
$\Rightarrow \phi(k)=0.5442 \Rightarrow k=\phi^{-1}(0.5442)=\underline{0.111}$

(b) $P(Z>k)=0.2743<0.5 \Rightarrow k>0$

$\Rightarrow 1-\phi(k)=0.273 \Rightarrow \phi(k)=1-0.2743=0.7257$

$\Rightarrow k=\phi^{-1}(0.7257)=\underline{0.600}$ ✓

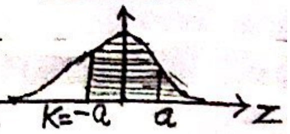
(c) $P(Z<k)=0.25<0.5 \Rightarrow k<0$, let $k=-a$



$\Rightarrow P(Z<-a)=0.25 \Rightarrow 1-\phi(a)=0.25 \Rightarrow \phi(a)=1-0.25=0.75$

$\therefore a=\phi^{-1}(0.75)=0.674 \Rightarrow \therefore k=-a=\underline{-0.674}$ ✓

(d) $P(Z>k)=0.648>0.5 \Rightarrow k<0 \rightarrow$ let $k=-a$



$\Rightarrow P(Z>-a)=0.648 \Rightarrow P(Z<a)=0.648$

$\Rightarrow a=\phi^{-1}(0.648)=0.380 \rightarrow k=-a=\underline{-0.380}$ ✓

(e) $P(-k<Z<k)=0.6994 \Rightarrow \phi(k)-\phi(-k)=0.6994$

$\Rightarrow \phi(k)-[1-\phi(k)]=0.6994$

$\Rightarrow 2\cdot\phi(k)-1=0.6994 \Rightarrow 2\cdot\phi(k)=1.6994 \Rightarrow \phi(k)=0.8497$

$k=\phi^{-1}(0.8497)=\underline{1.035}$ ✓

§ Standardising a normal distribution:

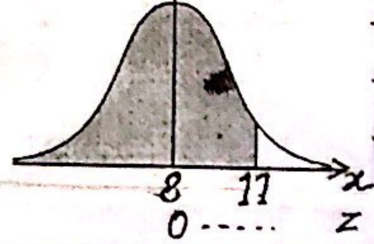
$$X \sim N(\mu, \sigma^2) \text{ to } Z \sim N(0,1)$$

using,
$$z = \frac{x-\mu}{\sigma}$$

Example 6: Standardise the approriate value(s) of the normal value of X represented in each diagram, and find the required probabilty.

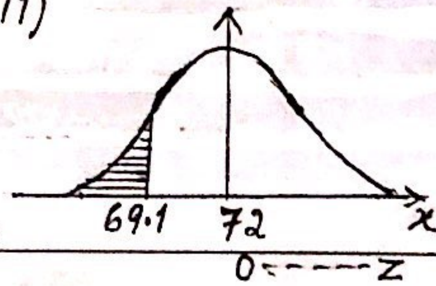
(a) Find $P(X \leq 11)$, given that $X \sim N(8, 25)$

$$P(X \leq 11) = P\left(Z \leq \frac{11-8}{\sqrt{25}}\right) = P(Z \leq 0.6)$$
$$= \phi(0.6) = \underline{0.726}$$



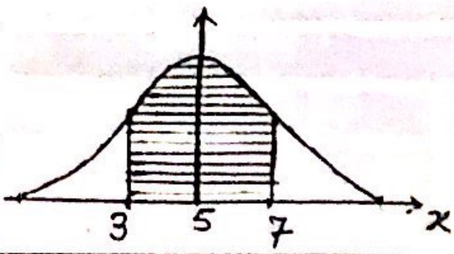
(b) Find $P(X < 69.1)$, given that $X \sim N(72, 11)$

$$P(X < 69.1) = P\left(Z < \frac{69.1-72}{\sqrt{11}}\right)$$
$$= P(Z < -0.874) = 1 - \phi(0.874)$$
$$= 1 - 0.8089 = \underline{0.191} \text{ (3sf)}$$



(c) Find $P(3 < X < 7)$, given that $X \sim N(5, 5)$

$$P(3 < X < 7) = P\left(Z < \frac{7-5}{\sqrt{5}} - Z < \frac{3-5}{\sqrt{5}}\right)$$
$$= P(Z < 0.894) - (Z < -0.894)$$
$$= \phi(0.894) - (1 - \phi(0.894))$$
$$= 2\phi(0.894) - 1$$
$$= 2 \times 0.81444 - 1$$
$$= 0.6288$$
$$= \underline{0.629} \text{ (3sf)}$$



Example 7: The time spent by shoppers in a large shopping centre has a normal distribution with mean 96 minutes and standard deviation 18 minutes. Find the probability that a shopper chosen at random spends between 85 and 100 minutes in the shopping centre. --[3]

[M-21/ 52/Q3(a)]

Solution: Given $X \sim N(96, 18^2)$

$$P(85 < X < 100) = P\left(\frac{85-96}{18} < Z < \frac{100-96}{18}\right) = P(-0.6111 < Z < 0.2222)$$

$$= \phi(0.2222) - \phi(-0.6111) = \phi(0.2222) - (1 - \phi(0.6111))$$

$$= 0.5879 + 0.7294 - 1 = \underline{0.317}$$

Example 8: The time in minutes, that Karli spends each day on social media are normally distributed with mean 125 and standard deviation 24.

(a) (i) On how many days of the year (365 days) would you expect Karli to spend more than 142 minutes on social media? --[5]

(ii) Find the probability that Karli spends more than 142 minutes on social media on fewer than 2 of 10 randomly chosen days. --[3]

[W-21/51/Q7(a)]

Solution: Given: $X \sim N(125, 24^2)$

(i) $P(X > 142) = P\left(Z > \frac{142-125}{24}\right) = P(Z > 0.7083) = 1 - \phi(0.7083)$

$$= 1 - 0.7604 = \underline{0.2396}$$

now for $n = 365$; $E(X) = np = 365 \times 0.2396 = 87.454$

$= \underline{87 \text{ days}}$ (or 88)

(ii) $P(X > 142) = 0.2396 \Rightarrow q = 0.7604,\ p = 0.2396,\ n = 10$

using Binomial prob distribution. $\{P(x=r) = {}^nC_r\, p^r \cdot q^{n-r}$

$$P(r < 2) = P(0, 1)$$

$$= (0.7604)^{10} + {}^{10}C_1 (0.2396)^1 (0.7604)^9$$

$$= \underline{0.268}$$

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Example 9: The time taken, in minutes, to complete a particular task by employees at a large company are normally distributed with mean 32.2 and standard deviation 9.6.

(a) Find the probability that a randomly chosen employee takes more than 28.6 minutes to complete the task. --[3]

(b) 20% of employees take longer than t minutes to complete the task. Find the value of t. --[3]

(c) Find the probability that the time taken to complete the task by a randomly chosen employee differs from the mean by less than 15 minutes.

[W-21/52/Q6] ---[4]

Solution: $X \sim N(32.2, 9.6^2)$.

(a) $P(X > 28.6) = P\left(Z > \frac{28.6 - 32.2}{9.6}\right) = P(Z > -0.375)$

$= P(Z < 0.375) = \phi(0.375) = 0.6462 = \underline{0.646}$ ✓ (3 s.f)

(b) $P(X > t) = 0.2 \Rightarrow P\left(Z > \frac{t - 32.2}{9.6}\right) = 0.2 \Rightarrow 1 - \phi\left(\frac{t - 32.2}{9.6}\right) = 0.2$

$(\Rightarrow t > \mu)$

$\Rightarrow \phi\left(\frac{t - 32.2}{9.6}\right) = 0.8 \Rightarrow \frac{t - 32.2}{9.6} = \phi^{-1}(0.8) = 0.842$

$\Rightarrow t - 32.2 = 9.6 \times 0.842 = 8.0832$

$\Rightarrow t = 40.2832 \Rightarrow \underline{t = 40.3}$ ✓ (3sf)

(c) $P(\mu - 15 < X < \mu + 15) = P\left(\frac{(\mu - 15) - \mu}{\sigma} < Z < \frac{(\mu + 15) - \mu}{\sigma}\right)$

$= P\left(\frac{-15}{9.6} < Z < \frac{15}{9.6}\right) = P\left(Z < \frac{15}{9.6}\right) - \left(Z < \frac{-15}{9.6}\right)$

$= \phi(1.5625) - (1 - \phi(1.5625))$

$= 2\phi(1.5625) - 1$

$= 2 \times 0.9409 - 1$

$= \underline{0.882}$ ✓

Example 8: Raj wants to improve his fitness, so every day he goes for a run. The times, in minutes, of his runs have a normal distribution with mean 41.2 and standard deviation 3.6.

(a) Find the prob. that on a randomly chosen day Raj runs for more than 43.2 minutes. ---[3]

(b) Find an estimate for the number of days in a year (365 days) on which Raj runs for less than 43.2 minutes. ---[2]

(c) On 95% of days, Raj runs for more than t minutes. Find the value of t :- [3]

$X \sim N(41.2, 3.6^2)$ [W-21/53/Q4]

Solution:
(a) $P(X > 43.2) = P\left(Z > \frac{43.2 - 41.2}{3.6}\right) = P(Z > 0.5556)$
$= 1 - \phi(0.5556)$
$= 1 - 0.7108 = \underline{0.289}$ ✓

(b) $P(X < 43.2) = 1 - P(X > 43.2) = 1 - 0.289$ (Part a)
$= 0.7108$ ✓

∴ Expection in 365 days $= np = 365 \times 0.7108 = \underline{259}$ days (or 260)

(c) $P(X > t) = 0.95 \Rightarrow t < \mu$ or $(t - \mu) < 0$

$\Rightarrow P\left(Z > \frac{t - 41.2}{3.6}\right) = 0.95$

$\Rightarrow P\left(Z > -\left(\frac{41.2 - t}{3.6}\right)\right) = 0.95$

$\Rightarrow P\left(Z < \left(\frac{41.2 - t}{3.6}\right)\right) = .95$ $[P(z > -a) = P(z < a)]$

$\Rightarrow \phi\left(\frac{41.2 - t}{3.6}\right) = 0.95$

$\Rightarrow \frac{41.2 - t}{3.6} = \phi^{-1}(0.95) = 1.645$

$\Rightarrow 41.2 - t = 1.645 \times 3.6 = 5.922$

$\Rightarrow t = 41.2 - 5.922$

$\underline{t = 35.3}$ ✓

Example 9: The weights of apples of a certain variety are normally distributed with mean 82 grams. 22% of the apples have a weight greater than 87 grams.

(a) Find the standard deviation of the weights of these apples. ---[3]

(b) Find the probability that the weight of a randomly chosen apple of this variety differs from the mean weight by less than 4 grams. [M-20/52/Q3] --[4]

Solution (a) Given $X \sim N(82, \sigma^2)$

$$P(X > 97) = P\left(z > \frac{87-82}{\sigma}\right) = 0.22$$

$$\Rightarrow 1 - \phi\left(\frac{5}{\sigma}\right) = 0.22 \Rightarrow \phi\left(\frac{5}{\sigma}\right) = 0.78$$

$$\Rightarrow \frac{5}{\sigma} = \phi^{-1}(0.78) = 0.772$$

$$\Rightarrow \sigma = \frac{5}{0.772} = \underline{6.48} \checkmark$$

(b) Given $|x-\mu| < 4 \Rightarrow -4 < x-\mu < 4$

$$\Rightarrow -\frac{4}{\sigma} < \frac{x-\mu}{\sigma} < \frac{4}{\sigma} \quad \left(z = \frac{x-\mu}{\sigma}\right)$$

$$P(|x-\mu| < 4) = P\left(-\frac{4}{\sigma} < z < \frac{4}{\sigma}\right)$$

$$= P\left(\frac{-4}{6.48} < z < \frac{4}{6.48}\right)$$

$$= P(-0.6176 < z < 0.6176)$$

$$= \phi(0.6176) - (1 - \phi(0.6176))$$

$$= 2\phi(0.6176) - 1$$

$$= 2 \times 0.7317 - 1$$

$$= \underline{0.463} \checkmark$$

Q 10 The lengths of male snakes have a normal distribution. A random sample of 200 snakes of this species is such that 32 have lengths less than 45 cm and 17 have lengths more than 56 cm.

Find the estimate of the mean and standard deviation of the lengths of male snakes of this species. --[5]

[S-20/ 51/ Q6]

$X \sim N(\mu, \sigma^2)$

Solution: $P(X < 45) = P\left(z < \frac{45-\mu}{\sigma}\right) = \frac{32}{200} = 0.16 < 0.5$

$P(z < -a) = 0.16 \Rightarrow P(z < a) = 1 - 0.16 = 0.84$ $\therefore \frac{45-\mu}{\sigma} = -a$ --- (1)

$\Rightarrow a = \phi^{-1}(0.84) = 0.994$ --- (2)

from (1) & (2) $\frac{45-\mu}{\sigma} = -0.994 \Rightarrow 45 - \mu = -0.994\sigma$ --- (3)

Again:

$P(x > 56) = P\left(z > \frac{56-\mu}{\sigma}\right) = \frac{17}{200} = 0.085 < 0.5$

$\Rightarrow P\left(z < \frac{56-\mu}{\sigma}\right) = 1 - 0.085 = 0.915$

$\Rightarrow \phi\left(\frac{56-\mu}{\sigma}\right) = 0.915 \Rightarrow \frac{56-\mu}{\sigma} = \phi^{-1}(0.915) = 1.372$

$\Rightarrow 56 - \mu = 1.372\sigma$ --- (4)

Solving (3) & (4) $\mu = 49.6$; $\sigma = 4.65$

Example 11: The times, in minutes, that Karli spends each day on social media are normally distributed with mean 125 and standard deviation 24.

On 90% of days, Karli spends more than t minutes on social media. Find the value of t. [W-21/ 51/ Q7(b)] --- [3]

Solution: $P(X > t) = 0.9 > 0.5 \Rightarrow t < \mu \Rightarrow \frac{t-\mu}{\sigma} < 0$ $\{X \sim N(125, 24^2)$

$\Rightarrow P\left(z > \frac{t-\mu}{\sigma}\right) = 0.9 \Rightarrow P(z > -a) = 0.9$ $\{$ Let $\frac{t-\mu}{\sigma} = -a$ --- (1)

$\Rightarrow P(z < a) = 0.9 \Rightarrow a = \phi^{-1}(0.9) = 1.282 \Rightarrow \frac{t-125}{24} = -1.282$

$\Rightarrow t - 125 = -1.282 \times 24 = -30.768$

$\Rightarrow t = 125 - 30.768 = 94.232 \Rightarrow t = 94.2$ (3sf)

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Example 12: The time spent by shoppers in a large shoping centre has a normal distribution with mean 96 minutes and standard deviation 18 minutes. 88% of shoppers spend more than t minutes in the shopping centre. Find the value of t. [M-21/52/Q3(b)] ---[3]

Solution: Given $P(X>t)=0.88>0.5 \Rightarrow t<\mu \Rightarrow \frac{t-96}{18}<0$ $\quad (X\sim N(96,18^2))$

$\Rightarrow P\left(Z>\frac{t-96}{18}\right)=0.88$ $\quad \Rightarrow \left(\text{Let } \frac{t-96}{18}=-a \text{ --- (1)}\right)$

$\Rightarrow P(Z>-a)=0.88 \Rightarrow P(Z<a)=0.88$

$\Rightarrow \phi(a)=0.88$

$\Rightarrow a=\phi^{-1}(0.88)=1.175$

for (1) $\frac{t-96}{18}=-1.175 \Rightarrow t-96=-1.175\times 18=-21.15$

$\Rightarrow t=96-21.15=\underline{74.85}$ ✓ (or 74.9)

Example 13: The lengths, in centimetres, of middle figures of women in Raneland have a normal distribution with mean 'μ' and standard deviation 'σ'. It is found that 25% of these women have fingers longer than 8.8 cm and 17.5% have fingers shorter than 7.7 cm. Find the vaules of μ and σ. [M-17/62/Q7(i)] ---[5]

Solution: $P(X>8.8)=0.25$ $\quad \{X\sim N(\mu,\sigma^2)$

$\Rightarrow P\left(Z>\frac{8.8-\mu}{\sigma}\right)=0.25 \Rightarrow P\left(Z<\frac{8.8-\mu}{\sigma}\right)=1-0.25$

$\Rightarrow \phi\left(\frac{8.8-\mu}{\sigma}\right)=0.75$

$\Rightarrow \frac{8.8-\mu}{\sigma}=\phi^{-1}(0.75)=0.674$

$\Rightarrow \frac{8.8-\mu}{\sigma}=0.674$

$\Rightarrow 8.8-\mu=0.674\sigma$ ------ (1)

Again, $P(X<7.7)=0.175<0.5 \Rightarrow X<\mu \Rightarrow \frac{X-\mu}{\sigma}<0$

$\Rightarrow P\left(Z<\frac{7.7-\mu}{\sigma}\right)=0.175$ $\quad \left\{\text{Let } \left(\frac{7.7-\mu}{\sigma}\right)=-a \text{ --- (2)}\right.$

$\Rightarrow P(Z<-a)=0.175$

$\Rightarrow 1-\phi(a)=0.175 \Rightarrow \phi(a)=1-0.175=0.825$

$\Rightarrow a=\phi^{-1}(0.825)=0.935$ --- (3)

from (2) and (3) $\Rightarrow \frac{7.7-\mu}{\sigma}=-0.935 \Rightarrow 7.7-\mu=-0.935\sigma$ --- (4)

Solving (1) and (4) $\left\{\underline{\mu=8.34 \text{ and } \sigma=0.684}\right.$ ✓

Example 14: The lengths of the leaves of a particular type of tree are modelled by normal distribution. A scientist measures the leaves of a random sample of 500 leaves from this type of tree and finds that 42 are less than 4 cm long and 100 are more than 10 cm long.

(a) Find estimates for mean and standard deviation of the lengths of leaves from this type of tree. ---[5]

The length, in cm, of the leaves of a different type of tree have the distribution $N(\mu, \sigma^2)$. The scientist takes a random sample of 800 leaves from this type of tree.

(b) Find how many of these leaves the scientist would expect to have lengths, in cm, between $\mu - 2\sigma$ and $\mu + 2\sigma$. [S-21/53/Q5] -[4]

Solution: (a) $P(X<4) = 42/500 = 0.084 < 0.5$ $\quad [n=500\ (X \sim N(\mu, \sigma^2))]$

$\Rightarrow P(Z < \frac{4-\mu}{\sigma}) = 0.084 \quad \} \Rightarrow 4<\mu \Rightarrow \frac{4-\mu}{\sigma} < 0 \Rightarrow$ let $\frac{4-\mu}{\sigma} = -a$ --- (1)

$\Rightarrow P(z < -a) = 0.084$

$\Rightarrow 1 - \phi(a) = 0.084 \Rightarrow \phi(a) = 0.916$

$\Rightarrow a = \phi^{-1}(0.916) = 1.378$ ------------ (2)

from (1) and (2) $\frac{4-\mu}{\sigma} = -1.378 \Rightarrow 4-\mu = -1.378\sigma$ --- (3)

Also,

$P(X>10) = 100/500 = 0.2 \Rightarrow P(Z > \frac{10-\mu}{\sigma}) = 0.2$

$\Rightarrow 1 - \phi(\frac{10-\mu}{\sigma}) = 0.2 \Rightarrow \phi(\frac{10-\mu}{\sigma}) = 0.8$

$\Rightarrow \frac{10-\mu}{\sigma} = \phi^{-1}(0.8) = 0.842 \Rightarrow 10-\mu = 0.842\sigma$ --- (4)

Solving (3) and (4) $\underline{\sigma = 2.70}$ ✓ and $\underline{\mu = 7.72}$ ✓

(b) To find $P(\mu - 2\sigma < x < \mu + 2\sigma) = P\left(\frac{(\mu-2\sigma)-\mu}{\sigma} < Z < \frac{(\mu+2\sigma)-\mu}{\sigma}\right)$

$= P(-2 < Z < 2) = P(z<2) - P(z<-2)$

$= \phi(2) - (1 - \phi(2))$

$= 2 \cdot \phi(2) - 1$

$p = 2 \times 0.9772 - 1 = 0.9544$ ✓

Given total number $= 800$

Expectation $= np = 800 \times 0.9544 = \underline{763}$ (or 764)

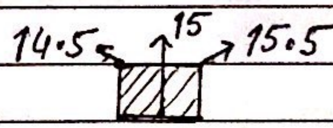
§ The normal distribution as an approximation to the binomial distribution:

For the binomial distribution $X \sim B(n, p)$, when $np > 5$ and $nq > 5$ $(q = 1 - p)$

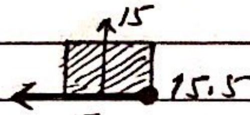
Then $X \sim B(n,p)$ appoximates to the normal distribution:
$Z \sim N(\mu, \sigma^2)$: $\mu = np$ and $\sigma^2 = npq$.
with a continuity correction.

§ Continuity correction:

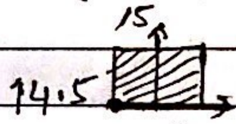
When binomial distribution (Discrete) approximates to normal distribution (continuous). A continity correction is required as follows:



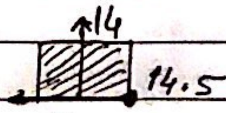
(i) $P(X = 15) \sim P(14.5 \leq Z < 15.5)$



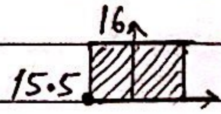
(ii) $P(X \leq 15) \sim P(Z \leq 15.5)$



(iii) $P(X \geq 15) \sim P(Z \geq 14.5)$



(iv) $P(X < 15) = P(X \leq 14) \sim P(Z \leq 14.5)$



(v) $P(X > 15) = P(X \geq 16) \sim P(Z \geq 15.5)$

(vi) $P(15 \leq X < 20) = P(15 \leq X \leq 19) \sim P(15.5 \leq Z \leq 19.5)$

(vii) $P(15 < X \leq 20) = P(16 \leq X \leq 20) \sim P(15.5 \leq Z \leq 20.5)$

(viii) $P(15 < X < 20) = P(16 \leq X \leq 19) \sim P(15.5 \leq Z \leq 19.5)$

Example 15: On average at all the schools in the country 30% of the students do not like any sports. 90 students are chosen are chosen at random.

Use an approximation to find the probability that fewer than 32 of them do not like any sports. [M-21|52|Q7(b)] - [5]

Solution: Given $X \sim B(n,p) = B(90, 0.3)$, $(np = 27 > 5,\ nq = 63 > 5)$

$p = 0.3,\ q = 0.7$

$\mu = \text{Mean} = np = 90 \times 0.3 = 27$

Variance $\sigma^2 = npq = 90 \times 0.3 \times 0.7$

$\sigma^2 = 18.9$

Binomial distribution approximates to the normal distribution: $X \sim N(27, 18.9)$ $\{N(\mu, \sigma^2)$

Continuity correction: $X < 32 \Rightarrow X \leq 31 \rightarrow X < 31.5$

$$\therefore P(X < 32) = P\left(Z < \frac{31.5 - 27}{\sqrt{18.9}}\right)$$

$$= P(Z < 1.035) = \phi(1.035) = \underline{0.850}$$

Example 16: In Greenton, 70% of adults own a car. A random sample of 120 adults from Greenton is chosen.

Use an approximation to find the probability that more than 75 of them own a car. [M-20|52|Q5(b)] - [5]

Solution: $p = 0.7,\ n = 120,\ X \sim B(120, 0.7)$; $\mu = np = 120 \times 0.7 = 84 > 5$

approximates to normal distribution $N(\mu, \sigma^2)$

$q = 1 - 0.7 = 0.3$; $nq = 120 \times 0.3 = 36 > 5$

Variance $\sigma^2 = npq = 120 \times 0.7 \times .3 = 25.2$

$= N(84, 25.2)$

$$P(X > 75) = P\left(Z > \frac{75.5 - 84}{\sqrt{25.2}}\right) = P(Z > -1.693)$$

$$= P(Z < 1.693)$$

$$= \phi(1.693)$$

$$= 0.9548$$

$$= \underline{0.955} \quad (3\text{ sf})$$

Example 17: On any given day, the probability that Moena messages her friend Pasha is 0.72.
Use an approximation to find the probility that in any period of 100 days Moena messages Pasha on fewer than 64 days.--[5]
[S-20|52|7(c)]

Solution: $p = 0.72,\ q = 0.28,\ n = 100;\ np = 100 \times 0.72 = 72 > 5$
Binomial distribution: $nq = 100 \times 0.28 = 28 > 0$
$X \sim B(100, 0.72)$ variance $\sigma^2 = npq = 100 \times 0.72 \times 0.28 = 20.16$
approximates to Normal distribution $N(\mu, \sigma^2) = N(72, 20.16)$
Hence $P(X < 64) = P\left(Z < \frac{63.5 - 72}{\sqrt{20.16}}\right)$ continuity correction $X < 64 \rightarrow X \leq 63 \rightarrow X = 63.5$

$= P(Z < -1.893) = 1 - \phi(1.893) = 1 - 0.9708 = \underline{0.0292}$ ✓

Example 18: In Quarendon, 66% of households are satisfied with the speed of their wifi connection. A random sample of 150 households in Quarendon is chosen. Use a suitable approximation to find the probability that more than 84 are satisfied with the speed of their wifi connection. [W-19|62|4(ii)] --[5]

Solution: $p = 0.66,\ q = 0.34,\ n = 150;\ \mu = np = 150 \times 0.66 = 99 > 5$
$X \sim B(150, 0.66)$ Variance $\sigma^2 = npq = 150 \times 0.66 \times 0.34 = 33.66$
Approximates to normal distribution $X \sim N(\mu, \sigma^2) = (99, 33.66)$
hence $P(X > 84) = P\left(Z > \frac{84.5 - 99}{\sqrt{33.66}}\right)$ continuity correction $x > 84 \rightarrow x \geq 85 \rightarrow x > 84.5$

$= P(Z > -2.499)$
$= \phi(2.499)$
$= 0.9938$
$= \underline{0.994}$ ✓ (3 sf)

Example 19: There is a probability of $\frac{1}{7}$ that Wenjie goes out with her friends on any particular day. 252 days are chosen at random:

(i) Use a normal approximation to find the probability that the number of days on which Wenjie goes out with her friends is less than 30, or more than 44. --[5]

(ii) Give a reason why the use of a normal approximation is justified.

[S-14/63/Q2] --[1]

Solution: $p = \frac{1}{7}, q = \frac{6}{7}, n = 252 \rightarrow \mu = np = 252 \times \frac{1}{7} = 36$ ✓

(i) Variance $\sigma^2 = npq = 252 \times \frac{1}{7} \times \frac{6}{7} = 30.857$

$B(n, p) \sim N(\mu, \sigma^2) = N(36, 30.857)$ ✓

Continuity correction: $X < 30 \rightarrow X \leq 29 \sim 29.5$; $X > 44 \rightarrow X \geq 45 \sim 44.5$

$P(X < 30)$ or $P(X > 44)$

$= P\left(Z < \frac{29.5 - 36}{\sqrt{30.857}}\right) + P\left(Z > \frac{44.5 - 36}{\sqrt{30.857}}\right)$

$= P(Z < -1.170) + P(Z > 1.530)$

$= (1 - \phi(1.170)) + (1 - \phi(1.530))$

$= 1 - 0.8790 + 1 - 0.9370$

$= \underline{0.184}$ ✓

(ii) $np = 252 \times \frac{1}{7} = 36 > 5$ and $nq = 252 \times \frac{6}{7} = 30.857 > 5$

$\therefore$ $np > 5$ and $nq > q$ $\therefore$ we can use normal approximation.

Example 20: The times in hours taken by a garage to fit a tow bar onto a car have the distribution $N(\mu, \sigma^2)$, where $\mu = 3\sigma$. Find the probability that it takes more than 0.6μ hours to fit a tow bar onto a randomly chosen car at this garage.

[M-16/62/Q7(vii)] - [3]

Solution: $P(\text{Time} > 0.6\mu) = P\left(Z > \frac{0.6\mu - \mu}{\mu/3}\right)$ $\left\{ \mu = 3\sigma \Rightarrow \sigma = \frac{\mu}{3} \right.$

$= P(Z > -1.2)$

$= P(Z < 1.2)$

$= \phi(1.2) = 0.8849$

$= \underline{0.885}$ ✓ (3sf)

Example 21: On trains in morning rush hour, each person is either a student with prob. 0·36, or an office worker with prob. 0·22, or a shop assistant with prob. 0·29, or none of these.
300 people, on a morning rush hour train, are chosen at random. Find the prob. that between 31 and 49 inclusive are neither students, nor office workers, nor shop assistants.
[W-13|62|Q5] ---[6]

Solution: $p = P($none of the student, or office worker or shop assistant$)$

$$p = 1-(0\cdot36+0\cdot22+0\cdot29) = 1-0\cdot87 = 0\cdot13 \quad ; n = 300$$

$$\therefore p = 0\cdot13, \quad q = 0\cdot87 \; ; \quad \mu = np = 300\times0\cdot13 = 39$$

Variance $\sigma^2 = npq = 300\times0\cdot13\times0\cdot87 = 33\cdot93$

$$X \sim B(n,p) \rightarrow X \sim N(\mu,\sigma^2) = N(39, 33\cdot93)$$

To find $P(31 \leq X \leq 49) = P(Z \leq 49) - P(Z < 31)$

continuity correction
$X \leq 31 \rightarrow X < 30\cdot5$
$X \leq 49 \rightarrow X < 49\cdot5$

$$P\left(Z < \frac{49\cdot5-39}{\sqrt{33\cdot93}}\right) - P\left(Z < \frac{30\cdot5-39}{\sqrt{33\cdot93}}\right)$$

$$= P(Z < 1\cdot8026) - P(Z < -1\cdot4592)$$

$$= \phi(1\cdot8026) - \{1-\phi(1\cdot4592)\}$$

$$= 0\cdot9643 + 0\cdot9278 - 1 = \underline{0\cdot892}$$

Example 22: On any day at noon, the probability Kersley is asleep is 0·2. Use an approximation to find the prob. that in any period of 100 days, Karsley is asleep at noon on at most 30 days. [W-16|62|Q3] --[5]

Solution: $p = 0\cdot2, \; q = 0\cdot8, \; n = 100, \; \mu = np = 100\times0\cdot2 = 20 > 5$

$B(n,p)$ Variance $\sigma^2 = npq = 100\times0\cdot2\times0\cdot8 = 16$

$X \sim B(100, 0\cdot2) \rightarrow$ approxiate to $N(\mu,\sigma^2) \rightarrow N(20, 16)$

$$P(X \leq 30) = P\left(Z < \frac{30\cdot5-20}{\sqrt{16}}\right)$$

Continuity correction:
$X \leq 30 \rightarrow X = 30\cdot5$

$$= P(Z < 2\cdot625)$$

$$= \phi(2\cdot625)$$

$$= 0\cdot9957$$

$$= \underline{0\cdot996} \quad (3 \text{ sf})$$

Example 23: The random variable Y is normally distributed with mean μ and standard deviation σ. It is given that $5\mu = 2\sigma^2$ and that $P(Y < \frac{1}{2}\mu) = 0.281$. Find the value of μ and σ. ---[4]

[W-13/63/Q5]

Solution: Given $P(Y < \frac{1}{2}\mu) = 0.281$ and $5\mu = 2\sigma^2 \Rightarrow \mu = \frac{2}{5}\sigma^2$ --- ①

$\Rightarrow P\left(Z < \frac{\frac{1}{2}\mu - \mu}{\sigma}\right) = 0.281 \Rightarrow P\left(Z < -\frac{\frac{1}{2}\mu}{\sigma}\right) = 0.281$

$\Rightarrow 1 - \phi\left(\frac{\mu}{2\sigma}\right) = 0.281 \Rightarrow \phi\left(\frac{\mu}{2\sigma}\right) = 0.719 \Rightarrow \frac{\mu}{2\sigma} = \phi^{-1}(0.719)$

$\Rightarrow \frac{\mu}{2\sigma} = 0.58 \Rightarrow \frac{1}{2\sigma} \times \frac{2\sigma^2}{5} = 0.58$ [from ① $\mu = \frac{2}{5}\sigma^2$

$\Rightarrow \frac{\sigma}{5} = 0.58 \Rightarrow \underline{\sigma = 2.9}$ ✓

fr ① $\mu = \frac{2}{5} \times (2.9)^2 = 3.364$ ✓

$\therefore \mu = 3.36$ ✓ ; $\sigma = 2.9$ ✓

Example 24: The volume of soup in super soup cartons has a normal distribution with mean μ millilitres and standard deviation 9 millilitres. Test have shown that 10% of cartons contain less than 440 millilitres of soup. Find the value of μ. ---[3]

[S-18/62/Q3]

Solution: $N(\mu, 9^2)$

Given $P(X < 440) = 0.1$

$\Rightarrow P\left(Z < \frac{440-\mu}{9}\right) = 0.1 < 0.5$

$\Rightarrow P(Z < -a) = 0.1$

$\therefore 440 < \mu$

$\frac{440-\mu}{9} < 0$

$\therefore$ let $\frac{440-\mu}{9} = -a$ --- ①

$\Rightarrow 1 - \phi(a) = 0.1 \Rightarrow \phi(a) = 0.9$

$\Rightarrow a = \phi^{-1}(0.9) = 1.282$ --- ②

fr ① & ② $\frac{440-\mu}{9} = -1.282 \Rightarrow 440 - \mu = -1.282 \times 9 = -11.538$

$\Rightarrow \mu = 440 + 11.538 = 451.538$

$\Rightarrow \underline{\mu = 452}$ ✓ (3sf)