DATE 10.08.21

P-1

Pure Maths 1

Coordinate Geometry Notes

Suresh Goel
(Former Director)
Alliance World School
Noida, Delhi. NCR
INDIA

(+91 9810444804)

Scanned with CamScanner

P-1

Coordinate-Geometry

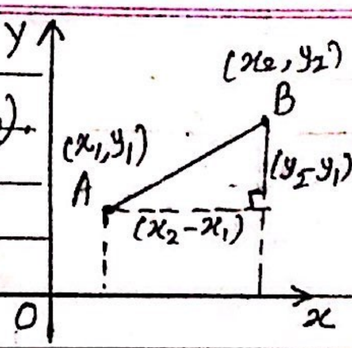
1. § Distance formula:

Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$.

$$AB^2 = (x_2-x_1)^2 + (y_2-y_1)^2$$

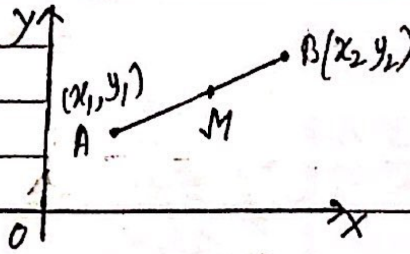
$$AB = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

Note: $OA = \sqrt{x_1^2 + y_1^2}$



2. § Mid point of segment AB:

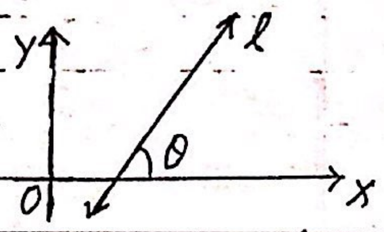
$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$



3. § The inclination of a line 'θ':

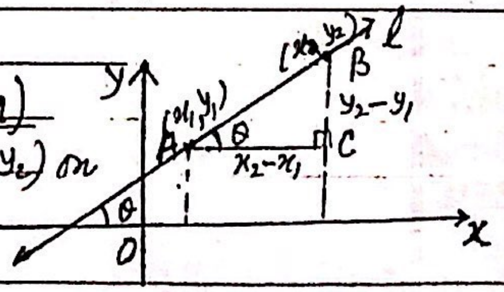
If a line 'l' is inclined at an angle 'θ' with the positive direction of x-axis.

Inclination of the line 'l' $= \theta$.



4. § Gradient of a line (denoted by m)

Given two point $A(x_1, y_1)$, $B(x_2, y_2)$ on line 'l'.

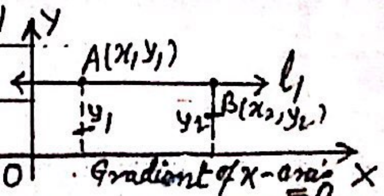


Gradient of line $m = \tan\theta$

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{y_2-y_1}{x_2-x_1}$$

(i) Gradient of line $l_1 \parallel X$-axis: $m = 0$

$$m = \frac{y_2-y_1}{x_2-x_1} = \frac{0}{(x_2-x_1)} = 0 \quad (\because y_1 = y_2)$$

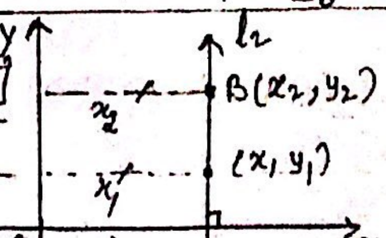


(ii) Gradient of a line $l_2 \parallel Y$-axis:

$$m = \frac{y_2-y_1}{x_2-x_1} = \frac{y_2-y_1}{0} \quad [\because x_1 = x_2]$$

$= \infty$

(Gradient of Y-axis $= \infty$ (not defined))



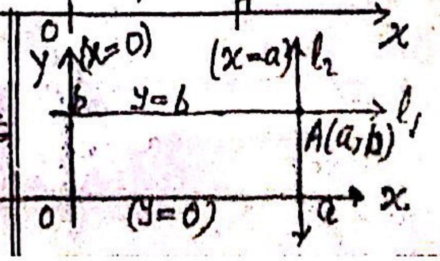
Note (i) Equation of line $l_1 \parallel x$-axis is $y = b$

(Equation of x-axis is $y = 0$)

(ii) Equation of line $l_2 \parallel Y$-axis is $x = a$

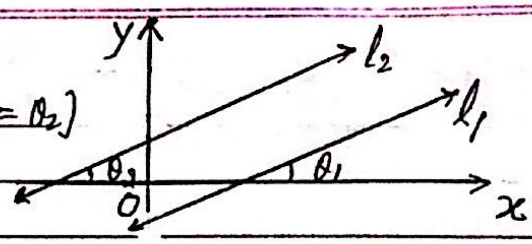
(Given a point $A(a, b)$ on the line)

(Equation of Y axis is $x = 0$)



§ **Gradient of parallel lines:**

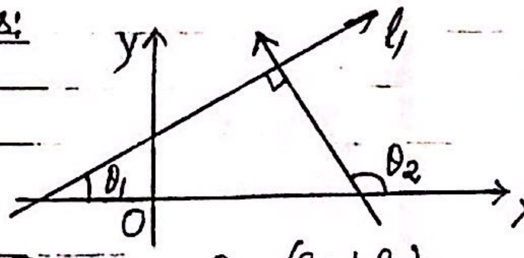
(iii) $l_1 \parallel l_2 \Leftrightarrow m_1 = m_2$ $(\because \theta_1 = \theta_2)$



(iv) **Gradient of perpendicular lines:**

$$l_1 \perp l_2 \Leftrightarrow m_1 \cdot m_2 = -1$$

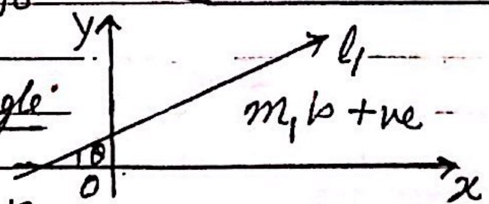
$$\text{or} \quad m_2 = -\frac{1}{m_1}$$



Note (i) Gradient of line l_1: $0 < \theta < 90$

Gradient $m_1 > 0$

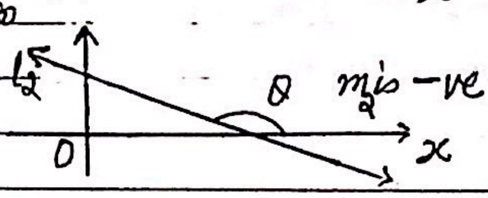
line l_1 is inclined at an acute angle with the +ve direction of x-axis



(ii) Gradient of line l_2: $90 < \theta < 180$

$m_2 < 0$

Line l_2 is inclined at an obtuse angle with the +ve direction of x-axis,



5. § (i) **General equation of line:**

$ax + by + c = 0$ [Here a & b both are not equal to zero]

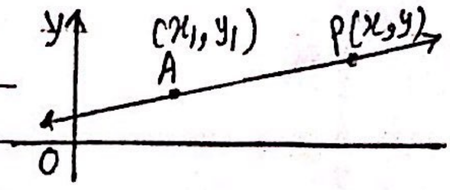
Example: (i) $2x - 3y + 10 = 0$

(ii) $3x + 7 = 0$ (line parallel to y-axis)

(iii) $5y - 8 = 0$ (line parallel to x-axis)

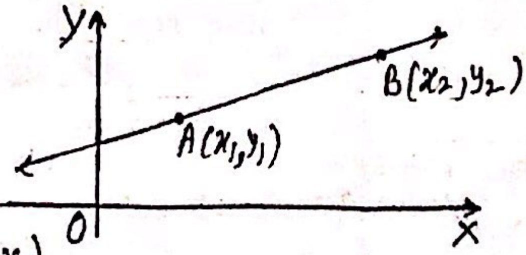
(ii) **Equation of a line passing through a point $A(x_1, y_1)$ and gradient m.**

$$y = m(x - x_1),$$



(iii) **Equation of line passing through two points $A(x_1, y_1)$, $B(x_2, y_2)$:**

Gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

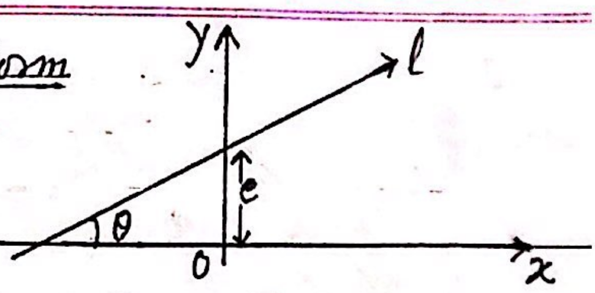


Equation of line: $y - y_1 = \dfrac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$

§5. (iv) Gradient & Y intercept form of equation of line:

Gradient $= m$; Y-Int $= c$

$$y = mx + c$$



§6. To reduce the general equation of line, $ax+by+c=0$ to gradient form $y=mx+c$.

$ax + by + c = 0$

$\Rightarrow by = -ax - c$

$y = -\frac{a}{b}x + \left(-\frac{c}{b}\right)$

$m = -\frac{a}{b}$ and Y-Int $= -\frac{c}{b}$

Example: Find the gradient of the line: $2x + 5y - 7 = 0$

$\Rightarrow 5y = -2x + 7$

$y = -\frac{2}{5}x + \frac{7}{5}$

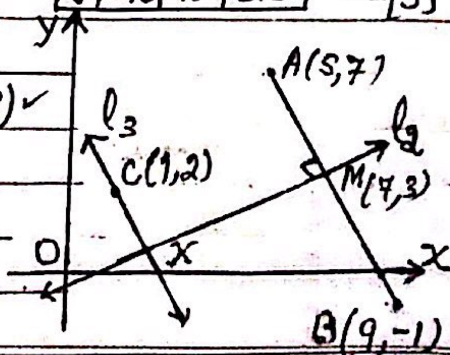
$\therefore$ Gradient $m = -\frac{2}{5}$; Y-Int $= \frac{7}{5}$

Example 1: Two point have coordinates $A(5,7)$ and $B(9,-1)$

(i) Find the equation of the perpendicular bisector of AB. --- [3]

The line through $C(1,2)$, parallel to AB meet the perp. bisector of AB at the point X.

(ii) Find, by calculation, the distance BX. [M-16/12/Q5] --- [5]



Solution (i) Mid point M of $AB = \left(\frac{5+9}{2}, \frac{7-1}{2}\right) = M(7,3)$ ✓

Gradient of $AB = \frac{-1-7}{9-5} = -\frac{8}{4} = -2$ --- ①

$\therefore$ Gradient of line perp to $AB = \frac{1}{2}$ --- ②

$\therefore$ Equation of perp bisector AB, l_2

$M(7,3)$, Grad $= \frac{1}{2}$; $y - 3 = \frac{1}{2}(x-7)$

l_2: or $y = \frac{1}{2}x - \frac{1}{2}$ --- ③ ✓

(ii) Equation of line parallel to AB, $m = -2$ through $C(1,2)$ is $y - 2 = -2(x-1)$

l_3: $\Rightarrow y = -2x + 4$ --- ④

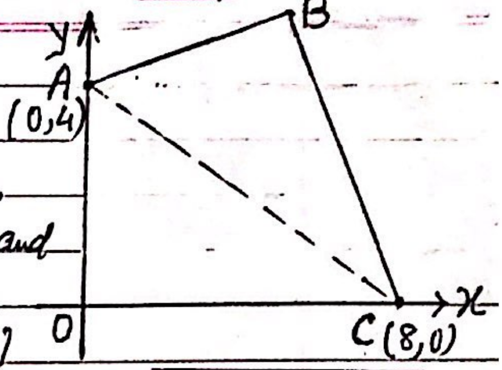
Lines l_2 & l_3 intersect at X.

Solving ③ & ④. $X\left(\frac{9}{5}, \frac{2}{5}\right)$ ✓

$\therefore BX^2 = \left(9 - \frac{9}{5}\right)^2 + \left(-1 - \frac{2}{5}\right)^2$

$= \frac{1345}{25} = 53.8$

$\therefore BX = \sqrt{53.8} = 7.33$ ✓



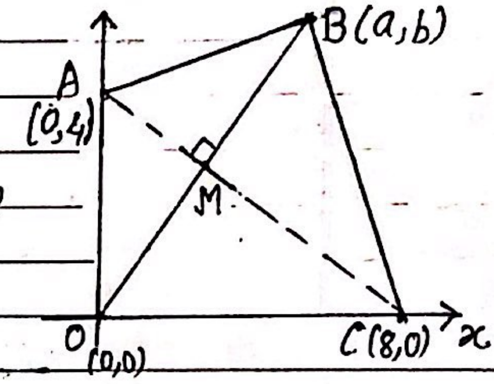
Examample 2:

The diagram shows a kite OABC, in which AC is the line of symmetry. The coordinates of A and C are (0,4) and (8,0) respectively at O is the origin.

(i) Find the equation of AC and OB. ---[4]

(ii) Find the coordinates of B. ---[3]

[S-18/11/Q5]



Solution: $A(0,4)$, $C(8,0)$

Gradient of AC, $m_1 = \frac{0-4}{8-0} = -\frac{1}{2}$ ✓

$\therefore$ Equation of AC. $A(0,4)$

$$y-4 = -\frac{1}{2}(x-0)$$

$$\Rightarrow y = -\frac{1}{2}x+4 \quad \text{---} (1)$$

Now OB is perp. to AC.

Grad. of OB $= 2$, passes through $O(0,0)$

Eqn of OB, $y-0 = 2(x-0)$

or $y = 2x$ --- (2)

AC is perp-bisector of OB, Solving (1) and (2) $M\left(\frac{8}{5}, \frac{16}{5}\right)$

Let the coordinates of $B(a,b)$.

M is the mid point of OB $\Rightarrow \left(\frac{0+a}{2}, \frac{0+b}{2}\right) = \left(\frac{8}{5}, \frac{16}{5}\right)$

$$\Rightarrow a = \frac{16}{5},\ b = \frac{32}{5}$$

$\therefore B(3.2, 6.4)$ ✓

Example 3: Two points A and B have coordinates $(3a, -a)$ and $(-a, 2a)$, respectively where a is a positive constant.

(i) Find the equation of line through origin, which is parallel to AB ---[2]

(ii) The length of AB is $3\frac{1}{3}$ units, Find the value of a. [W-18/11/Q3] ---[3]

Solution: Gradient of AB $= \frac{2a+a}{-a-3a} = -3/4$

(i) Equation of parallel line $= -3/4, (0,0)$

$\therefore$ Equation of line $y-0 = -\frac{3}{4}(x-0)$

$$y = -\frac{3}{4}x \checkmark$$

(ii) Given $AB = \frac{10}{3} \Rightarrow AB^2 = \frac{100}{9}$

$AB^2 = (3a+a)^2 + (2a+a)^2$

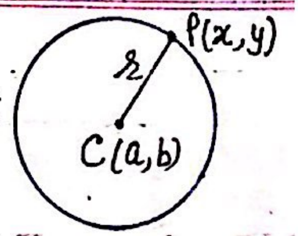
$= 16a^2 + 9a^2 = 25a^2 = \frac{100}{9}$

$\Rightarrow a^2 = \frac{100}{25\times 9} \Rightarrow a = \frac{10}{15} = \frac{2}{3}$ ✓

(Given a is positive).

Circles

§7. **Circle:** circle is the set of points, in a plane, such that each point is at a constant distance (radius) from a fixed point (centre).



§8. (i) **Equation of circle, centre at (a,b) and radius r:**

Let P (x, y) is any point on the circle. Centre C(a,b)

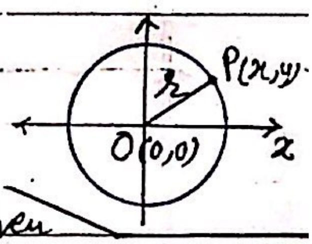
Given $CP = r$

or $CP^2 = r^2$

$\therefore$ $\boxed{(x-a)^2+(y-b)^2=r^2}$ is the equation of circle

(ii) Centre at origin O(0,0), radius r,

Equation of circle: $x^2+y^2=r^2$



Example 4: Find the equation of a circle, given Centre (2,−3) and radius 5 unit.

Solution: Centre (2,−3), r = 5

{ Equation of circle: $(x-a)^2+(y-b)^2=r^2$

Equ^n of circle:

$(x-2)^2+(y+3)^2 = 5^2$

$(x-2)^2+(y+3)^5 = 25$ ✓ ---- ①

may be simplified: $x^2-4x+4+y^2+6y+9=25$

or $x^2+y^2-4x+6y-12=0$ ✓ ---- ②

Example 5: Find the equation of circle, centre at origin and radius 4 units.

Solution: C(0,0), r = 4 $[x^2+y^2=r^2]$

Equ^n of circle: $x^2+y^2=16$. ✓

Scanned with CamScanner

§9. General Equation of a circle:

$$x^2+y^2+2gx+2fy+c=0 \quad \text{--- (1)}$$

Centre $(-g,-f)$

radius $= \sqrt{g^2+f^2-c}$

$$\begin{cases} g = \frac{1}{2}(\text{Coeff. of } x) \\ f = \frac{1}{2}(\text{Coeff of } y) \\ c = \text{Constant term} \end{cases}$$

$x^2+y^2+2gx+2fy = -c$

$x^2+2gx+g^2+y^2+2fy+f^2 = f^2+g^2-c$

$(x+g)^2+(y+f)^2 = g^2+f^2-c$

$[x-(-g)]^2+[y-(-f)]^2 = (\sqrt{g^2+f^2-c})^2$

comparing with

$[(x-a)^2+(y-b)=r^2]$

Centre $(-g,-f)$; $r=\sqrt{g^2+f^2-c}$ ✓

Note: (i) In general equation of circle (1): Coeff of x^2 = Coeff y^2 = 1.

(ii) If Eqn of circle is $ax^2+ay^2+bx+cy+d=0$ --- (2)

Divide (2) by 'a' $\rightarrow$ and then find 'g', 'f' and 'c'.

Example 6: Given the equation of circle:

$x^2+y^2-4x+6y-12=0$

find the coordinates of Centre and radius of the circle.

Solution: Centre $(-g,-f) = (-(-2), -3)$

$= C(2,-3)$ ✓

$$\begin{cases} g = \frac{1}{2}(-4) = -2 \\ f = \frac{1}{2} \times 6 = 3 \\ c = -12 \end{cases}$$

radius $r = \sqrt{g^2+f^2-c} = \sqrt{(-2)^2+3^2-(-12)} = \sqrt{25} = 5$ ✓

$\therefore$ Centre $(-2,3)$ and $r=5$.

Example 7: Find the Centre and radius of the circle:

$5x^2+5y^2+4x-8y-16=0$

Solution: $5x^2+5y^2+4x-8y-16=0$

Divide by 5. [Coeff x^2 = Coeff $y^2 \neq 1$

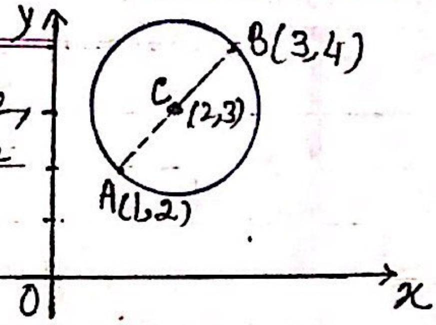
$x^2+y^2+\frac{4}{5}x-\frac{8}{5}y-\frac{16}{5}=0$

$$\begin{cases} g = \frac{1}{2} \times \frac{4}{5} = \frac{2}{5} \\ f = \frac{1}{2} \times \left(-\frac{8}{5}\right) = -\frac{4}{5} \\ c = -\frac{16}{5} \end{cases}$$

Centre $= (-g,-f) = \left(-\frac{2}{5}, \frac{4}{5}\right)$ ✓

$r = \sqrt{\left(\frac{2}{5}\right)^2+\left(-\frac{4}{5}\right)^2+\frac{16}{5}} = \sqrt{\frac{4}{25}+\frac{16}{25}+\frac{16}{5}} = \sqrt{\frac{100}{25}} = 2$ ✓

Scanned with CamScanner



Example 8: Find the equation of a circle, given the end points of diameter are $A(1,2)$ and $B(3,4)$.

Solution: End points of diameter are $A(1,2)$ and $B(3,4)$.

Centre C is the mid point of AB; $C\left(\frac{1+3}{2}, \frac{2+4}{2}\right) = (2,3)$

and radius = CA

$$r = \sqrt{(2-1)^2+(3-2)^2} = \sqrt{1+1} = \sqrt{2} \checkmark$$

$\therefore$ Equation circle:

$$(x-a)^2+(y-b)^2 = r^2$$

$$(x-2)^2+(y-3)^2 = (\sqrt{2})^2$$

$\left\{ \begin{array}{l} C(a,b) = (2,3) \\ r = \sqrt{2} \end{array} \right.$

or $x^2+y^2-4x-6y+11=0 \checkmark$

Example 9: Find the equation of the circle passing through three points, $A(1,1)$, $B(2,-1)$ and $C(3,2)$; also find the radius and coordinate of it centre.

Solution: Let the equation of circle is:

$x^2+y^2+2gx+2fy+c=0$ --- (1)

$A(1,1)$ lies on the circle (1)

$\Rightarrow 1^2+1^2+2g\times1+2f\times1+c=0$

or $2g+2f+c=-2$ --- (2)

$B(2,-1)$ lies on (1)

$2^2+(-1)^2+2g\times2+2f(-1)+c=0$

or $4g-2f+c=-5$ --- (3)

and $C(3,2)$ lies on (1)

$3^2+2^2+2g\times3+2f\times2+c=0$

or $6g+4f+c=-13$ --- (4)

Subtract (2)−(3) $\Rightarrow -2g+4f=3$ --- (5)

Subtract (2)−(4) $\Rightarrow -4g-2f=11$ --- (6)

Solving equations (5) and (6)

$g=-\frac{5}{2}$, $f=-\frac{1}{2} \checkmark$

Put the values of g and f in (2)

$2\times\left(-\frac{5}{2}\right)+2\left(-\frac{1}{2}\right)+c=-2$

$\Rightarrow c=4 \checkmark$

Put the values of g, f and c in (1)

Equation of circle:

$x^2+y^2+2\left(-\frac{5}{2}\right)x+2\left(-\frac{1}{2}\right)y+4=0$

or $x^2+y^2-5x-y+4=0 \checkmark$

Now

Centre $(-g,-f) = \left(\frac{5}{2}, \frac{1}{2}\right) \checkmark$

and radius $r = \sqrt{\left(-\frac{5}{2}\right)^2+\left(-\frac{1}{2}\right)^2-4}$

$= \sqrt{\frac{25}{4}+\frac{1}{4}-4} = \sqrt{\frac{10}{4}}$

$\therefore r = \frac{\sqrt{10}}{2} \checkmark$

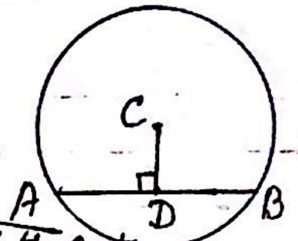
* [Refer example 14/ Page 12]

Scanned with CamScanner

"The line joining the centre of a circle to the mid point of a chord is perpendicular to the chord."

§10: Given A and B are any two points on a circle, centre C, D is the mid-point of AB. Then

CD is perpendicular to the chord AB.



Note: Perpendicular bisector of chord passes through the centre.

Example 10. The circle; $x^2+y^2+4x-2y-20=0$ has centre C, and passes through points A and B; $D(1\frac{1}{2}, 1\frac{1}{2})$ is the mid point of AB.

(a) State the coordinates of C. ---[1]

(b) Find the equation of AB, giving your answer in the form $y=mx+c$ ---[4]

(c) Find the x-coordinates of A and B. ---[3]

SP-20/1/Q10

Solution (a) Equation of circle; $x^2+y^2+4x-2y-20=0$ --- ①

$g=\frac{4}{2}=2,\ f=\frac{-2}{2}=-1 \Rightarrow$ Centre $(-g,-f)$

$= C(-2, 1)$ ✓

(b) Now $C(-2,1)$ and $D(3/2, 3/2)$

$\therefore$ Gradient of $CD = \frac{3/2-1}{3/2+2} = \frac{1}{7}$ ✓

Now AB is perpendicular to CD

$\therefore$ Gradient of $AB = -7$

$\therefore$ Equation of AB, (through $D(\frac{3}{2},\frac{3}{2})$)

$y-3/2 = -7(x-\frac{3}{2})$

or $y=-7x+12$ --- ②

(c) To get x-coordinates of A, B,

Solving ① and ②

$\Rightarrow x^2+(12-7x)^2+4x-2(12-7x)-20=0$

$\Rightarrow x^2+49x^2+144-168x+4x-24+14x-20=0$

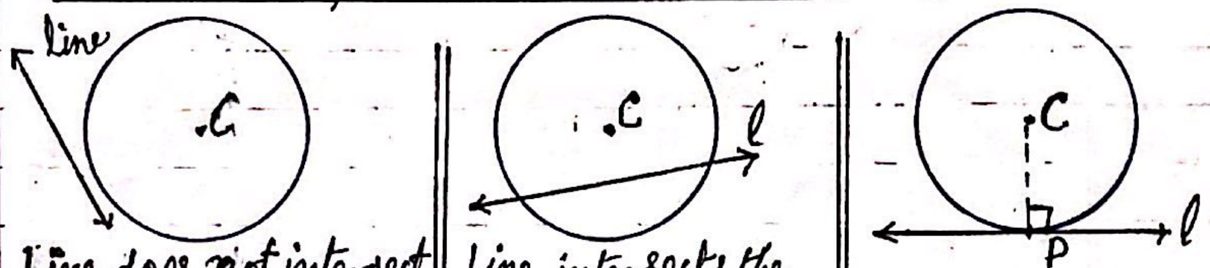
$\Rightarrow 50x^2-150x+100=0$

$\Rightarrow x^2-3x+2=0$

$\Rightarrow (x-2)(x-1)=0 \Rightarrow x=1; 2$

$\therefore$ x-coordinates of A and B are 1 and 2 respectively.

§ 11. Intersection of a line and a circle:



Line does not intersect the circle. | Line intersects the circle at two distinct points. | l is tangent to the circle at P.

Note: 1. If a line is tangent to the circle, it intersects the the circle at exactly one point P (point of contact).

2. The line joining the centre and the point of contact is perpendicular to the tangent. "CP perp. to l"

Example 11: Prove that the line $x+y=5$ touches the circle $x^2+y^2-2x-4y+3=0$, Find the point of contact.

Solution: Circle: $x^2+y^2-2x-4y+3=0$ --- (1)

line: $x+y=5 \Rightarrow y=(5-x)$ --- (2)

from (1) and (2) $\rightarrow x^2+(5-x)^2-2x-4(5-x)+3=0$

$\rightarrow 2x^2-8x+8=0$

$\Rightarrow x^2-4x+4=0$

$(x-2)^2=0 \Rightarrow x-2=0$

$x=2$ (Only one point)

As the line intersects the circle at only point, The line touches the circle,

$x=2$ from (2) $y=5-2=3$

$\therefore$ Point of contact is $(2,3)$ ✓

Scanned with CamScanner

Example 12: Given the equation of a circle $x^2+y^2-2x-4y+3=0$ and a line $x+y=k$, find the value of k, for which:

(i) Line a tangent to the circle,

(ii) Line intersects the circle at two different points.

(iii) Line does not intersects the circle.

Solution: Circle: $x^2+y^2-2x-4y+3=0$ ---- (1)

Line : $x+y=k \Rightarrow y=k-x$ --- (2)

for the intersection of line and circle,

Solving (1) and (2) $\rightarrow x^2+(k-x)^2-2x-4(k-x)+3=0$

$\Rightarrow 2x^2+2x(1-k)+(k^2-4k+3)=0$ --- (3)

(i) line will be tangent to the circle if it intersects the circle at exact one point $\rightarrow$ $B^2-4AC=0$ for eqn (3)

$B^2-4AC=[2(1-k)]^2-4\times 2\times(k^2-4k+3)$

$= -4[k^2-6k+5]$ --- (4)

$\left\{\begin{array}{l} B=2(1-k) \\ A=2 \\ C=(k^2-4k+3) \end{array}\right.$

for tangent $-4(k^2-6k+5)=0$

$\Rightarrow -4(k-5)(k-1)=0$

$\Rightarrow \underline{k=5;\ k=1}$ ✓

(ii) Line will intersect the circle at two different points if: $\longrightarrow B^2-4AC>0$

from (4) $-4(k-5)(k-1)>0$

$\Rightarrow (k-5)(k-1)<0$

$\Rightarrow \underline{1<k<5}$ ✓

$\left\{\text{critical values of } k \text{ are } 1, 5\right.$

(iii) Line will <u>not</u> intersect the circle for $B^2-4AC<0$

from (4) $-4(k-5)(k-1)<0$

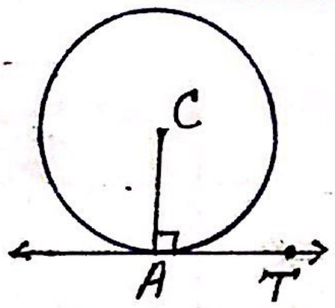
$\Rightarrow (k-5)(k-1)>0$

$\Rightarrow \underline{k<1;\ k>5}$ ✓

$\left\{\text{Critical values of } k \text{ are } 1, 5\right.$

Scanned with CamScanner

§12. To find the equation of tangent to a circle at a given point A on it.



Given a circle with centre C and a point A on it, Then the tangent 'AT' to the circle at A is perpendicular to 'CA'. ($AT \perp CA$)

Example 13: Find the equation of the tangent to the circle $x^2+y^2-30x+6y+109=0$ at a point $(4,-1)$ on it.

Solution: circle: $x^2+y^2-30x+6y+109=0$ --- ①

Centre $C(-g,-f)=(15,-3)$ $\left\{ \begin{array}{l} g=-\frac{30}{2}=-15 \\ f=\frac{6}{2}=3 \end{array} \right.$

given a point $A(4,-1)$ on the circle.

gradient of $CA=\frac{-3-(-1)}{15-4}=\frac{-2}{11}$

$\therefore$ gradient of tangent $=\frac{11}{2}$ [$\because$ tangent is perp to CA]

$\therefore$ Equation of the tangent to the circle at $A(4,-1)$ is

$$y+1=\frac{11}{2}(x-4)$$

or $2y+2=11x-44$

or $11x-2y-46=0$ ✓

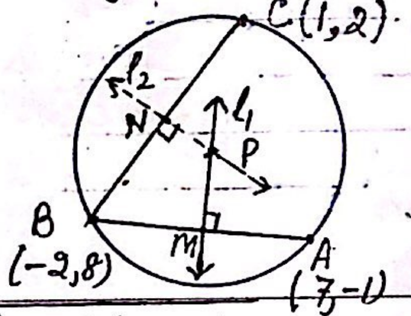
Scanned with CamScanner

Example 14. Given three non-collinear points $A(7,-1)$, $B(-2,8)$ and $C(1,2)$.

(i) Find the equation of the perpendicular bisector AB.

(ii) Show that the equation of the perpendicular bisector of BC is; $y=\frac{1}{2}x+\frac{21}{4}$

(iii) Find the equation of circle passing through the three points A, B and C.



Solution: $A(7,-1)$, $B(-2,8)$ and $C(1,2)$

(i) Mid point of AB, $M\left(\frac{7-2}{2},\frac{8-1}{2}\right)=\left(\frac{5}{2},\frac{7}{2}\right)$

Gradient of AB, $m_1=\left(\frac{8+1}{-2-7}\right)=-1$

$\therefore$ Gradient of line l_1 perp to $AB=1$

Eqn of line l_1, perp. bisector of AB; $y-\frac{7}{2}=1\left(x-\frac{5}{2}\right)\left(M\left(\frac{5}{2},\frac{7}{2}\right)\right)$

$\Rightarrow\ y=x+1$ ---- (1)

(ii) Mid point of BC, $N\left(\frac{-2+1}{2},\frac{8+2}{2}\right)$

$=N\left(-\frac{1}{2},5\right)$

Gradient of BC, $m_2=\frac{8-2}{-2-1}=\frac{6}{-3}=-2$

$\therefore$ gradient of l_2, the perp. bisector of $BC=\frac{1}{2}$

$\therefore$ Equation of l_2 $\quad N\left(-\frac{1}{2},5\right)$

$y-5=\frac{1}{2}\left(x+\frac{1}{2}\right)$

$\Rightarrow y=\frac{1}{2}x+\frac{21}{4}$ --- (2)

Note: Perpendicular bisector of chord of a circle passes through its centre.

Hence l_1 and l_2 both passes through the centre 'P' of the circle. To get the coordinates of P. Solve (1) & (2)

$x+1=\frac{1}{2}x+\frac{21}{4}$

$\Rightarrow \frac{1}{2}x=\frac{17}{4} \Rightarrow \quad x=\frac{17}{2}$

from (1) $y=\frac{17}{2}+1=\frac{19}{2}$

$P\left(\frac{17}{2},\frac{19}{2}\right)$ centre

radius of the circle: $r^2=PA^2$

$r^2=\left(\frac{17}{2}-7\right)^2+\left(\frac{19}{2}+1\right)^2$

$=\frac{450}{4}$

$\therefore$ Equation of the circle #

$\left(x-\frac{17}{2}\right)^2+\left(y-\frac{19}{2}\right)^2=\frac{450}{4}$

or

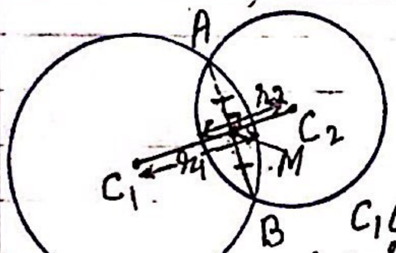
$x^2+y^2-17x-19y+50=0$

$(x-a)^2+(y-b)^2=r^2$

* (Refer example 9/ Page 7)

Scanned with CamScanner

§13 Intersection of two circles:



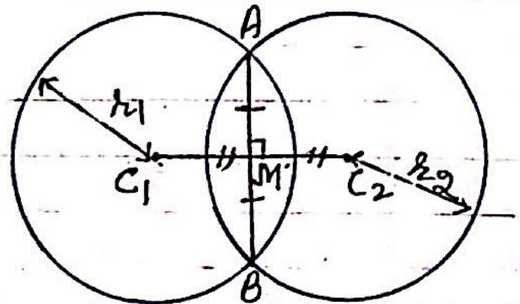
$r_1 \neq r_2$

$C_1C_2 < (r_1+r_2)$

$r_1 > r_2$

C_1C_2 is the perp bisector of AB.

$AM = BM$ (But $C_1M \neq C_2M$)



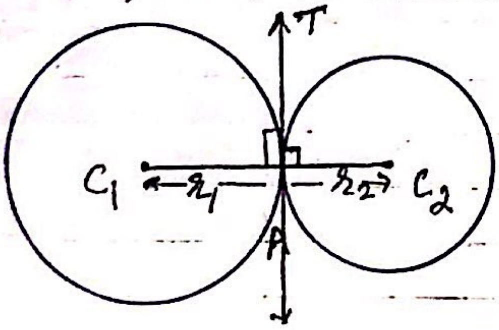
$r_1 = r_2$ & $C_1C_2 < r_1 + r_2$

$AM = BM$ and $C_1M = C_2M$

AB and C_1C_2 both are the perpendicular bisector of each other; M is the midpoint of AB and M is the mid point of C_1C_2.

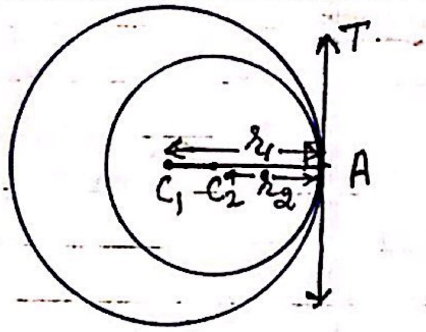
Note: Given the Equation of two circles as $S_1 = 0$ and $S_2 = 0$

Then the equation of the common chord AB is $S_1 - S_2 = 0$



$r_1 + r_2 = C_1C_2$

Two circles touch each other externally; AT is the common tangent.



$r_1 - r_2 = C_1C_2$

Two circles touch each other internally; AT is the common tangent.

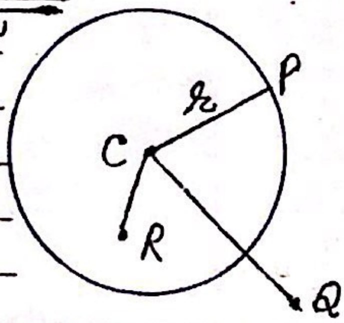
§ Point lies outside/inside of a circle:

Given a circle with centre at C and radius 'r'

(i) $CP = r$ (P lies on the circle)

(ii) $CQ > r$, Q lies outside the circle

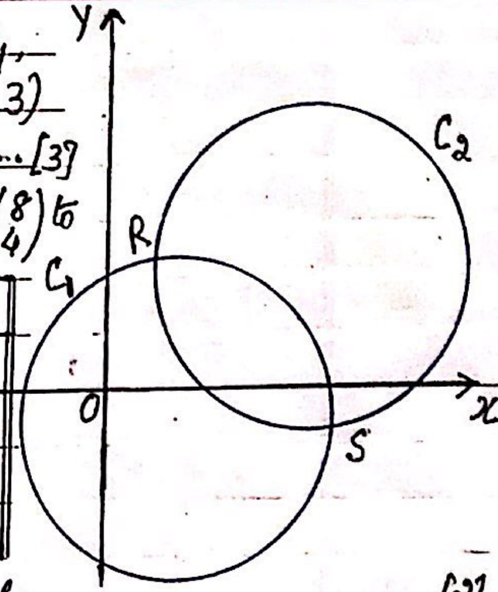
(iii) $CR < r$; R lies inside the circle.



Example 16: A diameter of a circle C_1, has end points at $(-3,-5)$ and $(7,3)$

(a) Find the equation of circle C_1. ....[3]

The circle C_1 is translated by $\begin{pmatrix}8\\4\end{pmatrix}$ to give circle C_2, as shown in the diagram.



(b) Find the equation of circle C_2 ....[2]

The two circle intersect at R and S.

(c) Show that the equation of line RS is;
$y = -2x + 13$ ....[4]

(d) Hence show that the x- coordinates of R and S satisfy the equation $5x^2 - 60x + 159 = 0$ ....[3] [M-20/12/Q12]

Solution: End points of the diameter of

(a) C_1 are $A(-3,-5)$, $B(7,3)$.

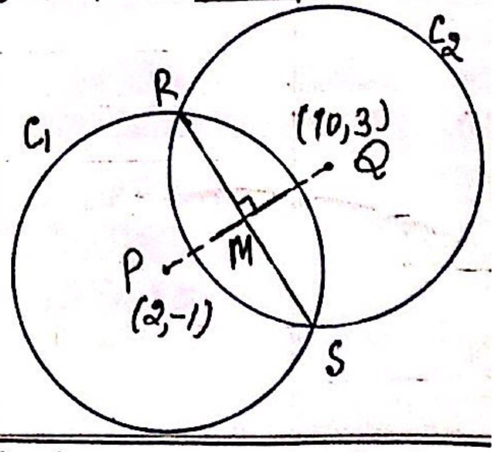
Centre of C_1 is the mid point of AB.

$P\left(\frac{-3+7}{2}, \frac{-5+3}{2}\right) = P(2,-1)$ ✓

and radius of C_1, $r_1^2 = CA^2$

$= \sqrt{(2-(-3))^2 + (-1+5)}^2$

$= \sqrt{5^2+4^2} = 41$ ✓



$\therefore$ Equation of circle C_1 is
$(x-2)^2 + (y+1)^2 = 41$ ✓
or C_1: $x^2+y^2-4x+2y-36=0$ --(1)

(b) Centre of C_2: $\begin{pmatrix}+2\\-1\end{pmatrix} + \begin{pmatrix}8\\4\end{pmatrix} = \begin{pmatrix}10\\3\end{pmatrix}$ Q ✓

and radius is same as of C_1; $r^2 = 41$ ✓

$\therefore$ Eqn of C_2: $(x-10)^2 + (y-3)^2 = 41$ ✓
or C_2: $x^2+y^2-20x-6y+68=0$ --(2)

(c) Let circle C_1: $S_1 = 0$ ---(1)
and circle C_2: $S_2 = 0$ ---(2)
Then the equation of the common

Chord RS is $S_1 - S_2 = 0$
or $(x^2+y^2-4x+2y-36) -$
from (1) & (2) $(x^2+y^2-20x-6y+68) = 0$
$\Rightarrow$ $16x + 8y - 104 = 0$ ✓
$\Rightarrow$ $y = -2x + 13 = 0$ --(3)

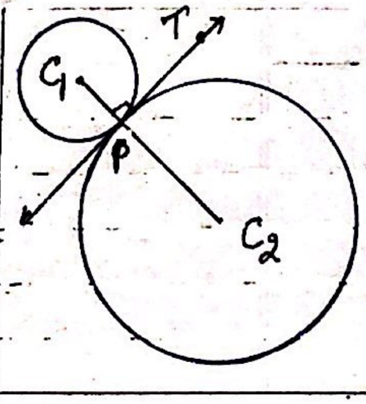
(d) To get the x-coord. of R_1 & S
Solving (2) and (3)
$x^2 + (13-2x)^2 - 20x - 6(13-2x) + 68 = 0$
or
$5x^2 - 60x + 159 = 0$ ✓

Scanned with CamScanner

Example 16: Given the equation of two circles; $x^2+y^2+2x+2y+1=0$ and $x^2+y^2-4x-6y-3=0$

(i) Show that the two circles touch each other.

(ii) Find the equation of common tangent and the point of contact.



Solution: Circle 1: $x^2+y^2+2x+2y+1=0$ --- ①

(i) Circle 2: $x^2+y^2-4x-6y-3=0$ --- ②

for circle ① $g=1, f=1, c=1$

$\therefore$ Centre $C_1(-1,-1)$, $r_1=\sqrt{1^2+1^2-1}=1$ ✓ -- ③

for circle ②, $g=-2, f=-3, c=-3$

Centre $C_2:(2,3)$, $r_2=\sqrt{2^2+3^2+3}=4$ ✓ -- ④

Distance $C_1C_2=\sqrt{(2+1)^2+(3+1)^2}=\sqrt{3^2+4^2}=5$ -- ⑤

$r_1+r_2=1+4=5=C_1C_2$ from ③, ④ & ⑤

$\therefore$ Two circle touch each other. $[r_1+r_2=C_1C_2]$

(ii) Equation of tangent will be $S_1-S_2=0$ [Equ of Circle ① $\Rightarrow S_1=0$; Equ of Circle ② $\Rightarrow S_2=0$]

$(x^2+y^2+2x+2y+1)-(x^2+y^2-4x-6y-3)=0$

$\Rightarrow 6x+8y+4=0$ or $3x+4y+2=0$

$\therefore$ Equation common tangent is $4y=-(3x+2)$

or $y=-\left(\frac{3x+2}{4}\right)$ --- ⑥

To find the coordinates of the point of contact;

Solving ① & ⑥ $\Rightarrow x^2+\left(\frac{3x+2}{4}\right)^2+2x+2x-\left(\frac{3x+2}{4}\right)+1=0$

Simplifying $\rightarrow$ $25x^2+20x+4=0$

$(5x+2)^2=0$

$\Rightarrow x=-2/5$ ✓

Put $x=-\frac{2}{5}$ in ⑥ $y=-\left(\frac{3x-\frac{2}{5}+2}{4}\right)=-\frac{1}{5}$ ✓

$\therefore$ The coordinate of the point of contact $P\left(-\frac{2}{5},-\frac{1}{5}\right)$ ✓

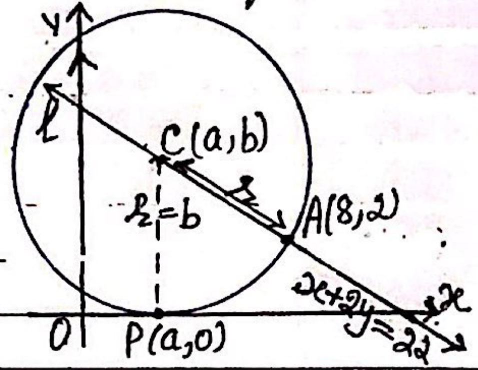
Scanned with CamScanner

Example 17: Two circles have the following properties;

- the x-axis is the common tangent to the circles.
- the point $(8,2)$ lies on both the circles.
- the centre of each circle lies on the line $x+2y=22$

(a) Find the equation of each circle.

(b) Prove that the line $4x+3y=88$ is a common tangent to these circles.



Solution: Let the circle has centre $C(a,b)$

(a) radius r; Equation of circle is:

$(x-a)^2+(y-b)^2=r^2$ ---- ①

as the circle passes through $(8,2)$ should lie on ①

$(8-a)^2+(2-b)^2=r^2$ ---- ②

Now as the circle touches x-axis, centre at (a,b)

$\Rightarrow r=b$ ---- ③

Put $r=b$ in ②, $(8-a)^2+(2-b)^2=b^2$ ---- ④

but centre $C(a,b)$ lies on the line 'l' $x+2y=22$

$\Rightarrow a+2b=22 \Rightarrow a=22-2b$ ---- ⑤

from ④ and ⑤ $[8-(22-2b)]^2+(2-b)^2=b^2$

$\Rightarrow (2b-14)^2+(2-b)^2=b^2$

$4b^2-56b+196+4-4b+b^2=b^2$

$4b^2-60b+200=0$

$b^2-15b+50=0$

$(b-5)(b-10)=0$

$b=5$; $b=10$

$a=12$; $a=2$ (from ⑤)

$\therefore$ Case I Centre $(12,5)$, $r=b=5$ ✓

Case II, Centre $(2,10)$, $r=10$ ✓

Case I Centre $(12,5)$, $r=5$

Circle① $(x-12)^2+(y-5)^2=5^2$

$x^2-24x+144+y^2-10y+25=25$

or $x^2+y^2-24x-10y+144=0$ ✓ ---- ⑥

Case II Centre $(2,10)$, $r=10$

Circle②: $(x-2)^2+(y-10)^2=10^2$ ✓

$x^2+y^2-4x-20y+4=0$ ---- ⑦

(b) Now given a line $4x+3y=88 \Rightarrow y=\frac{88-4x}{3}$

Put in ⑥ $x^2+\left(\frac{88-4x}{3}\right)^2-24x-10\left(\frac{88-4x}{3}\right)+144=0$

$\Rightarrow x^2-32x+256=0$

$\Rightarrow (x-16)^2=0$

$x=16$ only one point,

Hence the line $4x+3y=88$ is a tangent to Circle①

Similarly for Circle②

$\therefore$ is a common tangent.

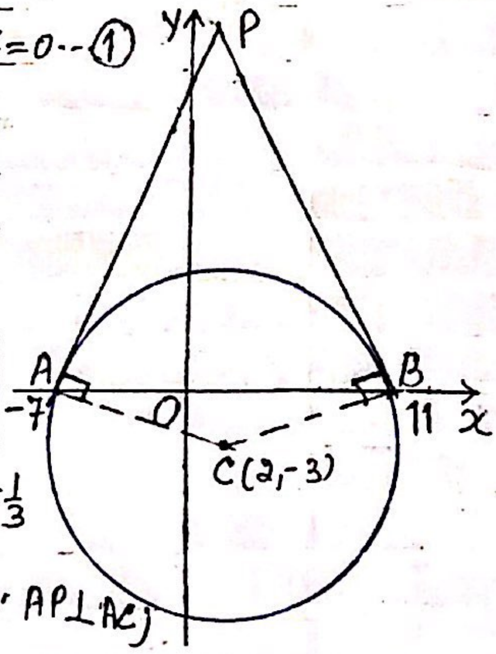
Scanned with CamScanner

Example 18: The equation of a circle is; $x^2+y^2-4x+6y-77=0$

(a) Find the x-coordinate of the points A and B where the circle intersect the x-axis. ---[2]

(b) Find the point of intersection of the tangents to the circle at A and B. --[6]

S-21 | 11 | Q10



Solution: Circle; $x^2+y^2-4x+6y-77=0$ --①

(a) Circle intersects x-axis, where $y=0$

from ① $x^2-4x-77=0$

$(x-11)(x+7)=0$

$x=-7;\ x=+11$

(b) $A(-7,0)$, $B(11,0)$

Centre of circle $C(-g,-f)=(2,-3)$

for the tangent to circle at A, PA,

Gradient of $CA=\left(\frac{-3-0}{2+7}\right)=-\frac{3}{9}=-\frac{1}{3}$

∴ Gradient of the tangent $PA = 3$ ($\because AP\perp AC$)

∴ the eqn of tangent AP, $A(-7,0)$

$y-0=3(x+7) \Rightarrow y=3x+21$ ---②

Now Gradient of $CB=\frac{-3-0}{2-11}=\frac{-3}{-9}=\frac{1}{3}$

∴ Gradient the tangent at B, $BP=-3$ ✓, $B(11,0)$

∴ Equation of the tangent BP,

$y-0=-3(x-11) \Rightarrow y=-3x+33$ ---③

Now to get the point of intersection of tangents PA and PB, solving ② and ③

$P(2,27)$ ✓

Scanned with CamScanner

Example 19: Points $A(-2,3)$, $B(3,0)$ and $C(6,5)$ lie on the circle with centre D.

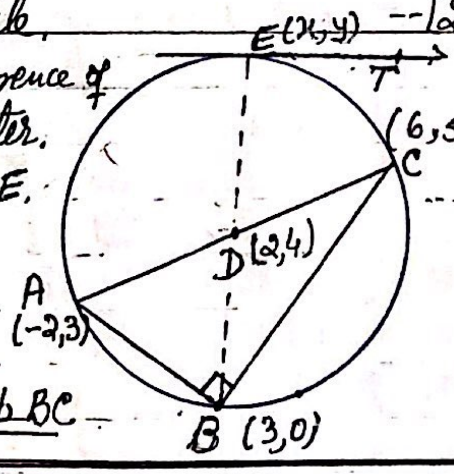
(a) Show that angle $ABC = 90°$ [S-21/13/Q10] ---[2]

(b) Hence state the coordinates of D. ---[1]

(c) Find an equation of the circle. ---[2]

The point E lies on the circumference of of the circle, such that BE is a diameter.

(d) Find the equation of tangent to circle at E. ---[5]



Solution: Gradient of AB, $m_1 = \frac{0-3}{3+2} = -\frac{3}{5}$ ✓

(a) Gradient of $BC = \frac{5-0}{6-3} = \frac{5}{3} = m_2$

$m_1 \times m_2 = -\frac{3}{5} \times \frac{5}{3} = -1$ ∴ AB perp BC

∴ angle $ABC = 90°$ ✓

§ Given a rt triangle, the circle drawn with hypotenuse as diameter passes through the opposite vertex. (Right angle)

(b) AC will be the diameter of the circle: Centre D will be Mid point of AC. $\left(\frac{-2+6}{2}, \frac{3+5}{2}\right)$

$D(2,4)$ ✓

(C) Radius of circle r: $r^2 = AD^2 = (2+2)^2 + (4-3)^2 = 16+1 = 17$ ✓

Centre $D(2,4)$; Equation of circle:

$(x-2)^2 + (y-4)^2 = 17$ $\left((x-a)^2 + (y-b)^2 = r^2\right)$

(d) Let the coordinates of $E(x,y)$, and $B(3,0)$, D is the mid point of BE (diameter)

$\Rightarrow \left(\frac{x+3}{2}, \frac{y+0}{2}\right) = (2,4) \Rightarrow \left\{\frac{x+3}{2} = 2;\ \frac{y}{2} = 4\right\} \Rightarrow \left\{\begin{array}{l} x=1 \\ y=8 \end{array}\right.$

∴ $E(1,8)$ ✓ and $D(2,4)$

Gradient of $DE = \frac{4-8}{2-1} = -4 \Rightarrow$ Gradient of tangent at $E = \frac{1}{4}$ ✓

∴ Equation of tangent to the circle at $E(1,8)$

$y - 8 = \frac{1}{4}(x-1)$

or $x - 4y + 31 = 0$ ✓

Scanned with CamScanner

Example 20: The point A has coordinates (1,5) and the line l has gradient $-\frac{2}{3}$ and passes through a point A. A circle has centre (5,11) and radius $\sqrt{52}$.

(a) Show that l is the tangent to the circle at A. ---[2]

(b) Find the equation of the other circle of radius $\sqrt{52}$ for which l is also the tangent at A. ---[3]

Solution: $A(1,5)$, Centre $C(5,11)$, $r=\sqrt{52}$ | S-21 | 12 | Q7 |

(a) distance $CA=\sqrt{(5-1)^2+(11-5)^2}$
$=\sqrt{4^2+6^2}=\sqrt{52}=r$

as distance $CA=r$

$\therefore$ point A lies on the circle,

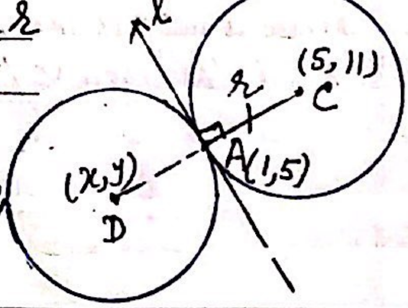
and gradient of $CA=\frac{11-5}{5-1}=\frac{6}{4}=\frac{3}{2}=m_1$

Gradient of line 'l' $m_2=-2/3$

$m_1\times m_2=\frac{3}{2}\times\left(-\frac{2}{3}\right)=-1$

$\Rightarrow$ line 'l' is perp. to CA.

$\therefore$ 'l' is tangent to the circle at A. ✓



(b) The centre 'D' of the other circle with the same radius $r=\sqrt{52}$ will be on the line CA produced such that (1,5) A is mid point of CD. Let $D(x,y)$

$\therefore \left(\frac{x+5}{2},\frac{y+11}{2}\right)=(1,5)\Rightarrow \begin{cases}\frac{x+5}{2}=1\Rightarrow x=-3\\ \frac{y+11}{2}=5\Rightarrow y=-1\end{cases}$

Centre $D(-3,-1)$

and radius $\sqrt{52}$ [here DA is perp to l hence l is tangent to the new circle]

$\therefore$ Equation of the other circle is

$(x+3)^2+(y+1)^2=(\sqrt{52})^2$

$(x+3)^2+(y+1)^2=52$ ✓

Scanned with CamScanner

1. Given two points A(−7,3) and B(3,−5). Find the distance AB.

Solution: $AB = \sqrt{(3+7)^2 + (-5-3)^2}$ $\quad (AB^2 = (x_2-x_1)^2 + (y_2-y_1)^2)$

$= \sqrt{10^2 + 8^2} = \sqrt{100+64} = \sqrt{164} = \sqrt{4 \times 41} = \underline{2\sqrt{41}}$ ✓

2. The distance between two points P(−3,−2), Q(b,2b), is 10. Find the two possible values of b.

Solution: $PQ^2 = (b+3)^2 + (2b+2)^2 = 10^2$ (Given)

$\Rightarrow b^2 + 6b + 9 + 4b^2 + 8b + 4 = 100$

$5b^2 + 14b - 87 = 0$

$5b^2 + 29b - 15b - 87 = 0$

$b(5b+29) - 3(5b+29) = 0$

$(5b+29)(b-3) = 0 \Rightarrow$

$5b+29 = 0$ or $b-3 = 0$

$b = -\frac{29}{5}$ or $b = 3$

$b = -5\frac{4}{5}$ ✓ ; $b = 3$ ✓

3. The point (−2,−3) is the mid point of the line segment joining P(−6,−5) and Q(a,b). Find the value of a and of b.

Solution: P(−6,−5), Q(a,b) $\quad$ $\{P(x_1,y_1),\ Q(x_2,y_2)$; Mid point $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Mid point of PQ is

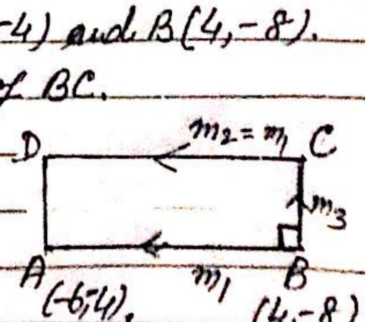
$\left(\frac{-6+a}{2}, \frac{-5+b}{2}\right) \equiv (-2,-3)$ (Given)

$\Rightarrow \frac{-6+a}{2} = -2$ and $\frac{-5+b}{2} = -3 \Rightarrow -6+a = -4$; $-5+b = -6$

$\Rightarrow a = 2,\ b = -1$ ✓

4. Two vertices of a rectangle, ABCD, are A(−6,−4) and B(4,−8). Find the gradient of CD and the gradient of BC.

Solution: Gradient of ABA, $m_1 = \frac{-8+4}{4+6} = \frac{-4}{10} = -\frac{2}{5}$



Now $CD \parallel BA \Rightarrow$ Grad of $CD = m_1 = -2/5$ ✓

Now $BC \perp BA$, $m_3 \times m_1 = -1$

$m_3 \times -2/5 = -1 \Rightarrow m_3 = 5/2$ = Grad of BC ✓

$\left[m = \frac{y_2-y_1}{x_2-x_1} = \text{Grad of line joining } (x_1,y_1) \text{ and } (x_2,y_2) \right]$

5. Find the equation of a line with gradient -3 and passing through the point $(1,-4)$.

Solution: Gradient $m = -3$, point $(1,-4)$ $\left\{ \begin{array}{l} \text{Point } (x_1, y_1), \text{ grad} = m \\ y - y_1 = m(x - x_1) \end{array} \right.$

Equation of line:

$$y-(-4) = -3(x-1)$$

$$y+4 = -3x+3 \Rightarrow \underline{y = -3x-1}$$

6. Find the equation of line passing through the points $(3,-5)$; $(-2,4)$.

Solution: Given two points; $(3,-5)$, $(-2,4)$ — Two Points (x_1, y_1), (x_2, y_2)

Equation of line: $y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$

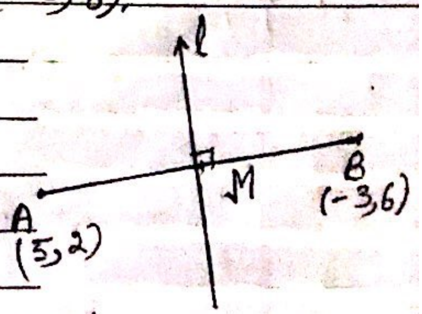
Eqn of line; $y-(-5) = \frac{4-(-5)}{-2-3}(x-3)$

$$y+5 = \frac{9}{-5}(x-3)$$

$$\Rightarrow -5(y+5) = 9(x-3) \Rightarrow -5y-25 = 9x-27$$

$$\Rightarrow \underline{9x+5y = 2}$$

7. Find the equation of the perpendicular bisector of the line segment joining the points $(5,2)$ and $(-3,6)$.



Solution: $A(5,2)$, $B(-3,6)$

Mid point of AB, $M\left(\frac{5+-3}{2}, \frac{2+6}{2}\right)$

$$= (1, 4)$$

Gradient of line AB, $m_1 = \frac{6-2}{-3-5} = \frac{4}{-8} = -\frac{1}{2}$

$\therefore$ Gradient of line l, perp. to AB, $m_2 = -\frac{1}{m_1} = 2$ $[m_1 \times m_2 = -1]$

$\therefore$ Equation of l, the perp. bisector of AB, $(m=2, (1,4))$

$$y - 4 = 2(x-1)$$

$$y - 4 = 2x - 2$$

$$\underline{y = 2x + 2}$$

Scanned with CamScanner