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P.1

Pure Maths_1

Circular Measure Notes

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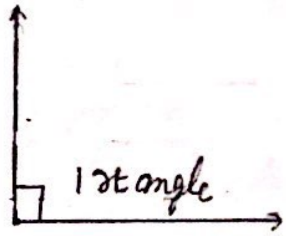
Circular measure

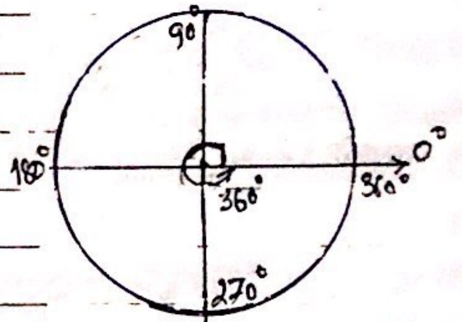
§ **Measurement of angles:**

English System (or degree system). Unit = degree (°)

1 right angle = 90°



Complete angle at the centre of a circle = 4 rt angles = 360°



§ $\dfrac{\text{Circumference of a circle}}{\text{diameter of the circle}} = \pi$ (constant) $= 3.14159\cdots$ (an irrational number)

Circumference of a circle $= \pi d$ (d = diameter of circle)

or $C = 2\pi r$ (r = radius of circle).

§ **Circular measure:**

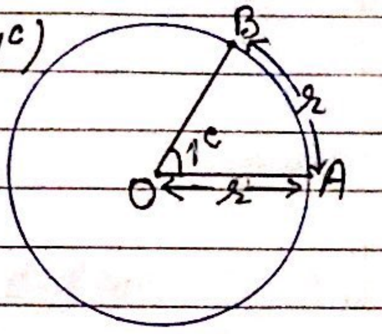
Unit = radian (or 1^c)

A radian is the angle subtended by an arc $\widehat{AB}$ of length 'r' at the centre of a circle of radius r.

length of arc $\widehat{AB} = r$

radius $OA = r$

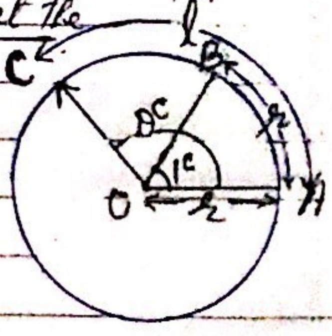
angle $AOB = 1$ radian (or 1^c) (^{c} for circular)



§ Angle subtended by an arc of length 'l' at the centre of a circle of radius 'r' is θ^c:

→ We know that the angles at the centre of a circle are proportional to the lengths of the corresponding arcs: $\dfrac{\angle AOC}{\angle AOB} = \dfrac{\text{arc } AC}{\text{arc } AB}$

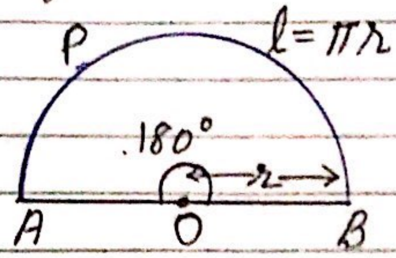
or $\boxed{\dfrac{\theta^c}{1^c} = \dfrac{l}{r}}$ or $\boxed{l = r\theta^c}$



Circular measure

§ **Relation between angle measure in "degree and Radian"**

Consider a semicircle $\widehat{APB}$ of radius 'r'.



Length of Semicircular arc $\widehat{APB}$

$$l = \pi r$$

angle in circular measure $\theta^c = \frac{l}{r} = \frac{\pi \cancel{r}}{\cancel{r}}$

$$\therefore \theta^c = \pi^c = 180° \text{ [straight angle]}$$

$$\boxed{\pi^c = 180°}$$

or $\boxed{180° = \pi^c}$

$$\pi^c = 180°$$

$$\therefore 1^c = \frac{180°}{\pi} = \frac{180°}{3.14159} = 57.2958°$$

$$\boxed{1° = \frac{\pi}{180}^c} \qquad \boxed{1^c = 57.2958°}$$

For some particular Angles:

$$30° = \frac{\pi}{3}^c \; ; \; 45° = \frac{\pi}{4}^c \; ; \; 60° = \frac{\pi}{3}^c \; , \; 90° = \frac{\pi}{2}^c$$

Example 1. Express angle $40°$ in radians.

$$40° = 40 \times \frac{\pi}{180}^c = 0.6981^c \checkmark$$

Example 2.

Express 1.4^c in degrees.

$$1.4^c = 1.4 \times \frac{180}{\pi}° = 80.214° \checkmark$$

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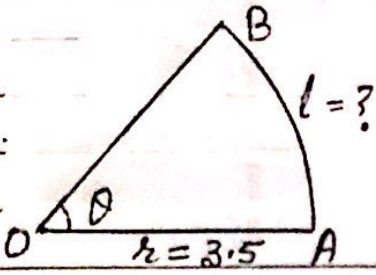
3. A sector has an angle of 0·65 radians and radius 3.5 cm. Find the arc length of the sector.



Solution: Sector angle $\theta = 0{\cdot}65$ radians

radius $= 3.5$ cm

$\therefore$ length of arc $\widehat{AB}$, $l = r\theta$

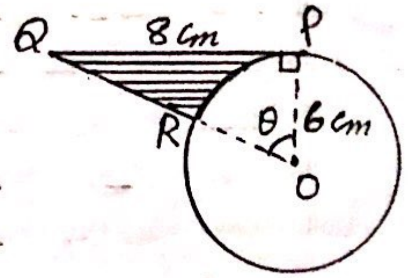
$= 3.5 \times 0{\cdot}65 = \underline{2{\cdot}275\text{ cm}}$ ✓

4. The circle has centre O and radius 6cm. PQ is a tangent to the circle at the point P. QRO is a straight line. Find:

(a) angle POQ, in radians.

(b) the length QR.

(c) the perimeter of the shaded region.



Solution: PQ is perpendicular to radius OP ⇒ angle OPQ = 1 rt angle

In rt ΔOPQ, $OQ^2 = OP^2 + PQ^2 = 6^2 + 8^2 = 36 + 64 = 100$

$\Rightarrow OQ = \sqrt{100} = 10$ ---- ①

(a) In rt ΔOPQ $\tan POQ = \frac{PQ}{OP} = \frac{8}{6} \Rightarrow$ angle $POQ = \tan^{-1}\frac{8}{6} = \underline{0{\cdot}927\text{ rad.}}$ ✓

(b) length $QR = OQ - OR$

$= 10 - 6 = \underline{4\text{ cm}}$ ✓ $\left\{\begin{array}{l}\text{from ① } OQ = 10\text{ cm} \\ \text{and } OR = \text{radius} = 6\text{ cm}\end{array}\right.$

(c) Perimeter of the shaded region $= PQ + QR + \text{arc } PR$ ---- ②

now length of arc $PR = r\theta = 6 \times 0{\cdot}927$

$= 5{\cdot}562$ cm, $\left\{\begin{array}{l}\text{from part (a)} \\ \theta = 0{\cdot}927\text{ rad}\end{array}\right.$

$\therefore$ from ① Perimeter of the shaded region:

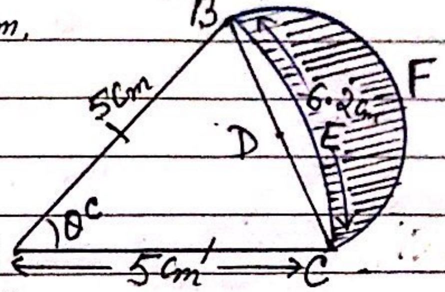
$= PQ + QR + \text{arc } PR$

$= 8 + 4 + 5{\cdot}56 = 17{\cdot}56$

or $\underline{17{\cdot}6\text{ cm}}$ ✓

Circular measure

Example 5: The diagram shows an isosceles triangle ABC, where $AB = AC = 5\,cm$. The arc BEC is part of the circle with centre A and has length $6.2\,cm$. The point D is the mid point of the line BC. The arc BFC is a semicircle, centre D.

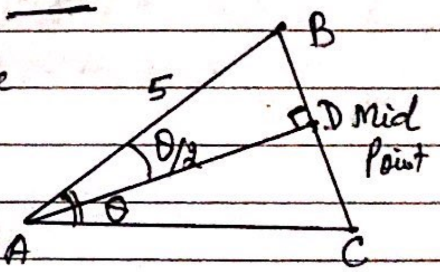


(i) Show that angle BAC is 1.24 radians. [1]

(ii) Find the perimeter of the shaded region. (BECFB) [3]

Solution: (i) angle $BAC = \theta = \frac{l}{r} = \frac{6.2}{5} = \underline{1.24^c}$ $[\because l = r\theta^c]$

(ii) $AD \perp BC$ (In isosceles triangle, line joining the vertex to the mid point of base is perpendicular to base)



In rt $\triangle ADB$, $\frac{BD}{AB} = \sin\theta/2 = \sin\left(\frac{1.24}{2}\right) = \sin(0.62)^c$
$= 0.581$

$\therefore BD = AB \times 0.581 = 5 \times 0.581 = 2.91\,cm$

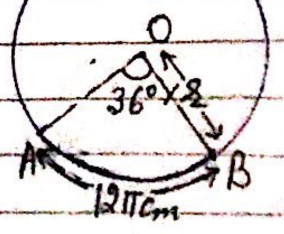
$\therefore$ length of arc $BFC = \pi r = \pi \times 2.91 = 9.13\,cm$ ✓ $[BD = r = 2.91]$

$\therefore$ Perimeter of shaded region $= \widehat{BEC} + \widehat{BFC}$ [Given $\widehat{BEC} = 6.2$]
$= 6.2 + 9.13$
$= \underline{15.3}$ ✓ cm

Example 6: Arc $\widehat{AB}$ of a circle, radius 'r' and centre O, subtends $36°$ at the centre. Find the radius 'r', given length of arc $AB = 12\pi\,cm$.

Solution: Length of arc $l = 12\pi$

angle $AOB = 36° = 36 \times \frac{\pi}{180}^c = \frac{\pi}{5}^c$



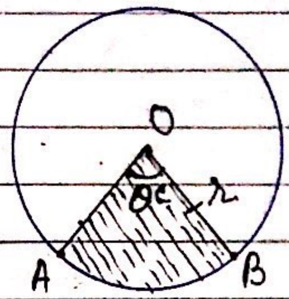
we know. $l = r\theta^c$

$\Rightarrow r = \frac{l}{\theta^c} = \frac{12\pi}{\frac{\pi}{5}} = 60\,cm.$ ✓

Circular measure

§ Area of a Sector of a circle of radius 'r', centre O, angle at the centre θ^c.

Area of a sector is proportional to the angle subtended at the centre:



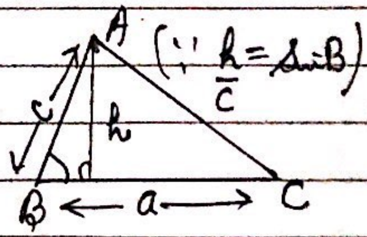
$$\frac{\text{area of Sector}}{\text{area of circle}} = \frac{\text{Sector angle}}{\text{Total angle at the centre O}}$$

or $$\frac{\text{Area of Sector}}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\Rightarrow \quad \text{Area of Sector} = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta$$

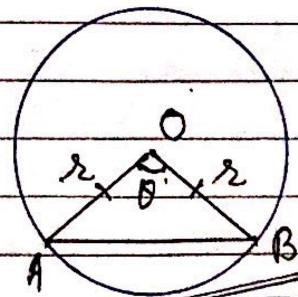
$$\boxed{\text{Area of Sector} = \frac{1}{2} r^2 \theta}$$

§ Area of a triangle (SAS)



$$\text{Area of triangle} = \frac{1}{2} ac \sin B$$

§ Area of a triangle formed by the Chord AB and radii OA and OB.
Given angle AOB $= \theta$. (SAS)

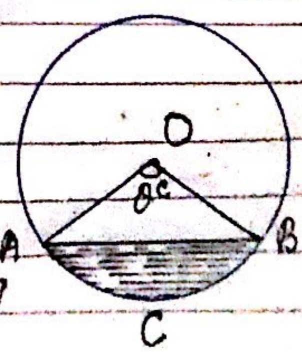


$$\text{Area of } \Delta AOB = \frac{1}{2} r \times r \times \sin\theta$$

$$\boxed{\text{ar of } \Delta AOB = \frac{1}{2} r^2 \sin\theta}$$

§ $\boxed{\text{Length of Chord } AB = 2r \sin\theta/2}$ $AM = r\sin\theta/2$, $AB = 2 \cdot AM = 2r \sin\theta/2$

§ Area of Segment of a circle:

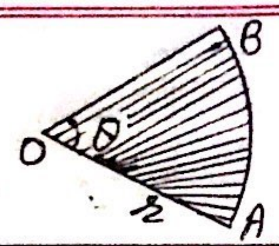


Area of Segment ACB = Area of Sector OACB − area of triangle OAB.

$$= \frac{1}{2} r^2 \theta^c - \frac{1}{2} r^2 \sin\theta$$

$$\therefore \boxed{\text{Area of segment of a circle} = \frac{1}{2} r^2 (\theta^c - \sin\theta)}$$

7. Find the area of a sector of radius 34cm, and sector angle 1.5 radians.



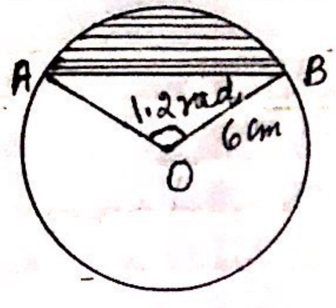
Solution: Area of sector $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \times 34^2 \times 1.5$

$= \underline{867\ cm^2}$ ✓

$\left\{ \begin{array}{l} r = 34cm \\ \theta = 1.5 \text{ radians} \end{array} \right.$

8. The circle has radius 6cm and centre O. AB is a chord angle AOB = 1.2 radians.
(a) Find the area of the sector AOB.
(b) Find the area of triangle AOB.
(c) Find the area of the shaded segent.



Solution (a) Area of the sector AOB $= \frac{1}{2} r^2 \theta = \frac{1}{2} \times 6^2 \times 1.2$

$= \underline{21.6\ cm^2}$ ✓

$\left\{ \begin{array}{l} r = 6cm \\ \theta = 1.2 \text{ radians} \end{array} \right.$

(b) Area of Δ AOB $= \frac{1}{2} r^2 \sin\theta$

$= \frac{1}{2} \times 6^2 \times \sin 1.2$

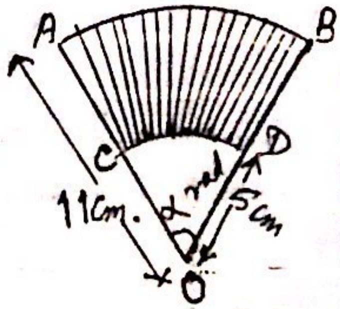
$= 16.776 = \underline{16.8\ cm^2}$ ✓

$\left\{ \begin{array}{l} \text{area of } \Delta = \frac{1}{2} ab \sin C \\ = \frac{1}{2} \times r \times r \times \sin\theta \\ = \frac{1}{2} r^2 \sin\theta \end{array} \right.$ ✓

(c) Area of the shaded segment of circle = area of sector OAB − ar ΔOAB

$= 21.6 - 16.78 = \underline{4.82\ cm^2}$ ✓

9. The diagram shows sector OAB with centre O, and radius 11cm. Angle AOB = α radians, points C and D lie on OA and OB respectively. Arc CD has centre O and radius 5cm.
(i) The area of the shaded region ABCD is k times the area of unshaded region. Find k ---[3]
(ii) The perimeter of the shaded region ABCD is twice the perimeter of the unshaded region OCD. Find the exact value of α. [W-13/13/Q6]



Solution: (i) $k = \dfrac{\frac{1}{2} \times 11^2 \times \alpha - \frac{1}{2} \times 5^2 \times \alpha}{\frac{1}{2} \times 5^2 \times \alpha}$

(Area $= \frac{1}{2} r^2 \times \theta$)

$k = \underline{3.84}$ ✓

(ii) $(11\alpha + 5\alpha + 6 + 6) = 2(5\alpha + 5 + 5)$

$\Rightarrow (16\alpha + 12) = 2(5\alpha + 10)$

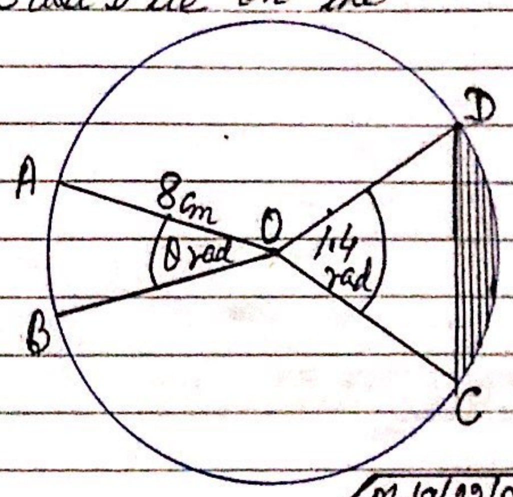
$\Rightarrow 6\alpha = 8$ $\quad$ $\boxed{l = r\theta}$

$\Rightarrow \alpha = \underline{4/3}$ radians

Circular measure

Example 7: The diagram shows a circle with centre O and radius 8 cm. The points A, B, C and D lie on the circumference of the circle.
Angle AOB = θ radians and angle COD = 1·4 rad. The area of sector AOB is 20 cm².

(i) Find angle θ ---[2]
(ii) Find the length of arc AB ---[2]
(iii) Find the area of the shaded segment. ---[3]



M-18/22/Q7

Solution: (i) area of sector AOB $= \frac{1}{2} r^2\theta = 20$

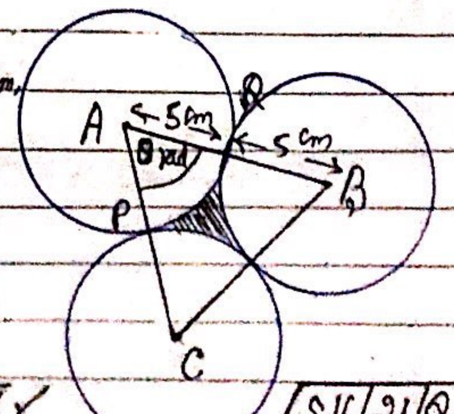
or $\frac{1}{2} \times 8^2 \times \theta = 20 \Rightarrow \theta = \frac{20}{32} = \frac{5}{8} = 0.625$ rad. ✓

(ii) length of arc AB $= r\theta = 8 \times \frac{5}{8}$ $[l = r\theta]$
$= 5$ cm ✓

(iii) Area of shaded segment $= \frac{1}{2} r^2 (\theta - \sin\theta)$
$= \frac{1}{2} \times 8^2 \times (1.4 - \sin 1.4)$
$= 32 \cdot (1.4 - 0.985)$
$= 13.26$ cm² ✓

Example 8. The diagram shows three circles with centres A, B and C, each of radius 5 cm. Each circle touches the other two circles. Angle BAC is θ radians.

i) Write down the value of θ ---[1]
ii) Find the area of the shaded region --[4]



S-16/21/Q4

Solution (i) $AB = BC = CA = 5 \times 2 = 10$ cm.
∴ Triangle ABC is equilateral ⇒ angle $\theta = \frac{\pi}{3}$ ✓

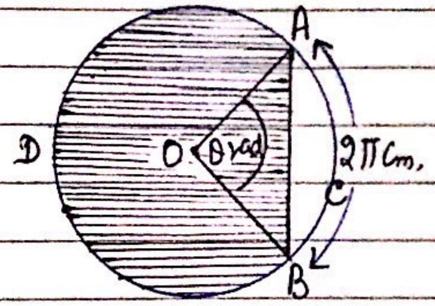
(ii) Area of Triangle ABC $= \frac{1}{2} \times 10^2 \times \sin\frac{\pi}{3} = 25\sqrt{3}$
Area of sector APQ $= \frac{1}{2} r^2\theta = \frac{1}{2} \times 5^2 \times \frac{\pi}{3} = \frac{25\pi}{6}$

∴ Area of shaded region = Area triangle − area of 3 sectors $= 25\sqrt{3} - 3 \times \frac{25\pi}{6}$
$= 4.03$ cm²

Circular measure

Example 9. The diagram shows a circle, centre O of radius r cm. and Chord AB, angle AOB = θ radians. The length of major arc AB is 5 times of the length of minor arc AB.
The minor arc AB has length 2π cm.

(i) Find the value of θ and of r. ---[2]
(ii) Calculate the exact perimeter of the shaded segment. ---[2]
(iii) Calculate the exact area of the shaded segment.



[S-17/23/Q8]

Solution: (i) Circumference $= 2\pi + 5\times 2\pi = 12\pi$

$\therefore 2\pi r = 12\pi$

$\Rightarrow r = 6$ cm. ✓

Now length of arc ACB $\cdot$ $l = r\theta$

or $2\pi = 6\times\theta \Rightarrow \theta = \frac{\pi}{3}$ ✓

(ii) Perimeter of the shaded segment

$= $ length of major arc ADC + AB

$= 5\times 2\pi + 2\times 6\times \sin\left(\frac{\pi}{3}\times\frac{1}{2}\right)$

$= 10\pi + 12\sin\frac{\pi}{6}$ $\left[\because \text{length Chord} = 2r\sin\frac{\theta}{2}\right]$

$= (10\pi + 6)$ cm.

(iii) Area of the shaded segment = area of major sector + area of triangle OAB

$= \frac{1}{2}r^2\times(2\pi - \theta) + \frac{1}{2}r^2\sin\theta$

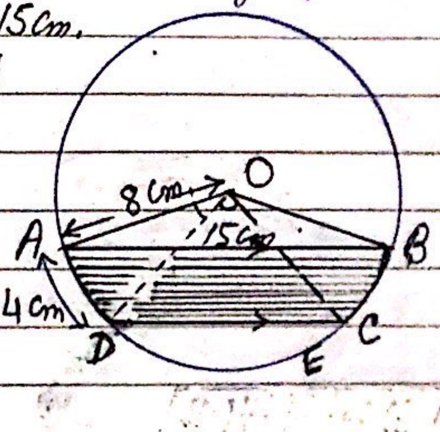
$= \frac{1}{2}\times 6^2\times\left(2\pi - \frac{\pi}{3}\right) + \frac{1}{2}\times 6^2\times\sin\frac{\pi}{3}$

$= 18\times\frac{5\pi}{3} + 18\times\frac{\sqrt{3}}{2}$

$= (30\pi + 9\sqrt{3})$ cm^2 ✓

Circular measure

Example 12. The diagram shows a circle, centre O, radius 8 cm. The points A, B, C and D lie on the circumference of the circle such that AB is parallel to DC. The length of arc AD is 4 cm and the length of chord AB is 15 cm.



(i) Find, in radians, angle AOD. --[1]

(ii) Hence show that DOC = 1.43 rad. correct to 2 decimal places. --[3]

(iii) Find the perimeter of the shaded region. --[3]

(iv) Find the area of the shaded region --[4]

Solution: Let angle AOB $= \theta^c$

length of chord $AB = 2r \sin \theta/2 = 2 \times 8 \sin \theta/2 = 15$ (given)

$\Rightarrow \sin \theta/2 = \frac{15}{16}$

$\therefore$ angle $AOB = 2 \sin^{-1} \frac{15}{16} = 2.43$ rad. $\longrightarrow$ (a)

(i) angle $AOD = \frac{l}{r} = \frac{\text{arc } AD}{r} = \frac{4}{8} = 0.5^c$ ✓ $\left[l = r\theta^c \Rightarrow \theta^c = \frac{l}{r} \right.$

(ii) angle $DOC =$ angle $AOB - 2$(angle AOD)

$= 2.43 - 2 \times 0.5$ [frm (a)

$= 1.43^c$ ✓

(iii) Lenth of Chord $DC = 2r \sin\left(\frac{DOC}{2}\right) = 2 \times 8 \times \sin\left(\frac{1.43}{2}\right)$

$= 16 \times \sin 0.715 = 10.49$

$\therefore$ Perimeter of the shaded region $= \overset{\frown}{AD} + DC + \overset{\frown}{BC} + AB$

$= 4 + 10.49 + 4 + 15$

$= 33.5$ cm ✓

(iv) Area of the shaded region = Area of segment AEB − Area segment DEC

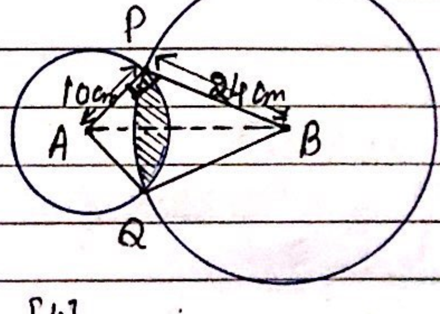
$= \frac{1}{2} \times 8^2(2.43 - \sin 2.43) - \frac{1}{2} \times 8^2(1.43 - \sin 1.43)$

$= 42.8 \text{ cm}^2$ ✓

[$\because$ Area of segment of a circle $= \frac{1}{2} r^2(\theta^c - \sin\theta)$]

Circular measure

Example 13. The diagram shows a circle, centre A, radius 10 cm, intersecting a circle, centre B, radius 24 cm. The two circles intersect at the points P and Q. The radii AP and AQ are tangents to the circle with centre B. The radii BP and BQ are tangents to the circle with centre A.



(i) Show that angle PAQ is 2.35 radians correct to 3 s.f. ----[2]

(ii) Find angle PBQ in radians ---[1]

(iii) Find the perimeter of the shaded region. ---[3]

(iv) Find the area of the shaded region. ---[4]

Solution: AP is tangent to the circle with centre B.

$\therefore$ $AP \perp PB$ (tangent $\perp$ radius segment)

$\therefore$ $\angle APB$ is a rt angle.

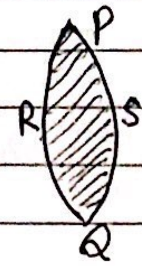
(i) $\therefore$ In rt triangle APB, $\tan PAB = \frac{24}{10} = 2.4$

angle $PAB = 1.176$

$\therefore$ Angle $PAQ = 2 \times$ angle $PAB = 2 \times 1.176 = 2.35$ rad. ✓

(ii) Following same steps as in part (i), angle $PBQ = 0.790^c$ ✓

(iii)



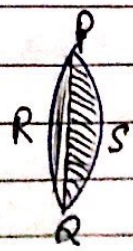
Perimeter of the shaded region = Lengths of arc PSQ + arc PRQ

$= 10 \times 2.352 + 24 \times 0.790$ $[\because l = r \times \theta^c]$

$= 42.5$ cm ✓

(iv) Area of the shaded region =

area of segment PRQ + area of segment PSQ



$= \frac{1}{2} \times 24^2 \times (0.79 - \sin 0.79)$

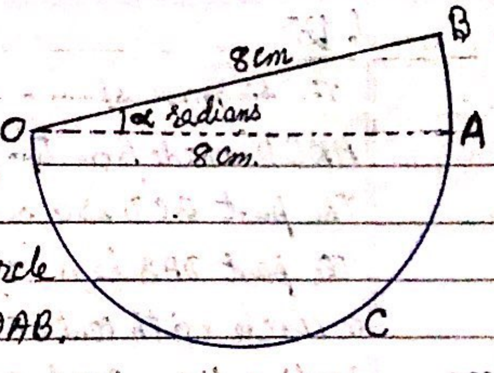
$+ \frac{1}{2} \times 10^2 \times (2.352 - \sin 2.352)$

$= 22.94 + 82.1$

$= 105$ cm^2 ✓

[$\because$ Area of segment of a circle $= \frac{1}{2} r^2 \times (\theta^c - \sin\theta)$]

Example 14: In the diagram, OAB is a sector of a circle with centre O, and radius 8 cm. Angle BOA is α radians. OAC is a semicircle with diameter OA. The area of the semicircle OAC is twice the area of the sector OAB.



(i) Find α in terms of π --[3]

(ii) Find the perimeter of the complete figure in terms of π --[2

[S-13/11/Q3]

Solution (i) Given: area of semicircle OAC = 2 × area of sector OAB

$$\frac{1}{2}\pi \times 4^2 = 2 \times \frac{1}{2} \times 8^2 \times \alpha \quad \left[\text{area of sector} = \frac{1}{2}r^2\theta\right]$$

$$\Rightarrow \quad 8\pi = 64\alpha \Rightarrow \alpha = \frac{\pi}{8}$$

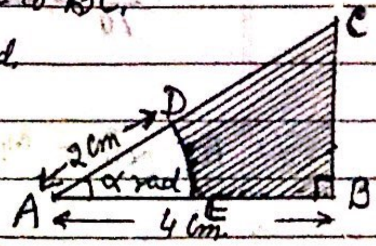
(ii) Perimeter of the figure = OB + arc AB + semicircle OAC

$$= 8 + 8 \times \frac{\pi}{8} + \pi \times 4$$

$$= 8 + \pi + 4\pi \quad \left[\text{length of arc } l = r\theta; \ \text{semicircle length} = \pi r\right.$$

$$= (8 + 5\pi)$$

Example 11: In triangle ABC, AB is perpendicular to BC. The length of AB is 4 cm and angle CAB is α rad. The arc DE with centre A and radius 2 cm, meets AC at D and AB at E. Find in terms of α.



(i) the area of the shaded region. --[3]

(ii) the perimeter of the shaded region. --[3]

[S-14/11/Q6]

Solution:

(i) Area of the shaded region = Area of rt ΔABC − area of sector ADE

$$= \frac{1}{2} \times 4 \times 4\tan\alpha - \frac{1}{2} \times 2^2 \times \alpha \quad (\because BC = 4\tan\alpha)$$

$$= (8\tan\alpha - 2\alpha)$$

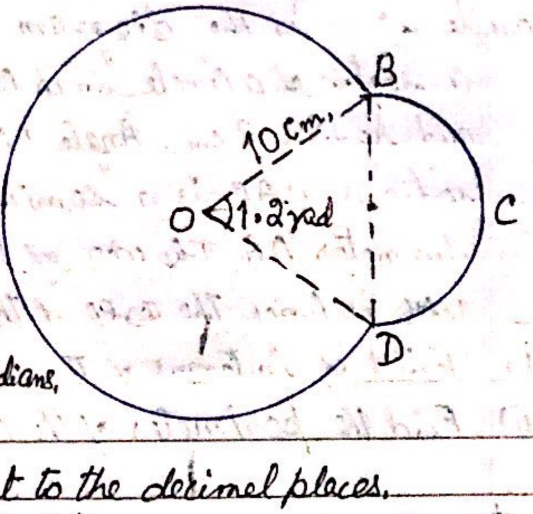
(ii) Perimeter of the shaded region = $\overset{\frown}{DE} + DC + BC + BE$

$$= 2\alpha + \left(\frac{4}{\cos\alpha} - 2\right) + 4\tan\alpha + 2 \quad \left\{\text{Arc } DE = r\theta = 2\alpha; \ AC = \frac{4}{\cos\alpha}\right.$$

$$= \left(2\alpha + \frac{4}{\cos\alpha} + 4\tan\alpha\right)$$

Example 15.

The diagram shows a metal plate ABCD made from two parts, The part BCD is a semicircle. The part DAB is a segment of a circle with centre O and radius 10cm. Angle BOD is 1·2 radians.



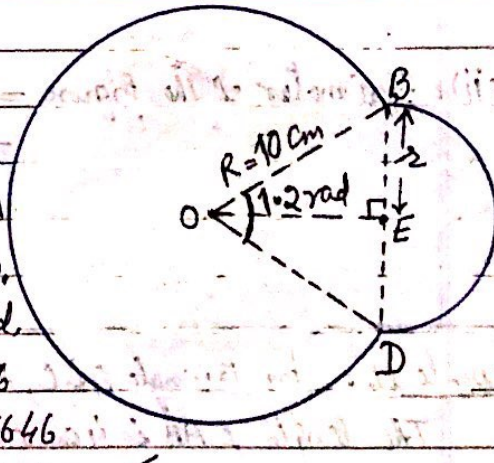
(i) Show that the radius of the semicircle is 5·646 cm, correct to the decimal places. --[2]

(ii) Find the perimeter of the metal plate. --[3]

(iii) Find the area of the metal plate. --[3]

[W-16/12/Q6]

Solution Draw OE ⊥ Chord BD,
E is the mid point of BD,
Given radius of segment of circle BAD, $R = OB = 10$ cm.
$r = BE$ is the radius of semicircle BCD.



(i) In rt Δ BOE, angle $BOE = \frac{1 \cdot 2}{2} = 0 \cdot 6$ rad.

$\frac{r}{10} = \sin 0{\cdot}6 \Rightarrow r = 10 \sin 0{\cdot}6$
$= 10 \times 0{\cdot}5646$
$\therefore r = 5{\cdot}646$ cm.

(ii) Perimeter = Major arc BAD + Semicircle BCD
$= R \cdot \theta + \pi r$
$= 10 \times (2\pi - 1{\cdot}2) + \pi \times 5{\cdot}646$
$= 50{\cdot}832 + 17{\cdot}737 = \underline{60{\cdot}6 \text{ cm}}$

(iii) Area = Area of Major sector OABD + ar of ΔOB + ar of semicircle BCD
$= \frac{1}{2} R^2 \theta + \frac{1}{2} R^2 \sin\theta, + \frac{1}{2} \pi r^2$
$= \frac{1}{2} \times 10^2 (2\pi - 1{\cdot}2) + \frac{1}{2} \times 10^2 \times \sin 1{\cdot}2 + \frac{1}{2} \pi \times (5{\cdot}646)^2$
$= 50 \times 5{\cdot}0832 + 50 \times 0{\cdot}9320 + 1{\cdot}57 \times 31{\cdot}877$
$= 254{\cdot}159 + 46{\cdot}602 + 50{\cdot}1$
$= \underline{351 \text{ cm}^2}$