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Pure Maths-1

Binomial Theorem
Revised Notes

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Binomial Theorem

§ Factorial notation:

Consider the product:

$$5\times4\times3\times2\times1 = 5! \text{ (Five Factorial)}$$

$$4\times3\times2\times1 = 4!$$

$$6! = 6\times5\times4\times3\times2\times1 = 6\times5!$$

Now $\frac{7!}{5!} = \frac{7\times6\times5!}{5!} = 7\times6 = 42$

Note (i) $1! = 1$

(ii) $\underline{0! = 1}$ (By definition).

$$\boxed{\text{In General: Factorial } n; \quad n! = n(n-1)(n-2)\cdots\times2\times1.}$$

Binomial coefficient: n is a positive Integer

$$\binom{n}{r} \text{ or } {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!} \; : \quad 0 \le r \le n$$

$${}^{n}C_{0} = 1 \; ; \; {}^{n}C_{1} = n \; ; \; {}^{n}C_{2} = \frac{n(n-1)}{2!}, \cdots, {}^{n}C_{n} = 1.$$

Example: (i) ${}^{1}C_{0} = 1$ and ${}^{1}C_{1} = 1$ (for $n=1$)

(ii) ${}^{2}C_{0} = 1$; ${}^{2}C_{1} = 2$; ${}^{2}C_{2} = 1$ (for $n=2$)

(iii) ${}^{3}C_{0} = 1$; ${}^{3}C_{1} = 3$; ${}^{3}C_{2} = 3$; ${}^{3}C_{3} = 1$ (for $n=3$)

(iv) ${}^{4}C_{0} = 1$; ${}^{4}C_{1} = 4$, ${}^{4}C_{2} = 6$; ${}^{4}C_{3} = 3$; ${}^{4}C_{4} = 1$ (for $n=4$)

(v) ${}^{5}C_{0} = 1$; ${}^{5}C_{1} = 5$; ${}^{5}C_{2} = 10$, ${}^{5}C_{3} = 10$, ${}^{5}C_{4} = 5$, ${}^{5}C_{5} = 1$

(vi) ${}^{6}C_{0} = 1$; ${}^{6}C_{1} = 6$; ${}^{6}C_{2} = 15$, ${}^{6}C_{3} = 20$; ${}^{6}C_{4} = 15$; ${}^{6}C_{5} = 6$; ${}^{6}C_{6} = 1$

Note: $\boxed{{}^{n}C_{r} = {}^{n}C_{n-r}}$

Example:

(i) ${}^{6}C_{0} = {}^{6}C_{6} = 1$; ${}^{6}C_{1} = {}^{6}C_{5} = 6$; ${}^{6}C_{2} = {}^{6}C_{4} = 15$; ${}^{6}C_{3} = {}^{6}C_{3} = 20$

Binomial Theorem:

(i) $(x+a)^1 = x + a = {}^1C_0 \cdot x^1 + {}^1C_1 x^0$

(ii) $(x+a)^2 = x^2 + 2xa + a^2 = {}^2C_0 x^2 + {}^2C_1 x^{2-1} \cdot a^1 + {}^2C_2 a^2$

(iii) $(x+a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$
$= {}^3C_0 x^3 + {}^3C_1 x^{3-1} \cdot a^1 + {}^3C_2 x^{3-2} \cdot a^2 + {}^3C_3 a^3$

(iv) $(x+a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$
$= {}^4C_0 x^4 + {}^4C_1 x^{4-1} \cdot a^1 + {}^4C_2 x^{4-2} a^2 + {}^4C_3 x^{4-3} \cdot a^3 + {}^4C_4 a^4$

(V) $(x+a)^5 = x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5x^1a^4 + a^5$
$= {}^5C_0 x^5 + {}^5C_1 x^{5-1} \cdot a + {}^5C_2 x^{5-2} \cdot a^2 + {}^5C_3 x^{5-3} \cdot a^3 + {}^5C_4 x^{5-4} \cdot a^4 + {}^5C_5 \cdot a^5.$

§ Binomial Theorem:

$(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} \cdot a + {}^nC_2 \cdot x^{n-2} \cdot a^2 + \cdots + \cdots + \cdots + {}^nC_r x^{n-r} \cdot a^r + \cdots + {}^nC_n a^n$

Note (i) General Term $= {}^nC_r \cdot x^{n-r} \cdot a^r$ $(r = 0, 1, 2, \ldots n)$

(ii) Number of terms in the expansion of $(x+a)^n$ is $(n+1)$

(iii) Coefficient of first term = Coeff of last term = 1
Coeff of second term = Coeff of second last term = n

(iv) (a) If n is even then the number of terms $(n+1)$ is odd.

(b) If n is odd then the number of terms $(n+1)$ is even

(V) The sum of the exponents of x and a = n

§ Binomial Theorem (Particular Case):

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_r x^r + \cdots + {}^nC_n x^n.$

1.(i) Find the first three terms in the expansion of $(2-x)^6$ in the ascending powers of x. ---[2]

(ii) Find the value of k for which there is no term in x^2, in the expansion of $(1+kx)(2-x)^6$. ---[3]

S-05	01	Q4

Solution (i) $(a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + ---$

$\therefore (2-x)^6 = {}^6C_0 2^6 + {}^6C_1 2^5 \cdot (-x) + {}^6C_2 \cdot 2^4 \cdot (-x)^2 + --$

$= 1\times 64 + 6\times 32(-x) + 15\times 16\cdot x^2 + -$

$= \underline{64 - 192x + 240x^2 ---}$ ✓ ---(1)

(ii) Now $(1+kx)(2-x)^6 = (1+kx)(64-192x+240x^2 ---)$ ---(2)

from (2) the terms containing $x^2 = 1\times 240x^2 + kx\cdot(-192x)$

$= (240-192k)x^2$ ---(3)

Given that there is no term in (2) containing x^2,

$\therefore$ from (3) $240-192k=0 \Rightarrow k = 240/192 = \underline{1.25}$ ✓

2. The coefficients of x^2 in the expansion of $(2+x/2)^6 + (a+x)^5$ is 330. Find the value of a. ---[5]

S-18	12	Q1

Solution: $(a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + -- + {}^nC_n x^n$.

$\therefore (a+x)^5 = ({}^5C_0 a^5 + {}^5C_1 a^4x + {}^5C_2 a^3x^2 + ---)$.

$= (a^5 + 5a^4x + 10a^3x^2 + --)$ ---(1)

and $(2+x/2)^6 = {}^6C_0 2^6 + {}^6C_1 \cdot 2^5 \cdot (x/2) + {}^6C_2 \cdot 2^4 \cdot (x/2)^2 + -$

from (1) & (2) $= 64 + 96x + 60x^2 + ---$ ---(2)

$\therefore (2+x/2)^6 + (a+x)^5 = (64+96x+60x^2+) + (a^5 + 5a^4x + 10a^3x^2 + -)$ (3)

$\therefore$ Coefficient of x^2 in (3) $= 60 + 10a^3 = 330$ given.

$\Rightarrow 10a^3 = 270$

$\Rightarrow a^3 = 27$

$\Rightarrow \underline{a = 3}$ ✓

3. (i) Find the coefficients of x^2 and x^3 in the expansion of $(1-2x)^7$ ---[3]
(ii) Hence find the coefficient of x^3 in the expansion of:
$(2+5x)(1-2x)^7$. ---[2]

[M-18/12/Q2]

Solution: $(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + ---$

$\therefore (1-2x)^7 = 1 + {}^7C_1(-2x) + {}^7C_2(-2x)^2 + {}^7C_3(-2x)^3 + ---$
$= 1 + 7(-2x) + 21\times(4x^2) + 35(-8x^3) + -$
$= 1 - 14x + 84x^2 - 280x^3 + ---$ — (1)

$\therefore$ In the expansion of $(1-2x)^7$ $\begin{cases} \text{Coeff of } x^2 = \underline{84} \checkmark \\ \text{Coeff of } x^3 = \underline{-280} \checkmark \end{cases}$

(ii) Consider $(2+5x)(1-2x)^7$
from (1) $= (2+5x)(1-14x+84x^2-280x^3+---)$ .... (2)
$\therefore$ Coefficient of x^3 in (2) $= 2\times(-280) + 5\times 84$
$= -560 + 420 = \underline{-140} \checkmark$

4 (i) Find the first three terms in the expansion of $(1+ax)^5$ in ascending powers of x.
(ii) Given that there is no term in x in the expansion of $(1-2x)(1+ax)^5$, find the value of the constant a.
(iii) For this value of a, find the coefficient of x^2 in the expansion of $(1-2x)(1+ax)^5$.

[S-10/12/Q6]

Solution (i) $(1+ax)^5 = 1 + {}^5C_1(ax) + {}^5C_2(ax)^2 + --$ $[(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + --]$
$= 1 + 5ax + 10a^2x^2 + --$ (1) ✓

(ii) $(1-2x)(1+ax)^5 = (1-2x)(1+5ax+10a^2x^2+-)$ -(2) from (1)
from (2) Term in $x = 1\times 5ax - 2x\times 1 = (5a-2)x$
But given that there is no term in x in (2) $\Rightarrow 5a-2=0 \Rightarrow \underline{a = \frac{2}{5}} \checkmark$ (3)

(iii) from (2) term in $x^2 = 1\times 10a^2x^2 + (-2x)(5ax)$
$= (10a^2 - 10a)x^2$
$\therefore$ Coeff. of $x^2 = 10a^2 - 10a = 10a(a-1)$ (from (3) $a = 2/5$)
$= 10\times\frac{2}{5}\times(\frac{2}{5}-1) = 4\times(-0.6) = \underline{-2.4} \checkmark$

5. The coefficient of x^4 in the expansion of $\left(2x^2+\frac{k^2}{x}\right)^5$ is a.
The coefficient of x^2 in the expansion of $(2kx-1)^4$ is b.

(a) Find a and b in terms of the constant k. ---[3]

(b) Given that $a+b=216$, find the possible values of k. ---[3]

[S-22 | 11 | Q3]

Solution: $\left(2x^2+\frac{k^2}{x}\right)^5 = {}^5C_0 (2x^2)^5 + {}^5C_1 (2x^2)^4 \cdot \left(\frac{k^2}{x}\right) + \underline{{}^5C_2 (2x^2)^3 \cdot \left(\frac{k^2}{x}\right)^2} + {}^5C_3 (2x^2)^2 \cdot \left(\frac{k}{x}\right)^3$

(a) term with $x^4 = {}^5C_2 \times 8x^6 \times \frac{k^4}{x^2} = 10 \times 8x^6 \times \frac{k^4}{x^2} = 80k^4 \cdot x^4$

$\therefore$ Coeff of $x^4 = \underline{a = 80k^4}$ ✓ ---(1)

Again, $(2kx-1)^4 = {}^4C_0 (2kx)^4 + {}^4C_1 (2kx)^3 \cdot (-1)^1 + {}^4C_2 (2kx)^2 \times (-1)^2 + \text{---}$

Term with $x^2 = {}^4C_2 \cdot (2kx)^2 (-1)^2 = 6 \times 4k^2x^2 \times 1$ ✓

$\therefore$ Coeff of $x^2 = b = 24k^2$ ——— (2)

(b) Given $a+b=216 \Rightarrow 80k^4+24k^2=216$ [from (1) & (2)]

$\Rightarrow 80k^4+24k^2-216=0$

$\Rightarrow 10k^4+3k^2-27=0 \Rightarrow (2k^2-3)(5k^2+9)=0$

$\Rightarrow k^2 = 3/2$ or $k^2 = -9/5$ ✗

$\therefore k = \pm\sqrt{3/2}$ ✓

6. The coefficient of x^4 in the expansion of $(3+x)^5$ is equal to the coeffient of x^2 in the expansion of $\left(2x+\frac{a}{x}\right)^6$.
Find the value of the positive constant a. ---[4]

[S-22 | 12 | Q1]

Solution: $(3+x)^5 = {}^5C_0 \cdot 3^5 + {}^5C_1\, 3^4 \cdot x^1 + {}^5C_2 \cdot 3^3 \cdot x^2 + {}^5C_3 \cdot 3^2 \cdot x^3 + {}^5C_4\, 3^1 \cdot x^4 + \text{---}$

Coeff of x^4 in $(3+x)^5 = {}^5C_4 \cdot 3^1 = 5 \times 3 = 15$ ——— (1)

Now $\left(2x+\frac{a}{x}\right)^6 = (2x)^6 + {}^6C_1 (2x)^5 \cdot \left(\frac{a}{x}\right)^1 + {}^6C_2 (2x)^4 \left(\frac{a}{x}\right)^2 + \text{---}$

Term of x^2 in $\left(2x+\frac{a}{x}\right)^6 = {}^6C_2 \cdot (2x)^4 \cdot \left(\frac{a}{x}\right)^2 = 15 \times 16x^4 \times \frac{a^2}{x^2} = 240x^2a^2$

$\therefore$ Coeff of $x^2 = 240a^2$ ——— (2)

Given coeff of x^4 in $(3+x)^5$ = coeff of x^2 in $\left(2x+\frac{a}{x}\right)^6$

$\Rightarrow 15 = 240a^2$ (from (1) & (2))

$\Rightarrow a^2 = \frac{15}{240} = \frac{1}{16}$

$\Rightarrow \underline{a = \frac{1}{4}}$ ✓ $(a>0)$

7. The coefficient of x^3 in the expansion of $(p+\frac{1}{p}x)^4$ is 144. Find the possible values of the constant p. ---[4]

[S-22|13|Q1]

<u>Solution:</u> $(p+\frac{1}{p}x)^4 = {}^4C_0 \cdot p^4 + {}^4C_1 \cdot p^3 \cdot (\frac{1}{p}x)^1 + {}^4C_2\, p^2 \cdot (\frac{1}{p}x)^2 + \underline{{}^4C_3 \cdot p \cdot (\frac{1}{p}x)^3} + \cdots$

Term containing $x^3 = {}^4C_3 \cdot p \cdot \frac{1}{p^3} \cdot x^3 = \frac{4}{p^2} \cdot x^3$

$\therefore$ Coefficient of x^3: $\frac{4}{p^2} = 144$ (Given)

$\Rightarrow p^2 = 4/144 = \frac{1}{36}$

$\Rightarrow p = \pm\frac{1}{6}$ ✓

8. (i) Find the term independent of x in the expansion of $(\frac{2}{x} - 3x)^6$ ---[2]

(ii) Find the value of a for which there are no term independent of x in the expansion of $(1+ax^2)(\frac{2}{x} - 3x)^6$ ---[3]

[W-17|13|Q3]

<u>Solution</u> (i) $(\frac{2}{x} - 3x)^6 = {}^6C_0 (\frac{2}{x})^6 + {}^6C_1 (\frac{2}{x})^5 (-3x)^1 + {}^6C_2 (\frac{2}{x})^4 \cdot (-3x)^2 + \underline{{}^6C_3 (\frac{2}{x})^3 \cdot (-3x)^3} + \cdots$

The term independent of $x = {}^6C_3 \cdot (\frac{2}{x})^3 \cdot (-3x)^3$

$= 20 \times \frac{8}{x^3} \times -27x^3 = \underline{-4320}$ ✓

(ii) $(1+ax^2)(2/x - 3x)^6$

$= (1+ax^2)[{}^6C_0 (\frac{2}{x})^6 + {}^6C_1 (\frac{2}{x})^5 \cdot (-3x)^1 + {}^6C_2 (\frac{2}{x})^4 \cdot (-3x)^2 + {}^6C_3 (\frac{2}{x})^3 (-3x)^3$ ---- (1)

Term independent of x in (1) $= 0$

$\Rightarrow 1 \times {}^6C_3 \cdot (\frac{2}{x})^3 \cdot (-3x)^3 + ax^2 \times {}^6C_2 (\frac{2}{x})^4 (-3x)^2 = 0$ ⊗⊗

$\Rightarrow 20 \times 8/x^3 \times -27x^3 + ax^2 \times 15 \times \frac{16}{x^4} \times 9x^2 = 0$

$\Rightarrow -4320 + 2160a = 0$

$a = \frac{4320}{2160} = 2$

$\therefore \underline{a = 2}$ ✓

(⊗⊗ Remove the terms independent of x)

9. Find the term independent of x in the expansion of:

(i) $\left(x-\frac{2}{x}\right)^6$ --[2]

(ii) $\left(2+\frac{3}{x^2}\right)\left(x-\frac{2}{x}\right)^6$ --[4]

[S-16/12/Q4]

<u>Solution</u> (i) $(x-{}^2/x)^6 = {}^6C_0\,x^6 + {}^6C_1\,x^5\cdot\left(-\frac{2}{x}\right)^1 + {}^6C_2\,x^4\cdot\left(-\frac{2}{x}\right)^2 + {}^6C_3\,x^3\cdot\left(-\frac{2}{x}\right)^3 + \cdots$

The term independent of $x = {}^6C_3\cdot x^3\cdot\left(-\frac{2}{x}\right)^3 = 20\cdot\cancel{x^3}\times\frac{(-8)}{\cancel{x^3}} = \underline{-160}$ ✓ --(1)

(ii) $\left(2+\frac{3}{x^2}\right)\left(x-\frac{2}{x}\right)^6$

$= \left(2+{}^3/x^2\right)\cdot\left({}^6C_0x^6 + {}^6C_1\,x^5\cdot\left(\frac{-2}{x}\right) + {}^6C_2\,x^4\cdot\left(-\frac{2}{x}\right)^2 + {}^6C_3\,x^3\cdot\left(-\frac{2}{x}\right)^3 + \cdots\right)$

The term independent of $x = 2\times{}^6C_3\,x^3\cdot\left(-\frac{2}{x}\right)^3 + \frac{3}{x^2}\cdot{}^6C_2\cdot x^4\cdot\frac{4}{x^2}$

$= 2\times(-160) + \frac{3}{x^2}\times 15\times x^4\times {}^4/x^2$

$= -320 + 180 = \underline{-140}$ ✓

10. In the expansion of $(3-2x)(1+{}^x/_2)^n$, the coefficient of x is 7. Find the value of the constant n, and hence find the coefficient of x^2. ---[6]

[W-16/12/Q4]

<u>Solution</u>: $(3-2x)(1+{}^x/_2)^n = (3-2x)\left(1+n\cdot{}^x/_2 + \frac{n(n-1)}{2}\left(\frac{x}{2}\right)^2 + \cdots\right)$ --- (1)

Term in $x = 3\times n\,{}^x/_2 - 2x\times 1 = \left(\frac{3n}{2}-2\right)x$

$\therefore$ Coefficient of $x = \frac{3n}{2} - 2 = 7$ (Given)

$\Rightarrow \frac{3n}{2} = 9 \Rightarrow \underline{n=6}$ ✓

from (1)

Now terms of $x^2 = 3\times\frac{n(n-1)}{2}\frac{x^2}{4} - 2x\times n.\frac{x}{2}$

$= 3\times\frac{6\times5}{2}\frac{x^2}{4} - 6\cdot x^2$ [for $n=6$]

$= \left(\frac{45}{4} - 6\right)x^2$

$\therefore$ Coefficient of $x^2 = \frac{45}{4} - 6$

$= \underline{\frac{21}{4}}$ ✓