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P.1

# Pure Maths 1

## Functions and transformations Notes

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P-1

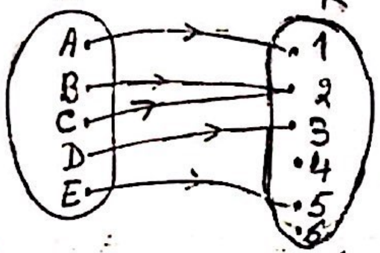
# Functions and Transformations
## Notes



§ **Function:** Given two sets A and B.
A function from set A to set B, denoted by $f: A \to B$, is a mapping (or association) such that each element of set A is associated to a unique (only and only one) element of set B.

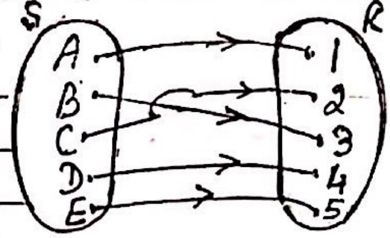
Example 1: Some students who got admission in a "Residential School" are seeking hostel accomodation;
'S' denotes the set of students.
'R' denotes the set of hostel room.

Case (i) $f: S \to R$



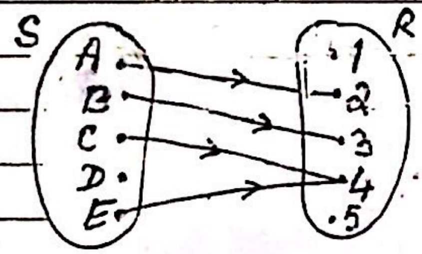

f is many one function

Case (ii) $g: S \to R$



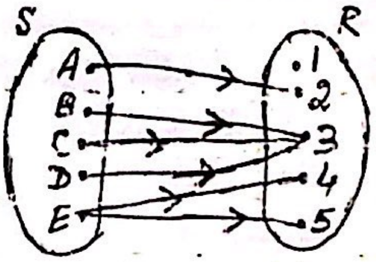

g is one-one function

Case (iii) $h: S \to R$




h is not a function ?

Case (iv) $p: S \to R$




p is not a function ?

§ Domain of a function $f: A \to B$ is set A. (set of input values)
Range of a function $f: A \to B$ is the set of those element B which correspond to the element of set A.

In example 1 (i) Domain of $f = S = \{A, B, C, D, E\}$
Range of $f \subseteq R = \{1, 2, 3, 4, 5\}$

Example 2 (i) Express $x^2-2x-15$ in the form $(x+a)^2+b$ ---[2]

The function $f: x \to x^2-2x-15$; for $p \leq x \leq q$, where $p$ and $q$ are positive constants.

The range of $f$ is given by $c \leq f(x) \leq d$, where $c$ and $d$ are constants.

(ii) State the smallest value of $c$. ---[1]

for the case where $c=9$ and $d=65$

(iii) find $p$ and $q$. ---[4]

$\boxed{W\text{-}14 / 11 / Q\ 10}$

Solution (i) $x^2-2x-15 = x^2-2x+1^2-1-15$

$= \underline{(x-1)^2-16}$ ✓ ---(i) ---[2]

(ii) $f(x) = x^2-2x-15$

$= (x-1)^2-16$ ...(ii) from Part (i)

Now given $c \leq f(x) \leq d$

from (i) the smallest value of $\underline{c=-16}$ ✓

$\left\{\begin{array}{l}\text{as } (x-1)^2 \geq 0 \\ (x-1)^2-16 \text{ has} \\ \text{smallest value} \\ = -16 \text{ at } x=1\end{array}\right.$

(iii) Now given $c \leq f(x) \leq d$

$\Rightarrow 9 \leq f(x) \leq 65$ (for $c=9$ and $d=65$)

$\Rightarrow 9 \leq (x-1)^2-16 \leq 65$ from (ii)

$25 \leq (x-1)^2 \leq 81$

$5^2 \leq (x-1)^2 \leq 9^2$

$5 \leq (x-1) \leq 9$

$\Rightarrow 6 \leq x \leq 10$ $\left[\ p \leq x \leq q\ \right]$

$\therefore p=6$ and $q=10$ ✓ $\left[\ p \text{ and } q \text{ are positive.}\right.$

Example 3: Function $g: x \to 2x^2-6x+5$ for $0 \leq x \leq 4$

(i) Express $g(x)$ in the form $a(x+b)^2+c$, where $a, b, c$ are constants. --[3]

(ii) Find the range of $g$. $\boxed{S\text{-}15 / 12 / Q\ 11}$ --[2]

Solution (i) $2x^2-6x+5$

$= 2[x^2-3x]+5$

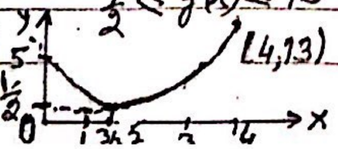
$= 2\left[x^2-3x+\left(\frac{3}{2}\right)^2\right]-\frac{9}{2}+5$

$= \underline{2\left(x-\frac{3}{2}\right)^2+\frac{1}{2}}$ ✓

(ii) Now $g(x) = 2\left(x-\frac{3}{2}\right)^2+\frac{1}{2}$ for $0 \leq x \leq 4$

Range of $g(x)$

$\therefore \frac{1}{2} \leq g(x) \leq 13$




$\left\{\begin{array}{l}\left(x-\frac{3}{2}\right)^2 = 0 \text{ at } x=\frac{3}{2} \\ g(x)=\frac{1}{2} \\ \text{and} \\ g(x)=13 \text{ at } x=4 \\ g(x)=5 \text{ at } x=0\end{array}\right.$



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**Example 4:** The function: $f: x \to 4\sin x - 1$, for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. State the range of $f$.

**Solution:** $f(x) = 4\sin x - 1$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$$\left.\begin{array}{l} -1 \le \sin x \le 1 \text{ for } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ -4 \le 4\sin x \le 4 \\ -5 \le 4\sin x - 1 \le 3 \end{array}\right.$$

Hence $-5 \le 4\sin x - 1 \le 3$

Range of $f(x)$: $-5 \le f(x) \le 3$ ✓

---

**Example 5:** Find the largest possible domain for each of the following function and state the corresponding range.

(a) $f(x) = 4x+1$
(b) $f(x) = x^2 - 5x + 6$
(c) $f(x) = 3^x$
(d) $f(x) = \sqrt{x-4} - 2$
(e) $f(x) = \frac{1}{x-3}$
(f) $f(x) = \frac{1}{x^2-5x+6}$
(g) $f(x) = \cos x$.

**Solution:** (a) $f(x) = 4x+1$
domain $= R$ (set of all real numbers)
Range $= R$.

(b) $f(x) = x^2 - 5x + 6 =$
$= \left(x - \frac{5}{2}\right)^2 - \frac{1}{4}$
Domain $= x \in R$
Range $= f(x) \ge -\frac{1}{4}$

(c) $f(x) = 3^x$
Domain $= R$
Range $= f(x) > 0$

(d) $f(x) = \sqrt{x-4} - 2$
Domain: $x \ge 4$
Range: $f(x) \ge -2$

(e) $f(x) = \frac{1}{(x-3)}$
Domain $= R - \{3\}$
Range $= f(x) \in R,\ f(x) \ne 0$

(f) $f(x) = \frac{1}{(x^2-5x+6)}$
$= \frac{1}{(x-2)(x-3)}$
Domain $= R - \{2,3\}$
Range $= f(x) \in R,\ f(x) \ne 0$

(g) $f(x) = \cos x$
Domain $= x \in R$
Range. $-1 \le f(x) \le 1$.



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## § To check a relation is a function/or not Using its graph:

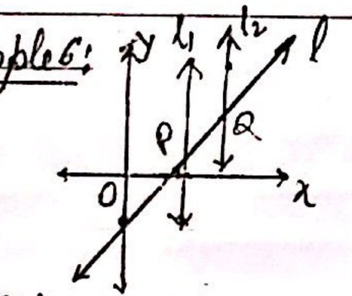
### • Vertical line (line parallel to y-axis) test:

(i) If vertical line intersect the graph exactly at one point then the relation is a function.

(ii) If any line intersects the graph in more than one point then the relation is not a function.

**Example 6:**

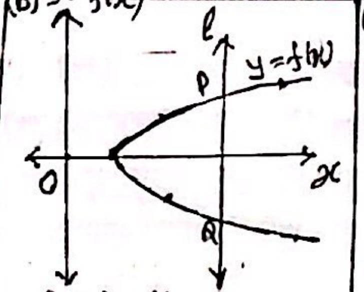
(a)




$f(x) = y = 2x - 1$

The vertical lines $l_1$, $l_2$ intersect the graph of $f(x) \to$ line $l$ at one point only, hence is a function.

(b) $y = f(x)$

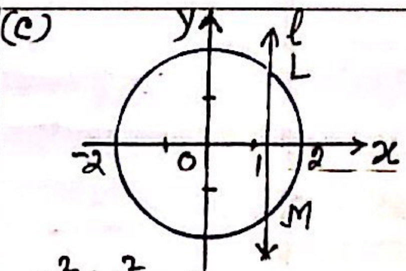



$f(x) = y$.

$y^2 = (x-1)$

vertical line $l$ intersects the graph at two points P and Q; hence is not a function.

(c)

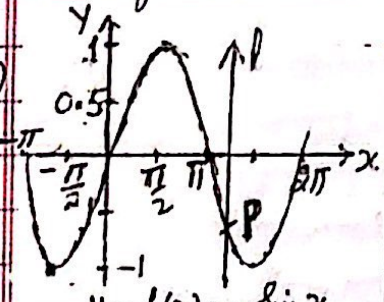



$x^2 + y^2 = 4$

circle $y = \pm\sqrt{4 - x^2}$

Vertical line '$l$' intersects the graph at two points, hence not a function.

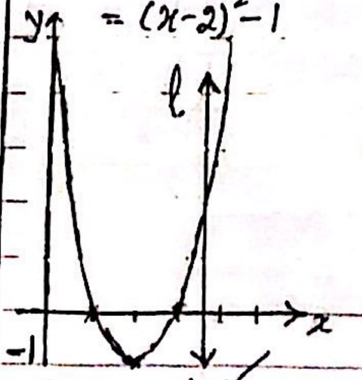
(d)




$y = f(x) = \sin x$

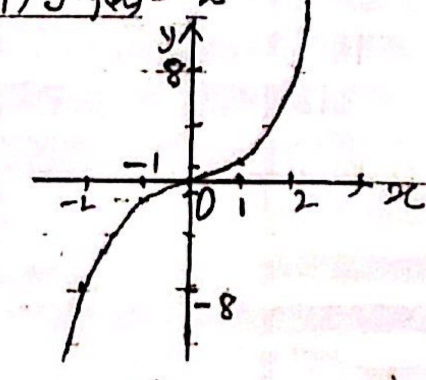
is a function as the vertical line '$l$' intersects the sine-curve at only one point.

(e) $y = f(x) = x^2 - 4x + 3$
$= (x-2)^2 - 1$




is a function as any vertical line intersects the curve at one point.

(f) $y = f(x) = x^3$




is a function



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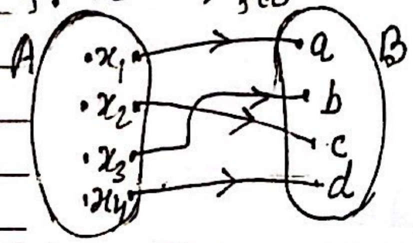
## § One-One functions and Many-one functions:

**One-one function:**

Given a function $y = f(x)$, such that:

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

$f: x \longrightarrow f(x)$




$f(x_1) = a$

f-Image of $x_1 = a$

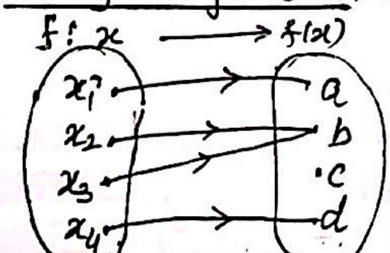
f-Image of $x_2 = b$

Preimage of $a = x_1$

Preimage of $b = x_2$

**Many-one function:**

$f: x \longrightarrow f(x)$




$f(x_2) = f(x_3) = b$

∴ Preimage of $b = x_2$ & $x_3$

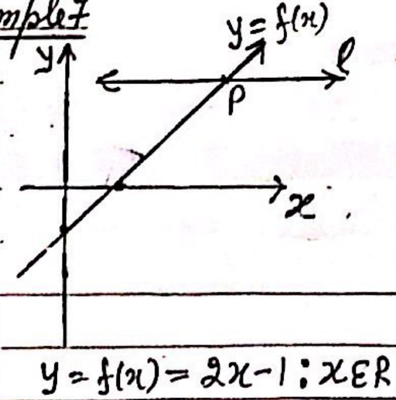
Many-one function.

## § Horizontal line test for one-one function / many-one functions:

If a line parallel to x-axis intersects the graph of $y = f(x)$ at one point then it is a one-one function / otherwise a many-one function.

**Example 7**

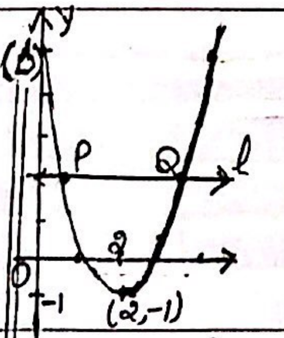
(a)




$y = f(x) = 2x - 1 : x \in R$

If we take any line '$\ell$' parallel to x-axis it intersect the graph at only one point, hence One-One function.

(b)




$y = f(x) = x^2 - 4x + 3 : x \in R$

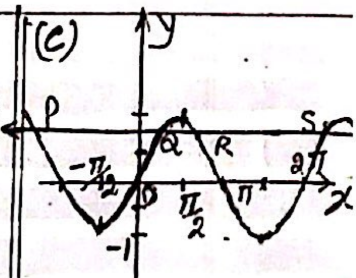
$= (x-2)^2 - 1$

Many-One function.

as a line '$\ell$' parallel to x-axis intersect it at two points P and Q.

Note: If the domain of the function is $x \geq 2$, Then it will be one-one.

(c)




$y = f(x) = \sin x : x \in R$

Many-one function

line '$\ell$' parallel to x-axis intersects the curve at many points P, Q, R, S ----.

Note: If the domain of the function $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ it will be one-One.

Note: An increasing / or decreasing function is one-one in that interval.



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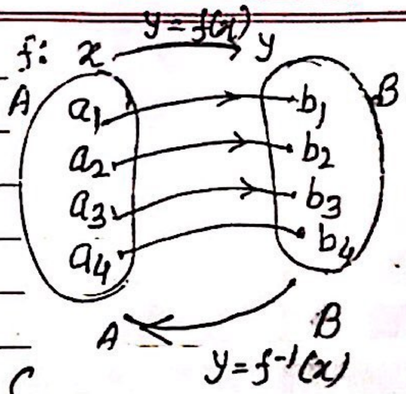
## § Inverse of a function:




Given a function $f: A \to B$
$y = f(x)$

Let $f(a_1) = b_1$
$f(a_2) = b_2$
$f(a_3) = b_3$
$f(a_4) = b_4$

Domain of $f(x) = \{a_1, a_2, a_3, a_4\}$
Range of $f(x) = \{b_1, b_2, b_3, b_4\}$

$$\left\{\begin{array}{l} f^{-1}(b_1) = a_1 \\ f^{-1}(b_2) = a_2 \\ f^{-1}(b_3) = a_3 \\ f^{-1}(b_4) = a_4 \end{array}\right.$$

The inverse of a function $y = f(x)$ (denoted by $f^{-1}(x)$) exists only when the function $y = f(x)$ is One-One function.

Note 1: The domain of $f^{-1}(x)$ is the range of $f(x)$ and conversely.

Note 2: Graph of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ in the line $y = x$.

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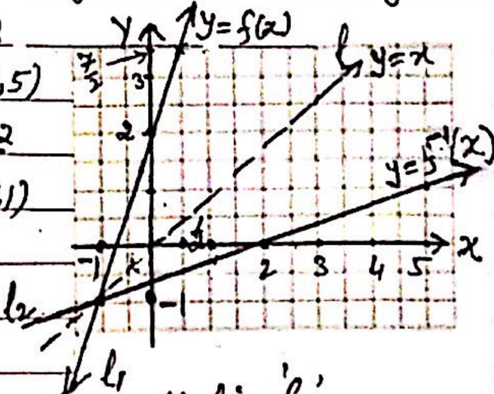
**Example 8:** (i) $f(x) = 3x + 2$ for $x \in R$; find $f^{-1}(x)$.
(ii) Check the graph of $f^{-1}(x)$ is the reflect of the graph of $y = f(x)$ in line $y = x$

**Solution:** Let $y = f(x) = 3x + 2$ is a linear function → one-one function

$y = 3x + 2$
Interchange $x$ & $y$
$x = 3y + 2$
$\Rightarrow y = \dfrac{x-2}{3}$
or $f^{-1}(x) = \dfrac{(x-2)}{3}$

(i) $y = f(x) = 3x + 2$
$(-1,-1), (0,2), (1,5)$
(ii) $y = f^{-1}(x) = \dfrac{x-2}{3}$
$(-1,-1), (2,0), (5,1)$




Check: $\left\{\begin{array}{l} f(1) = 3\times1+2 = 5 \\ f^{-1}(5) = \frac{5-2}{3} = 1 \end{array}\right\}$

The graph of $y = f(x)$ is the line '$l_1$'
The graph of $y = f^{-1}(x)$ is the line '$l_2$'
and the graph of $y = x$ is line '$l$'
Here '$l_2$' is the reflection of '$l_1$' in line '$l$'.



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Example 9: The function f defined by $f(x)=4x^2-24x+11$, for $x\in\mathbb{R}$.

(i) Express $f(x)$ in the form $a(x-b)^2+c$ and hence state the coordinates of the vertex of the graph of $y=f(x)$. ---[4]

The function g is defined by $g(x)=4x^2-24x+11$, for $x\leq 1$

(ii) State the range of g. ---[2]

(iii) Find an expression for $g^{-1}(x)$ and state the domain of $g^{-1}$ ---[4]

[N-12 | 11 | Q10]

Solution: $f(x)=4x^2-24x+11$ ; $x\in\mathbb{R}$

(i)
$$=4[x^2-6x\quad]+11$$
$$=4[x^2-6x+3^2-9]+11$$
$$=4(x-3)^2-36+11$$
$$f(x)=4(x-3)^2-25\text{ ---(i)}$$
Vertex of the graph $(3,-25)$ ✓ ---[4]

(ii) $g(x)=4x^2-24x+11$, for $x\leq 1$
$$=4(x-3)^2-25, \text{ for } x\leq 1$$
Range of g is : $g(x)\geq -9$ ✓ ---[2]

as $[g(1)=-9,\ g(0)=11]$

(iii) To find $g^{-1}(x)$.

Let $y=g(x)=4(x-3)^2-25$

Interchange x and y

$$x=4(y-3)^2-25$$
$$\Rightarrow (y-3)^2=\frac{x+25}{4}$$
$$y-3=\pm\sqrt{\frac{x+25}{2}}$$
$$y=3\pm\frac{1}{2}\sqrt{x+25}\text{ --(ii)}$$
$$\therefore g^{-1}(x)=3-\frac{1}{2}\sqrt{x+25}$$
$$g^{-1}(x)=3-\frac{1}{2}\sqrt{x+25}\ ✓$$
---[4]

Domain of $g^{-1}$ is $x\geq -9$ ✓

To Choose + or −

Domain of $g(x)$ is $x\leq 1$

$\Rightarrow$ "Range of $g^{-1}(x)$ is $x\leq 1$" ⊛

Domain of $g^{-1}(x)$ = Range of $g(x)$
$= g(x)\geq -9$

at $x=-9$

$g^{-1}(x)=3\pm\frac{1}{2}\sqrt{-9+25}=3\pm\frac{1}{2}\cdot 4$
$=3\pm 2$

$g^{-1}(x)=3+2$ ✗ , $3-2=1$ ⊛ $g^{-1}(-9)\leq 1$

---[4]



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**Example 10:** Function $f$ and $g$ are defined by.

$f : x \to 2x^2 - 8x + 10$ for $0 \leq x \leq 2$

$g : x \to x$ for $0 \leq x \leq 10$

(i) Express $f(x)$ in the form $a(x+b)^2 + c$; where $a, b, c$, are Constants. --[3]

(ii) State the range of $f$. ---[1]

(iii) State the domain of $f^{-1}$ --[1]

(iv) Sketch on the same diagram the graphs of $y = f(x)$, $y = g(x)$ and $y = f^{-1}(x)$, making clear the relationship between the graph. ...[4]

(v) Find an expression for $f^{-1}(x)$. [W-11/11/Q11] --[3]

**Solution** (i) $f(x) = 2x^2 - 8x + 10$; for $0 \leq x \leq 2$

$= 2(x^2 - 4x + 5)$

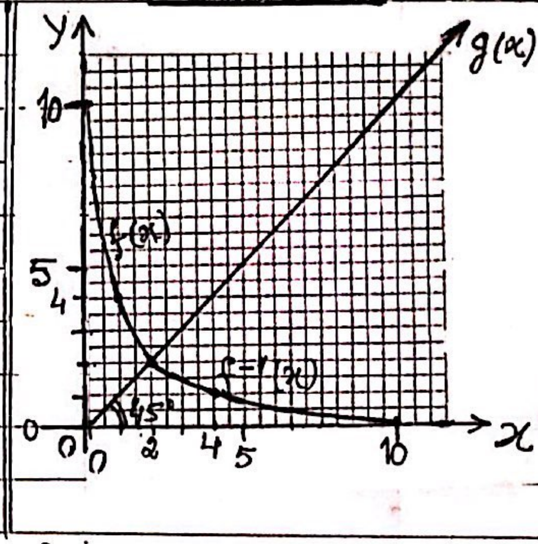
$= 2[(x-2)^2 + 1]$

$= 2(x-2)^2 + 2$. for $0 \leq x \leq 2$

Vertex $(2,2)$ ---[3]

(ii) Range of $f$: $2 \leq f(x) \leq 10$ ---[1]

(iii) Domain of $f^{-1}$: $2 \leq x \leq 10$ ---[1]




(iv) $f(x)$: half parabola from $(0,10)$ to $(2,2)$

$g(x)$: line the line through $O$ at $45°$

$f^{-1}(x)$: reflection of $f(x)$ in $g(x)$. ---[4]

(v) To find $f^{-1}(x)$.

Let $f(x) = y = 2(x-2)^2 + 2$

Interchange $x$ & $y$

$x = 2(y-2)^2 + 2$

$\Rightarrow (y-2)^2 = \frac{x-2}{2}$

$y - 2 = \pm\sqrt{\frac{x-2}{2}}$

$y = 2 \pm \sqrt{\frac{x-2}{2}}$

$f^{-1}(x) = 2 - \sqrt{\frac{x-2}{2}}$ ✓

$2 \leq x \leq 10$

$0 \leq f^{-1}(x) \leq 2$

To choose + or − ve sign

Domain of $f^{-1}$ = range of $f$ = $2 \leq x \leq 10$

Range of $f^{-1}$ = domain $f$ : $0 \leq f^{-1}(x) \leq 2$

Now $f^{-1}(2) = 2 - 0 = 2$

$f^{-1}(10) = 2 - 2 = 0$

$\therefore$ − ve sign. $0 \leq f^{-1}(x) \leq 2$

---[3]



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**Example 11:** The function $f: x \to x^2 - 4x + k$, for $x \geq p$ where $k$ and $p$ are constants.

(i) Express $f(x)$ in the form $(x+a)^2 + b + k$, where $a$ and $b$ are constants. ...[2]

(ii) State the range of $f$ in term of $k$. ...[1]

(iii) State the smallest value of $p$ for which $f$ is one-one. ...[1]

(iv) For the value of $p$ found in part (iii), find an expression for $f^{-1}(x)$ and state the domain of $f^{-1}$, giving your answers in terms of $k$. ...[4]

[S-12/11/Q8]

**Solution:** $f(x) = x^2 - 4x + k$ for $x \geq p$

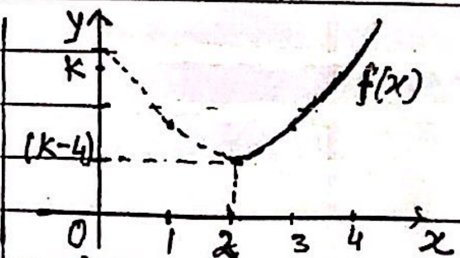
(i) $f(x) = x^2 - 4x + 4 - 4 + k$

$= (x-2)^2 + (k-4)$ ....(i) ....[2]

(ii) Range of $f$: $f(x) \geq (k-4)$ $\quad (\because (x-2)^2 \geq 0)$

(iii) from (i) vertex of parabola $(2, k-4)$

$\therefore$ Smallest value of $p = 2$.




(iv) $f(x) = (x-2)^2 + (k-4)$ ; $x \geq 2$ $\quad$ {from part (iii) $p = 2$}

To find $f^{-1}(x)$.

let $y = (x-2)^2 + (k-4)$

Interchange $x$ and $y$.

$x = (y-2)^2 + (k-4)$

$\Rightarrow (y-2)^2 = x - (k-4)$

$y - 2 = \pm\sqrt{x-(k-4)}$

$\Rightarrow y = 2 \pm \sqrt{x-(k-4)}$

$\therefore f^{-1}(x) = 2 + \sqrt{x-(k-4)}$ ✓

{ To choose + or − sign.
Domain of $f^{-1}$ = Range of $f(x)$
$x \geq (k-4)$
Range of $f^{-1}$ = Domain of $f(x)$
$\rightarrow f^{-1}(x) \geq 2$
$\therefore$ + sign. }

**Domain of $f^{-1}(x)$ is $x \geq (k-4)$** ✓

---[4]



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## § Composite Function:

Given two functions, $f: A \rightarrow B$

Case I: and $g: C \rightarrow D$

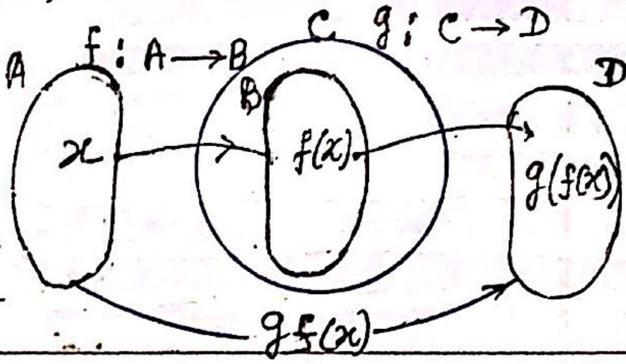
Now the composite function $gf(x)$ is $g[f(x)]$

is defined as $g$ of $f(x)$.

In this case "the range of $f(x)$ is subset of domain of $g$"

Case II: $fg(x) = f[g(x)]$

"here the range of $g(x)$ should be the subset of the domain of $f(x)$"




Note: In general: $gf(x) \neq fg(x)$

---

Example 12: Functions $f$ and $g$ are defined by:

$f(x) = 10 - 3x$ for $x \in R$.

and $g(x) = \dfrac{10}{3-2x}$, $x \in R$, $x \neq 3/2$

solve the equation: $ff(x) = gf(2)$ ---[3]

[S-16/12/ Q1]

Solution: $ff(x) = f(10-3x) = 10 - 3(10-3x) = (9x-20)$ --(i)

and $gf(2) = g[f(2)] = g(10-3\times2) = g(4) = \dfrac{10}{3-2\times4} = \dfrac{10}{-5} = -2$ --(ii)

Now Given $ff(x) = gf(2)$

$\Rightarrow 9x - 20 = -2$ (from (i) and (ii))

$\Rightarrow 9x = 18$

or $x = \underline{2}$ ✓ ---[3]



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**Example 13:** The function f is defined as, $f(x) = 3x+1$, for $x \leqslant a$, where $a$ is a constant.
The function g is defined by. $g(x) = -1-x^2$ for $x \leqslant -1$.
(i) Find the largest value of $a$ for which the composite function $gf$ can be formed. ---[2]
for the case where $a=-1$
(ii) Solve the equation $fg(x) + 14 = 0$ ---[3]
(iii) Find the set of values of $x$ which satisfy the identity $gf(x) \leqslant -50$. ---[4]

[W-15/13/Q8]

**Solution** (i) To form the function $gf$, the range of $f$ shuld be the subset of the domain of $g$. ---[2]

Given
$$\begin{cases} f(x) = 3x+1, & x \leqslant a \\ g(x) = -1-x^2, & x \leqslant -1 \end{cases}$$

Range of $f(x)$: $3a+1 \leqslant -1$
$\Rightarrow a \leqslant -\frac{2}{3}$ ---(i)
$\therefore$ Largest value of $a = -2/3$

(ii) Now for $a=-1 \Rightarrow f(x) = 3x+1$ for $a \leqslant -1$ ---(ii)

Solve $fg(x) + 14 = 0$
$\Rightarrow f(-1-x^2) + 14 = 0$
$\Rightarrow 3(-1-x^2) + 1 + 14 = 0$
$\Rightarrow 3x^2 = 12 \Rightarrow x^2 = 12$
$\Rightarrow x^2 = -2$ or $\cancel{+2}$ (as domain of $g(x), x \leqslant -1$)

(iii) $gf(x) \leqslant -50$
$\Rightarrow g(3x+1) \leqslant -50$
$-1-(3x+1)^2 \leqslant -50$
$-(3x+1)^2 \leqslant -49$
$\Rightarrow (3x+1)^2 \geqslant 49$
$(3x+1)^2 \geqslant 7^2$
$\Rightarrow 3x+1 \leqslant -7$ or $3x+1 \geqslant 7$
$\Rightarrow x \leqslant -\frac{8}{3}$ or $\cancel{x \geqslant 2}$ ($\because a \leqslant -\frac{2}{3}$
Domain of $f(x)$
$x \leqslant -2/3$)



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Example 14: The functions $f$ and $g$ are such that, $f(x)=2x+3$ for $x\geq 0$
and $g(x)=ax^2+b$ for $x\leq q$

where $a$, $b$ and $q$ are constants.

The function $fg$ is such that: $fg(x)=6x^2-21$ for $x\leq q$

(i) Find the value of $a$ and of $b$. ---[3]

(ii) Find the greatest possible value of $q$. ---[2]

It is given that $q=-3$

(iii) Find the range of $fg$ ---[1]

(iv) Find an expression for $(fg)^{-1}(x)$ and state the domain of $(fg)^{-1}$ --[3]

[S-16/13/Q10]

Solution: $f(x)=2x+3$ for $x\geq 0$ ; and $g(x)=ax^2+b$ for $x\leq q$
and $fg(x)=6x^2-21$ for $x\leq q$ ---(1)

(i) $fg(x)=f(ax^2+b)=2(ax^2+b)+3=6x^2-21$ from (1)

$\Rightarrow 2ax^2+(2b+3)=6x^2-21$

$\Rightarrow \begin{cases} 2a=6 \\ 2b+3=-21 \end{cases} \Rightarrow \begin{cases} a=3 \\ b=-12 \end{cases}$ ...[3]

(ii) Now $g(x)=ax^2+b=3x^2-12$

{as $fg(x)$ is defined only when range of $g(x)$ is subset of the domain of $f(x)$ or $x\geq 0$

$\Rightarrow 3x^2-12\geq 0$

$\Rightarrow x^2\geq 4$

$\Rightarrow x^2\geq 2^2$

$\Rightarrow x\leq -2$ or $x\geq 2$

$\therefore$ greatest value of $q=-2$

[Now $fg(x)=6x^2-21$ for $x\leq q$
$x\leq q$

---[2]

(iii) For range of $fg$, now $q=-3$

$fg(x)=6x^2-21$ for $x\leq -3$

$6x^2-21\geq 6(-3)^2-21$

$(=54-21)$

$(=33)$

Range of $fg$.

$\therefore fg(x)\geq 33$ ---[1]

(iv) for $(fg)^{-1}(x)$

Let $y=6x^2-21$ ...[3]

Interchange $x$ & $y$

$x=6y^2-21\Rightarrow 6y^2=x+21$

$y=\pm\sqrt{\dfrac{x+21}{6}}$

$y=-\sqrt{\dfrac{x+21}{6}}$

$\therefore (fg)^{-1}x=-\sqrt{\dfrac{x+21}{6}}$

{Domain of $(fg)^{-1}$ = range of $fg\geq 33$
and range of $(fg)^{-1}$ = is domain of $fg$; $x\leq -3$
$\therefore (fg)^{-1}(x)\leq -3$
$\therefore$ -ve sign



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15. A function $f$ is defined by, $f: x \mapsto 3 - 2\tan\left(\frac{1}{2}x\right)$ for $0 \le x \le \pi$

(i) State the range of $f$. --- [1]

(ii) State the exact value of $f\left(\frac{2}{3}\pi\right)$ -- [1]

(iii) Sketch the graph of $y = f(x)$ --- [2]

(iv) Obtain an expression, in terms of $x$, for $f^{-1}(x)$ -- [3]

[W-10/11/Q7]

Solution (i) $f(x) = 3 - 2\tan\left(\frac{x}{2}\right)$ for $0 \le x \le \pi$

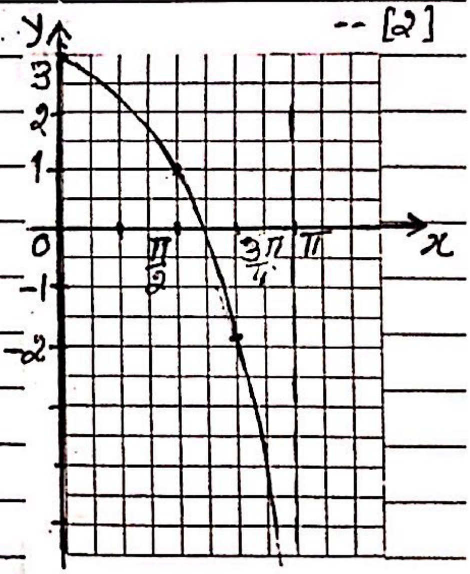
Range of $f(x)$: $f(x) \le 3$ --- [1]

(ii) $f\left(\frac{2}{3}\pi\right) = 3 - 2\tan\left(\frac{\pi}{3}\right) = 3 - 2\sqrt{3}$ --- [1]

(iii) $y = 3 - 2\tan\left(\frac{1}{2}x\right)$

| $x$ | 0 | $\pi/2$ | $3\pi/4$ | $\pi$ |
|---|---|---|---|---|
| $y$ | 3 | 1 | $\approx -1.8$ | $-\infty$ |

-- [2]




(iv) To find $f^{-1}(x)$.

Let $y = 3 - 2\tan\left(\frac{1}{2}x\right)$

Interchange $x$ & $y$

$x = 3 - 2\tan\left(\frac{1}{2}y\right)$

$\Rightarrow \tan\frac{y}{2} = \left(\frac{3-x}{2}\right)$

$\Rightarrow y = 2\tan^{-1}\left(\frac{3-x}{2}\right)$

$\Rightarrow f^{-1}(x) = 2\tan^{-1}\left(\frac{3-x}{2}\right)$ ✓ --- [3]



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16. The function $f: x \rightarrow 5+3\cos(\frac{1}{2}x)$, for $0 \leq x \leq 2\pi$

(i) Solve the equation $f(x)=7$, giving your answer correct to decimal places. ---[3]

(ii) Sketch the graph of $y=f(x)$ ---[2]

(iii) Explain why $f$ has an inverse. --[1]

(iv) Obtain an expression for $f^{-1}(x)$. ---[3]

Solution: $f(x) = 5+3\cos(\frac{1}{2}x)$, for $0 \leq x \leq 2\pi$

(i) Solve the equation $f(x)=7$

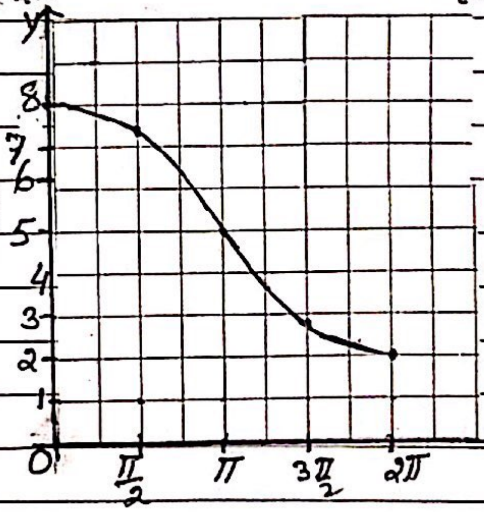
$$\Rightarrow 5+3\cos(\tfrac{1}{2}x)=7$$

$$\Rightarrow \cos\tfrac{1}{2}x = \tfrac{2}{3} \quad : 0 \leq \tfrac{x}{2} \leq \pi$$

$$\tfrac{1}{2}x = \cos^{-1}2/3 = 0.841 \Rightarrow \underline{x = 1.68 \text{ radians}}$$

(ii) Graph. $y=5+3\cos(\frac{1}{2}x)$, for $0 \leq x \leq 2\pi$ --[2]

| $x$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| $y$ | 8 | 7.4 | 5 | 2.87 | 2 |




(iii) $f$ is one-one (decreasing function in the given interval)

(iv) To find $f^{-1}(x)$

Let $y = 5+3\cos(\frac{1}{2}x)$

Interchange $x$ & $y$

$$x = 5+3\cos(\tfrac{1}{2}y)$$

$$\cos\left(\frac{y}{2}\right) = \left(\frac{x-5}{3}\right)$$

$$y = 2\cos^{-1}\left(\frac{x-5}{3}\right)$$

$$\therefore \underline{f^{-1}(x) = 2\cos^{-1}\left(\frac{x-5}{3}\right)}$$ ---[3]



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# Transformations of functions and Graphs:



§

## 1 Translation transformation:

The shape remains the same.

### (i) Translation along x- axis:

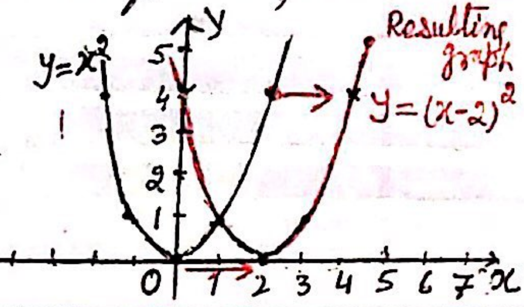
Given a function $y = f(x)$, then $y = f(x-a)$ is a translation of the given graph by the vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$

Example: The graph of $y = x^2$ is translated to the right by 2 units (or by vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$). Find the equation of the resulting graph.

Solution: Given $y = x^2$ and a translation by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

The equation of the new graph is $y = (x-2)^2$

or $y = x^2 - 4x + 4$ ✓




Shift on the right by 2 units

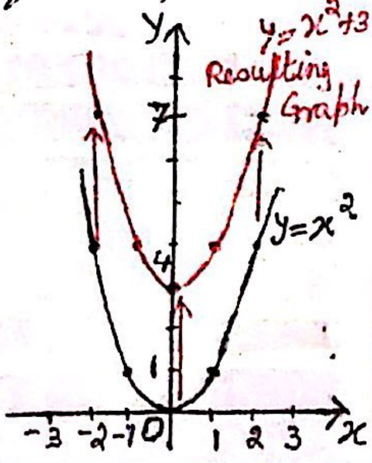
### (ii) Translation along y- axis:

Given a function $y = f(x)$, then $y = f(x) + b$ represent the translation of the given graph by vector $\begin{pmatrix} 0 \\ b \end{pmatrix}$.

Example: The graph of $y = x^2$ is translated upwards along y- axis by 3 units (or by vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$). Find the equation of the resulting graph.

Solution: Given $y = x^2$ and a translation by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

The equation of the new graph is $y = x^2 + 3$ ✓



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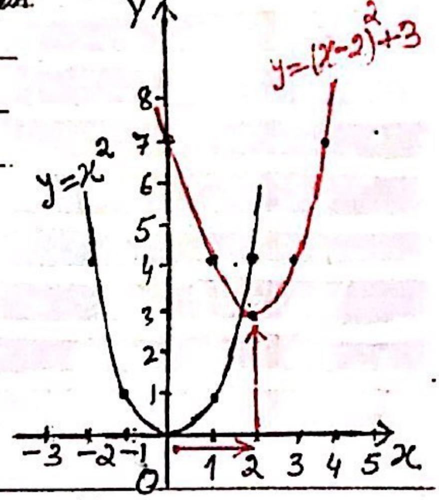
1(iii) Translation along x-axis and y-axis both:

Given a function $y = f(x)$ and the graph is translated by a units along x-axis and b units along y-axis is given by
$y = f(x-a) + b$. [Translation: $\begin{pmatrix} a \\ b \end{pmatrix}$]

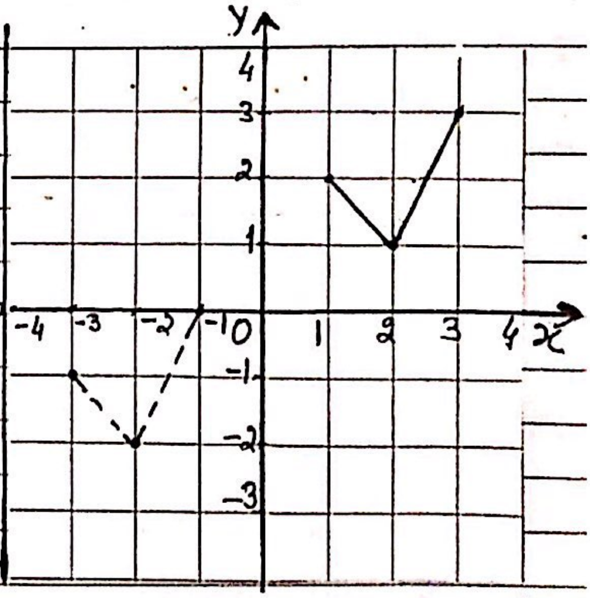



Example: Given a curve $y = x^2$ and a translation $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, the equation of the new graph is given by:

$$y = (x-2)^2 + 3$$

Example 17:

In the graph the solid line shows the graph of $y = f(x)$.
The graph shown with a broken line is a transformation of $y = f(x)$.
State in terms of f, the equation of the graph with broken line. [2]
[S-20/13/Q3(c)]




Solution: The broken line graph is a translation of $y=f(x)$ graph by 4 units in negative direction of x-axis and -3 units along y-axis.
∴ Eqn of the broken line graph is

[Translation by $\begin{pmatrix} a \\ b \end{pmatrix}$ is $y = f(x-a)+b$.]

$$y = f(x-(-4)) + (-3)$$

or $y = f(x+4) - 3$ or $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$ vector

Example 18: A cubic graph has equation, $y = (x+3)(x-2)(x-5)$.
Write the equation of the graph after a translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Solution: The equation of the graph after translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, is; $y = (x+3-2)(x-2-2)(x-5-2)$

or $y = (x+1)(x-4)(x-7)$ ✓

{as $y = f(x)$ → $y = f(x-a)$ after translation $\begin{pmatrix} a \\ 0 \end{pmatrix}$}

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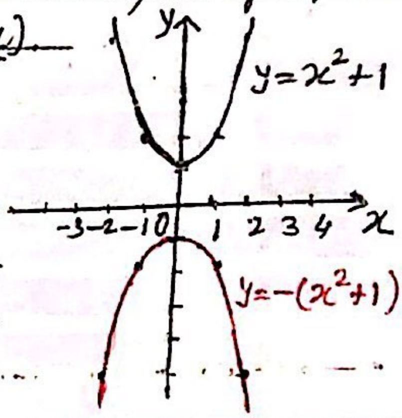
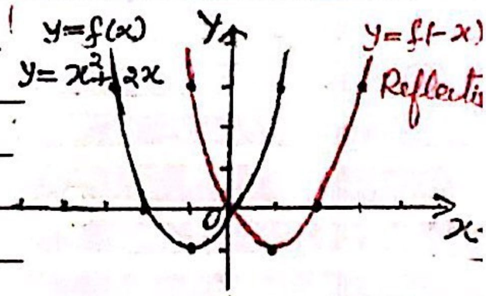
## § Reflection Transformation:

(i) Given a function $y = f(x)$; Then the equation of the graph of reflection in $x$-axis of $y = f(x)$ is $y = -f(x)$.




Example: Given the equation of a curve $y = x^2 + 1$. Then the equation of its reflection in X-axis is
$$y = -(x^2+1)$$

(ii) The equation of the reflection of $y = f(x)$ in Y-axis is $y = f(-x)$.




Example: Given a function $y = x^2 + 2x$ write the equation of the curve which is refection of $y = x^2 + 2x$ in Y-axis.

The required equation is $y = f(-x)$

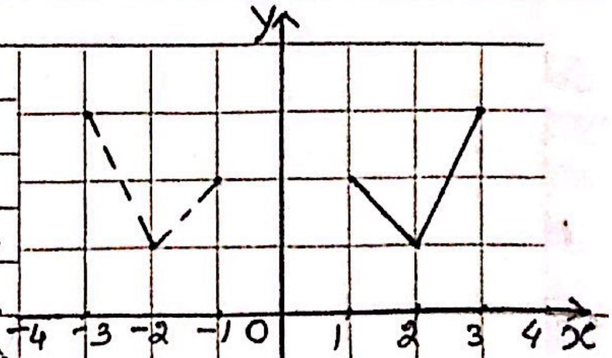
$\Rightarrow y = (-x)^2 + 2(-x)$

$y = x^2 - 2x$

For Graph: $\left\{ \begin{array}{l} y = x^2 + 2x = (x+1)^2 - 1 \\ \text{Vertex} (-1, -1) \end{array} \right.$

Reflection $\left\{ \begin{array}{l} y = x^2 - 2x = (x-1)^2 - 1 \\ \text{Vertex} (1, -1) \end{array} \right.$

Example 19: The graph shown with solid lines has equation $y = f(x)$. The graph shown with broken lines is a transformation of $y = f(x)$. State in terms of $f$, the equation of the graph shown with broken lines.




Solution: The equation of broken line is
$y = f(-x)$ ---[1] [S-20/13/Q 13(a)]
(Reflection in Y-axis)



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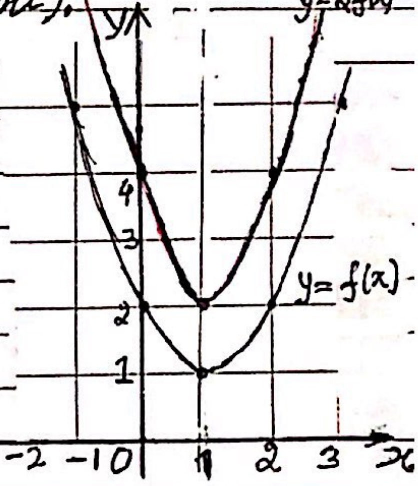
## § Stretch transformation:

(i) Vertical stretch (Stretch along y-axis), x-axis invariant.

Given a function $y = f(x)$, and its graph.
The graph of $y = a \cdot f(x)$ denotes the stretch of the graph of $y = f(x)$ along y-axis (x-axis invariant).
Stretch factor 'a'.

Example: $y = x^2 - 2x + 2 = (x-1)^2 + 1$
Vertex $(1,1)$

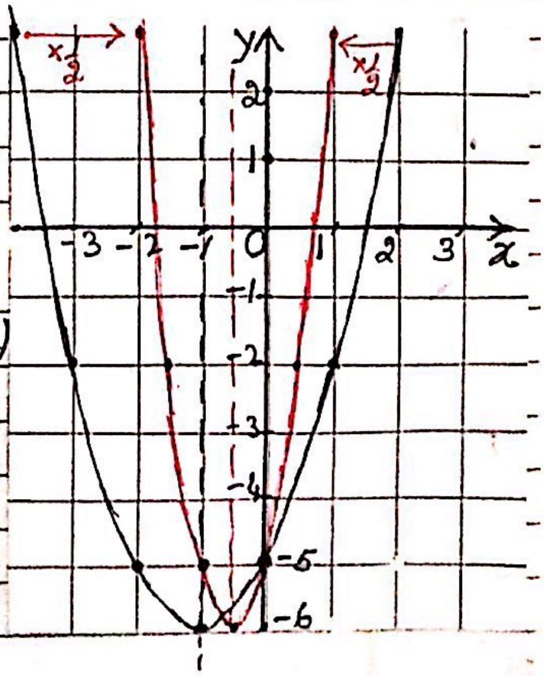
Then $y = 2[x^2 - 2x + 2]$ denotes the stretch along y-axis (x-axis invariant)
or $y = 2(x-1)^2 + 2$ vertex $(1,2)$




(ii) Stretch parallel to x-axis (y-axis invariant):
Given a function $y = f(x)$ and it graph,
The graph of $y = f(\frac{1}{a}x)$ is a stretch of the graph $y = f(x)$ with stretch factor $a$ parallel to the x-axis.

Example:
Given $y = x^2 + 2x - 5$
Find the equation after a stretch parallel to the x-axis with stretch factor $\frac{1}{2}$.

Solution: $y = x^2 + 2x - 5$ & $y = (x+1)^2 - 6$.
Eqn after the stretch parallel to the x-axis, stretch factor $\frac{1}{2}$ i.e. $y = f(2x)$
∴ Required eqn.
$y = (2x)^2 + 2 \cdot (2x) - 5$
$y = 4x^2 + 4x - 5$ ✓
$y = 4[(x + \frac{1}{2})^2 - \frac{6}{4}]$



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# Combined Transformations



Example 20(a) Express $x^2+6x+5$ in the form $(x+a)^2+b$, where $a$ and $b$ are constant. ---[2]

(b) The curve with equation $y=x^2$ is transformed to the curve with equation $y=x^2+6x+5$. Describe fully the transformations involved. ---[2]

[W-20/13/Q1]

Solution (a) $x^2+6x+5 = x^2+6x+3^2-9+5$

$= (x+3)^2-4$ ----(i) ✓ --[2]

(b) Given curve $y=x^2$, transformed to $y=x^2+6x+5$

or $y=(x+3)^2-4$ from (i)

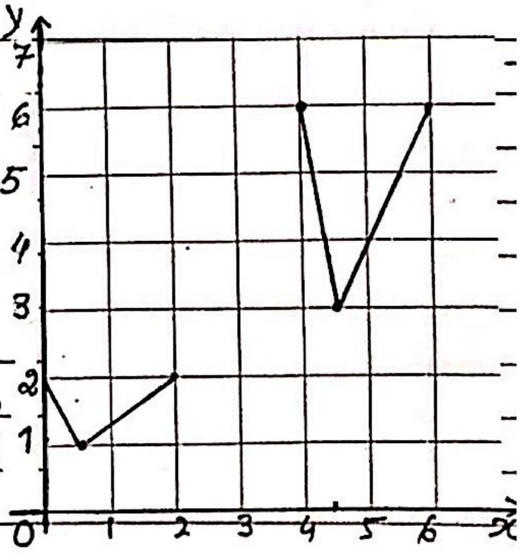
The transformation is translation by vector $\begin{pmatrix}-3\\-4\end{pmatrix}$ ✓

(or Translation of $-3$ units along $x$-axis
and translation of $-4$ units in $y$-direction)

Example 21: In the diagram, the graph of $y=f(x)$ is shown with the solid lines. The graph shown with broken lines is a transformation of $y=f(x)$

(a) Describe two single transformations of $y=f(x)$, that have been combined to give the resulting transformation. --[4]

(b) State in terms of $y$, $f$ and $x$, the equation of the graph shown with broken lines --[2]




[M-21/12/Q5]

Solution (a) Translation by $\begin{pmatrix}4\\0\end{pmatrix}$, followed by stretch parallel to $y$-axis by scale factor 3. ---[4]

(b) $y = 3f(x-4)$ --[2]



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Example 22(a) The curve $y=x^2+3x+4$ is translated by $\begin{pmatrix}2\\0\end{pmatrix}$.
find and simplify the equation of the translated curve. ---[2]
(b) The graph of $y=f(x)$ is translated to the graph of $y=3f(-x)$.
Describe fully the two single transformations which have been combined to give the resulting transformation. ---[3]
[SP-20/01/Q5]

Solution (a) $y=x^2+3x+4$ is translated by $\begin{pmatrix}2\\0\end{pmatrix}$.
The equation of the translated curve is:

$$y=(x-2)^2+3(x-2)+4$$

$$y=x^2-4x+4+3x-6+4$$

or $\underline{y=x^2-x+2}$ ✓ ---[2]

(b) Now $y=3f(-x)$ represents two transformations:
(i) Reflection in y-axis.
(ii) Stretch in y-direction, stretch factor 3. ---[3]

Example 23: The graph of $y=f(x)$ is transformed to the graph of $y=1+f\left(\frac{x}{2}\right)$.
Describe fully the two single transformations which have been combined to give the resulting transformation. ---[4]
[M-20/12/Q2]

Solution: $y=1+f\left(\frac{x}{2}\right)$ has two single transformations
(i) Stretch in x-direction (y-axis invariant) with stretch factor 2.
(ii) Translation in y-direction by 1 unit (or $\begin{pmatrix}0\\1\end{pmatrix}$)



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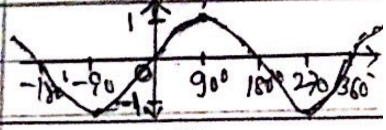
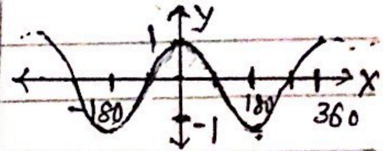
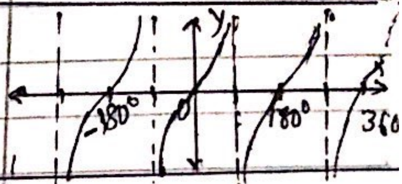
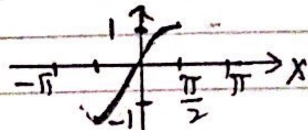
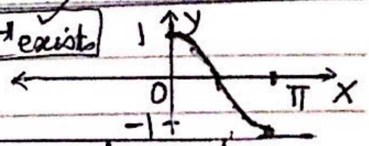
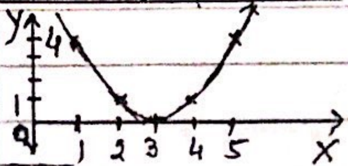
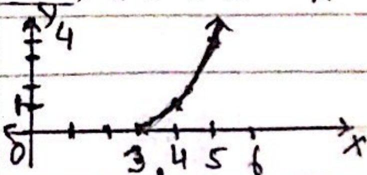
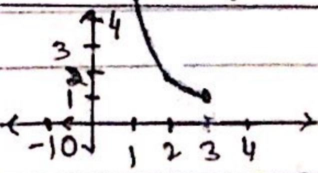
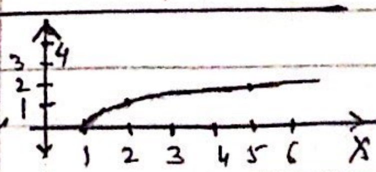
P1

## Functions and Graph:

| | Function | Domain | Range | One-One / Many-One | Graph |
|---|---|---|---|---|---|
| 1. | $f(x) = \sin x^\circ$ | $x \in R$ | $-1 \leq f(x) \leq 1$ | Many one<br>$f(30^\circ) \to \frac{1}{2}$<br>$f(150^\circ) \nearrow$ |  |
| 2. | $f(x) = \cos x^\circ$ | $x \in R$ | $-1 \leq f(x) \leq 1$ | Many-one<br>$f(30) = \frac{\sqrt{3}}{2}$<br>$\cos(330^\circ) \nearrow$ |  |
| 3. | $f(x) = \tan x^\circ$ | $R - \{180n \pm 90^\circ\}$, $n \in I$ | $-\infty < f(x) < \infty$ | Many one |  |
| 4. | $f(x) = \sin x$ | $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ | $-1 \leq f(x) \leq 1$ | One-One<br>$\boxed{f^{-1} \text{ exist}}$ ✓<br>(Increasing function) |  |
| 5. | $f(x) = \cos x$ | $0 \leq x \leq \pi$ | $-1 \leq f(x) \leq 1$ | One-One<br>(decreasing function)<br>$\boxed{f^{-1} \text{exist}}$ ✓ |  |
| 6. | $f(x) = \tan x$ | $-\frac{\pi}{2} < x < \frac{\pi}{2}$ | $-\infty < f(x) < \infty$ | One-One<br>$\boxed{f^{-1} \text{ exist}}$ ✓<br>(Increasing funct) |  |
| 7(i) | $f(x) = (x-3)^2$ | $x \in R$ | $f(x) \geq 0$ | Many-one |  |
| 7(ii) | $f(x) = (x-3)^2$ | $x \geq 3$ | $f(x) \geq 0$ | One-One<br>(Increasing funct)<br>$\boxed{f^{-1} \text{ exist}}$ ✓ |  |
| 7(iii) | $f(x) = (x-3)^2 + 1$ | $x \leq 3$ | $f(x) \geq 1$ | One-One<br>(decreasing fn)<br>$\boxed{f^{-1} \text{exist}}$ ✓ |  |
| 8(i) | $f(x) = \sqrt{x-1}$ | $x \geq 1$ | $f(x) \geq 0$ | One-One<br>(Increasing fn)<br>$\boxed{f^{-1} \text{ exists}}$ ✓ |  |
| (ii) | $y^2 = x$<br>$y = \pm\sqrt{x}$ | $x \geq 0$ | $f(x) \in R$ | |  |

$\boxed{\text{Not a function}}$

as $f(4) \to 2, -2$

$\boxed{\text{One-to Many}}$